Research Article

Noufe H. Aljahdaly, Amjad F. Alyoubi, and Aly R. Seadawy*

Solitary wave solutions of the ionic currents along microtubule dynamical equations via analytical mathematical method

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Abstract: In this article, a new generalized exponential rational function method (GERFM) is employed to extract new solitary wave solutions for the ionic currents along microtubules dynamical equations, which is very interested in nanobiosciences. In this article, the stability of the solutions is also studied. As a result, a variety of solitary waves are obtained with free parameters such as periodic wave solution and dark and bright solitary wave solutions. The solutions are plotted and used to describe physical phenomena of the problem. The work shows the power of GERFM. We found that the proposed method is reliable and effective and gives analytical and exact solutions.

Keywords: ionic currents along microtubule dynamical, GERFM, exact solitary wave solutions, stability analysis

1 Introduction

A wide range of physical phenomena and engineering, biological, and medical applications are described by non-linear evaluation equations (NLEEs). In order to well understand the dynamics of these applications, it is important to find analytical solutions [1–10]. Recently, analytical computational methods have attracted attention of researchers and mathematicians [11–20]. They have developed very powerful and useful methods such as the (G'/G) -expansion method [21,22], and (G'/G^2) -expansion method [23], the

sine-cosine method [24], the tanh method [25], the direct algebraic method [26–28], the modified Jacobi elliptic function method [29], the extended modified auxiliary equation mapping method [30], the modified Kudryashov method [31], and the $\exp(-\phi(\eta))$ -expansion method [32]. In 2018, Ghanbari and Inc provided a novel technique that is the generalized exponential rational function method (GERFM). The method introduces a variety of new travelling wave solutions for some NLEEs. In this study, GERFM is used to examine the ionic currents along microtubule dynamical equations describing microtubule dynamic.

Microtubules are the largest component of the cellular structure cylindrical polymers and they are formed from tubulin protein. A microtubule contains 13 parallel protofilaments. It plays an important role in cellular movement and transport [33,34]. The dynamic system of ion currents in microtubes involves ion transportation throughout the environment within cells. The function of microtubule based on their ability to switch between states of growth and shrinkage [35,36]. GTP caps at their ends stabilizes growing microtubules. Nanowire cable in electrostatics of nanosystems is considered as one of the applications of microtubules, and it plays a vital role in morphogenesis [37,38].

In this article, the transmission line models of nano-ionic currents along microtubules are studied that is given by ref. [39]

$$2\frac{\partial f}{\partial x} + \frac{L^2}{3}\frac{\partial^3 f}{\partial x^3} + \frac{ZC_0}{L}\frac{\partial f}{\partial t} + \left(\frac{\beta Z^{3/2}G_0}{L} - \frac{2bZ^{3/2}C_0}{L}\right)f\frac{\partial f}{\partial t} = 0, \quad (1)$$

where $L = 8 \times 10^{-9}$ m is the length of an elementary ring (ER), $C_0 = 1.8 \times 10^{-15}$ F is the total maximal capacitance of the ER, $Z = 5.56 \times 10^{10}$ Ω is the characteristic impedance of the system, and $G_0 = 1.1 \times 10^{-13}$ Si is the conductance of pertaining nano-pores (NPs). The parameters β and b describe the nonlinearity of conductance of NPs in ER and ER capacitor, respectively. Our objective is studying the effect of positive and negative nonlinear term and dispersion term travelling wave solution.

* **Corresponding author: Aly R. Seadawy**, Department of Mathematics, Faculty of science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia, e-mail: aabdelalim@taibahu.edu.sa, Aly742001@yahoo.com

Noufe H. Aljahdaly, Amjad F. Alyoubi: Department of Mathematics, Faculty of Sciences and Arts - Rabigh Campus, King Abdulaziz University, Jeddah, Saudi Arabia

The article is arranged as follows: In Section 2, the steps of GERFM are described briefly and for more details, the readers are referred to in ref. [40,41]. In Section 3, the exact solutions of the transmission line models of nano-ionic currents along microtubules via GERFM are derived. In Section 4, the new solutions are discussed and compared with the results in previous work via different methods. In Section 5, the stability of the system is investigated. Section 6 is a short summary.

2 Description of the GERFM

This section briefly explains the fundamental steps of the GERFM:

1. Consider the nonlinear partial differential equation expressed by

$$N(f, f_t, f_x, f_{xx}, \dots) = 0. \quad (2)$$

2. Transforming the function $f(x, t)$ to $f(\xi)$ by applying the travelling wave transformation $\xi = \frac{x}{L} - \frac{k}{\tau}t$, where k is the velocity. The system of equation (2) is transferred to the following ODE:

$$N\left(f, \frac{-k}{\tau}f_\xi, \frac{1}{L}f_\xi, \frac{1}{L^2}f_{\xi\xi}, \dots\right) = 0. \quad (3)$$

3. Suppose that the solution of equation (3) has the following form:

$$f(\xi) = A_0 + \sum_{i=1}^N A_i \phi(\xi)^i + \sum_{i=1}^N B_i \phi(\xi)^{-i}, \quad (4)$$

where the coefficients A_0, A_i, B_i ($1 \leq i \leq N$) are constants and ϕ is expressed as follows:

$$\phi(\xi) = \frac{p_1 \exp(\xi q_1) + p_2 \exp(\xi q_2)}{p_3 \exp(\xi q_3) + p_4 \exp(\xi q_4)}, \quad (5)$$

where p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4 are real (or complex) constants. In addition, the positive integer N can be determined by the balancing principle between the higher derivative and the nonlinear terms in equation (3).

4. Substituting function (4) into equation (3) and collecting all terms. Then, the left-hand side of equation (3) is converted to an algebraic equation $P(G_1, G_2, G_3, G_4) = 0$ in terms of $G_n = e^{q_n \xi}$ for $n = 1, \dots, 4$.
5. Equating every coefficient of P to zero to construct a system of nonlinear equations.
6. Solving the algebraic equations in step 3 with the help of symbolic computation, such as Maple or Mathematica, to determine the values of p_n, q_n ($1 \leq n \leq 4$), and $\frac{1}{L}, \frac{k}{\tau}, A_0, A_1, A_2, B_1, B_2$.

3 Application of GERFM to the ionic currents along microtubule dynamical equation

The GERFM is applied to find new traveling wave solutions of equation (1). First step, we apply the following travelling wave transformation:

$$f(x, t) = f(\xi), \quad \xi = \frac{x}{L} - \frac{k}{\tau}t, \quad (6)$$

where $\tau = R_1 C_0 = 0.6 \times 10^{-6} \text{ s}$ and k is the dimensionless velocity of the wave.

Equation (1) transforms into the following nonlinear ordinary differential equation (ODE):

$$\frac{(3k)(2bC_0 Z^{3/2} - \beta G_0 Z^{3/2})}{2\tau} f(\xi)^2 - 3\left(\frac{C_0 k Z}{\tau} - 2\right) f(\xi) + \frac{\partial^2 f(\xi)}{\partial \xi^2} + D = 0, \quad (7)$$

where D is the constant. Balancing $f^2(\xi)$ with $f''(\xi)$ in equation (7) gives $N = 2$. Therefore, the solution of aforementioned equations takes the form

$$f(\xi) = A_0 + A_1 \phi(\xi) + A_2 \phi(\xi)^2 + \frac{B_1}{\phi(\xi)} + \frac{B_2}{\phi(\xi)^2}, \quad (8)$$

where $\phi(\xi)$ is given by equation (5). Substituting equation (8) into equation (7) and following the method described in Section 2 leads to the following results.

Family 1: For $p = [i, -i, 1, 1]$ and $q = [i, -i, i, -i]$, in equation (5) which gives

$$\phi(\xi) = -\frac{\sin(\xi)}{\cos(\xi)}, \quad (9)$$

we obtain the following solutions:

Case 1.

$$\begin{aligned} A_0 &= \frac{3C_0 k Z - 14\tau}{3kZ^{3/2}(2bC_0 - \beta G_0)}, \quad A_1 = 0, \\ A_2 &= -\frac{4\tau}{kZ^{3/2}(2bC_0 - \beta G_0)}, \quad B_1 = 0, \quad B_2 = 0, \\ D &= \frac{1}{6} \left(\frac{20\tau}{kZ^{3/2}(2bC_0 - \beta G_0)} + \frac{9C_0^2 k \sqrt{Z}}{\tau(2bC_0 - \beta G_0)} - \frac{36C_0}{\sqrt{Z}(2bC_0 - \beta G_0)} \right). \end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (9), the exact solution of equation (1) is obtained as follows:

$$\phi_1(x, t) = \frac{3C_0 k Z - 2\tau \left(6 \tan^2 \left(\frac{x}{L} - \frac{kt}{\tau} \right) + 7 \right)}{3kZ^2(2bC_0 - \beta G_0)}.$$

Case 2.

$$\begin{aligned}
A_0 &= \frac{-\frac{24bC_0\tau\sqrt{Z}}{2bC_0-\beta G_0} + \frac{12\beta G_0\tau\sqrt{Z}}{2bC_0-\beta G_0} + 3C_0kZ^{3/2} - 2\tau\sqrt{Z}}{3(2bC_0kZ^2 - \beta G_0kZ^2)}, \\
A_1 &= 0, A_2 = 0, B_1 = 0, B_2 = -\frac{4\tau}{kZ^{3/2}(2bC_0 - \beta G_0)}, \\
D &= \frac{1}{6\tau} \left(\frac{9C_0^2k^2Z^{5/2}}{2bC_0kZ^2 - \beta G_0kZ^2} - \frac{120\tau^2}{kZ^{3/2}(2bC_0 - \beta G_0)} \right. \\
&\quad + \frac{20\tau^2\sqrt{Z}}{2bC_0kZ^2 - \beta G_0kZ^2} \\
&\quad + \frac{240bC_0\tau^2\sqrt{Z}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad - \frac{120\beta G_0\tau^2\sqrt{Z}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad \times \frac{36C_0k\tau Z^{3/2}}{2bC_0kZ^2 - \beta G_0kZ^2} \\
&\quad - \frac{72bC_0^2k\tau Z^{3/2}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad + \frac{36\beta C_0G_0k\tau Z^{3/2}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad \left. + \frac{36C_0\tau}{\sqrt{Z}(2bC_0 - \beta G_0)} \right).
\end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (9), the exact solution of equation (1) is obtained:

$$\phi_2(x, t) = \frac{3C_0kZ - 2\tau \left(6 \cot^2 \left(\frac{x}{L} - \frac{kt}{\tau} \right) + 7 \right)}{3kZ^2(2bC_0 - \beta G_0)}.$$

Case 3.

$$\begin{aligned}
A_0 &= \frac{1}{3} \left(-\frac{28bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} + \frac{14\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right. \\
&\quad \left. + \frac{3C_0k\sqrt{Z}}{2bC_0kZ - \beta G_0kZ} \right), \\
A_1 &= 0, B_1 = 0, \\
A_2 &= 4 \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right), \\
B_2 &= 4 \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right), \\
D &= \frac{1}{6} \left(-\frac{9C_0^2k^2Z}{(2bC_0kZ - \beta G_0kZ)^2} \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right) \right. \\
&\quad - \frac{440bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} + \frac{220\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \\
&\quad \left. - \frac{36C_0k\sqrt{Z}}{2bC_0kZ - \beta G_0kZ} \right).
\end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (9), the exact solution of equation (1) is obtained:

$$\begin{aligned}
\phi_3(x, t) &= \frac{3C_0kZ + 2\tau \left(-6 \tan^2 \left(\frac{x}{L} - \frac{kt}{\tau} \right) - 6 \cot^2 \left(\frac{x}{L} - \frac{kt}{\tau} \right) - 7 \right)}{3kZ^2(2bC_0 - \beta G_0)}.
\end{aligned}$$

Family 2: For $p = [1, -1, 1, 1]$ and $q = [-1, 1, -1, 1]$, in equation (5), we have

$$\phi(\xi) = -\tanh(\xi) \quad (10)$$

and the solutions take the following cases:

Case 1.

$$\begin{aligned}
A_0 &= \frac{3C_0kZ + 2\tau}{3kZ^{3/2}(2bC_0 - \beta G_0)}, A_1 = 0, \\
A_2 &= -\frac{4\tau}{kZ^{3/2}(2bC_0 - \beta G_0)}, B_1 = 0, B_2 = 0, \\
D &= \frac{1}{6} \left(\frac{20\tau}{kZ^{3/2}(2bC_0 - \beta G_0)} + \frac{9C_0^2k\sqrt{Z}}{\tau(2bC_0 - \beta G_0)} \right. \\
&\quad \left. - \frac{36C_0}{\sqrt{Z}(2bC_0 - \beta G_0)} \right).
\end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (10), the exact solution of equation (1) is obtained:

$$\phi_4(x, t) = \frac{3C_0kZ + 2\tau \left(1 - 6 \tanh^2 \left(\frac{x}{L} - \frac{kt}{\tau} \right) \right)}{3kZ^2(2bC_0 - \beta G_0)}.$$

Case 2.

$$\begin{aligned}
A_0 &= \frac{\frac{24bC_0\tau\sqrt{Z}}{2bC_0-\beta G_0} - \frac{12\beta G_0\tau\sqrt{Z}}{2bC_0-\beta G_0} + 3C_0kZ^{3/2} - 10\tau\sqrt{Z}}{3(2bC_0kZ^2 - \beta G_0kZ^2)}, \\
A_1 &= 0, A_2 = 0, B_1 = 0, \\
B_2 &= -\frac{4\tau}{kZ^{3/2}(2bC_0 - \beta G_0)}, \\
D &= \frac{1}{6\tau} \left(\frac{9C_0^2k^2Z^{5/2}}{2bC_0kZ^2 - \beta G_0kZ^2} + \frac{24\tau^2}{kZ^{3/2}(2bC_0 - \beta G_0)} \right. \\
&\quad + \frac{20\tau^2\sqrt{Z}}{2bC_0kZ^2 - \beta G_0kZ^2} - \frac{48bC_0\tau^2\sqrt{Z}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad + \frac{24\beta G_0\tau^2\sqrt{Z}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad - \frac{36C_0k\tau Z^{3/2}}{2bC_0kZ^2 - \beta G_0kZ^2} + \frac{72bC_0^2k\tau Z^{3/2}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad - \frac{36\beta C_0G_0k\tau Z^{3/2}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\
&\quad \left. - \frac{36C_0\tau}{\sqrt{Z}(2bC_0 - \beta G_0)} \right).
\end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (10), the exact solution of equation (1) is obtained:

$$\phi_5(x, t) = \frac{3C_0kZ + 2\tau\left(1 - 6\coth^2\left(\frac{x}{L} - \frac{kt}{\tau}\right)\right)}{3kZ^2(2bC_0 - \beta G_0)}.$$

Case 3.

$$\begin{aligned} A_0 &= \frac{1}{3} \left(\frac{4bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right. \\ &\quad \left. + \frac{3C_0k\sqrt{Z}}{2bC_0kZ - \beta G_0kZ} \right), \\ A_1 &= 0, \quad A_2 = 4 \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right), \\ B_1 &= 0, \quad B_2 = 4 \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right), \\ D &= \frac{1}{6} \left(- \frac{9C_0^2k^2Z}{(2bC_0kZ - \beta G_0kZ)^2 \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right)} \right. \\ &\quad - \frac{440bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} + \frac{220\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \\ &\quad \left. - \frac{36C_0k\sqrt{Z}}{2bC_0kZ - \beta G_0kZ} \right). \end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (10), the exact solution of equation (1) is obtained:

$$\begin{aligned} \phi_6(x, t) &= \frac{3C_0kZ + 2\tau\left(-6\tanh^2\left(\frac{x}{L} - \frac{kt}{\tau}\right) - 6\coth^2\left(\frac{x}{L} - \frac{kt}{\tau}\right) + 1\right)}{3kZ^2(2bC_0 - \beta G_0)}. \end{aligned}$$

Family 3: For $p = [1, 1, 1, -1]$ and $q = [1, -1, 1, -1]$, in equation (5) and

$$\phi(\xi) = \frac{\cosh(\xi)}{\sinh(\xi)}. \quad (11)$$

The obtained solutions are:

Case 1.

$$\begin{aligned} A_0 &= \frac{3C_0kZ + 2\tau}{3kZ^{3/2}(2bC_0 - \beta G_0)}, \quad A_1 = 0, \\ A_2 &= -\frac{4\tau}{kZ^{3/2}(2bC_0 - \beta G_0)}, \quad B_1 = 0, \quad B_2 = 0, \\ D &= \frac{1}{6} \left(\frac{20\tau}{kZ^{3/2}(2bC_0 - \beta G_0)} + \frac{9C_0^2k\sqrt{Z}}{\tau(2bC_0 - \beta G_0)} \right. \\ &\quad \left. - \frac{36C_0}{\sqrt{Z}(2bC_0 - \beta G_0)} \right). \end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (11), the exact solution of equation (1) is obtained:

$$\phi_7(x, t) = \frac{3C_0kZ + 2\tau\left(1 - 6\coth^2\left(\frac{x}{L} - \frac{kt}{\tau}\right)\right)}{3kZ^2(2bC_0 - \beta G_0)}.$$

Case 2.

$$\begin{aligned} A_0 &= \frac{\frac{24bC_0\tau\sqrt{Z}}{2bC_0 - \beta G_0} - \frac{12\beta G_0\tau\sqrt{Z}}{2bC_0 - \beta G_0} + 3C_0kZ^{3/2} - 10\tau\sqrt{Z}}{3(2bC_0kZ^2 - \beta G_0kZ^2)}, \\ A_1 &= 0, \quad A_2 = 0, \quad B_1 = 0, \\ B_2 &= -\frac{4\tau}{kZ^{3/2}(2bC_0 - \beta G_0)}, \\ D &= \frac{1}{6\tau} \left(\frac{9C_0^2k^2Z^{5/2}}{2bC_0kZ^2 - \beta G_0kZ^2} + \frac{24\tau^2}{kZ^{3/2}(2bC_0 - \beta G_0)} \right. \\ &\quad + \frac{20\tau^2\sqrt{Z}}{2bC_0kZ^2 - \beta G_0kZ^2} \\ &\quad - \frac{48bC_0\tau^2\sqrt{Z}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\ &\quad + \frac{24\beta G_0\tau^2\sqrt{Z}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\ &\quad - \frac{36C_0k\tau Z^{3/2}}{2bC_0kZ^2 - \beta G_0kZ^2} \\ &\quad + \frac{72bC_0^2k\tau Z^{3/2}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\ &\quad - \frac{36\beta C_0G_0k\tau Z^{3/2}}{(2bC_0 - \beta G_0)(2bC_0kZ^2 - \beta G_0kZ^2)} \\ &\quad \left. - \frac{36C_0\tau}{\sqrt{Z}(2bC_0 - \beta G_0)} \right). \end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (11), the exact solution of equation (1) is obtained:

$$\phi_8(x, t) = \frac{3C_0kZ + 2\tau\left(1 - 6 \tanh^2\left(\frac{x}{L} - \frac{kt}{\tau}\right)\right)}{3kZ^2(2bC_0 - \beta G_0)}.$$

Case 3.

$$\begin{aligned} A_0 &= \frac{1}{3} \left(\frac{4bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right. \\ &\quad \left. + \frac{3C_0k\sqrt{Z}}{2bC_0kZ - \beta G_0kZ} \right), \quad A_1 = 0, \\ A_2 &= 4 \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right), \quad B_1 = 0 \\ B_2 &= 4 \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right), \\ D &= \frac{1}{6} \left(-\frac{9C_0^2k^2Z}{(2bC_0kZ - \beta G_0kZ)^2} \left(\frac{\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} - \frac{2bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \right) \right. \\ &\quad - \frac{440bC_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} + \frac{220\beta G_0k\tau\sqrt{Z}}{(2bC_0kZ - \beta G_0kZ)^2} \\ &\quad \left. - \frac{36C_0k\sqrt{Z}}{2bC_0kZ - \beta G_0kZ} \right). \end{aligned}$$

Using the aforementioned values and substituting equation (8) and equation (11), the exact solution of equation (1) is obtained:

$$\begin{aligned} \phi_9(x, t) &= \frac{3C_0kZ + 2\tau\left(-6 \tanh^2\left(\frac{x}{L} - \frac{kt}{\tau}\right) - 6 \coth^2\left(\frac{x}{L} - \frac{kt}{\tau}\right) + 1\right)}{3kZ^2(2bC_0 - \beta G_0)}. \end{aligned}$$

Figure 1: (a) 2D plot of the solution of $\phi_1(x, t)$ with $k = -\frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_1(x, t)$ with $k = -\frac{\pi}{4}$.

4 Dissection and comparison

In order to study the obtained solutions, the figures of the solutions are plotted and the parameters take the following value: $\tau = 0.6 \times 10^{-6}$, $\beta = 16.36 \times 10^{-3}$, $b = 0.5$, $L = 8 \times 10^{-9}$, $G_0 = 1.1 \times 10^{-13}$, $C_0 = 1.8 \times 10^{-15}$, $Z = 5.56 \times 10^{10}$, and k , which is the arbitrary constant [39]. The figures in the current study are new and unique compared to the solutions in previous works that are found by methods of boundary conditions [39,42], the $\exp(-\phi(\xi))$ -expansion method [43,44], modified extended tanh-function method, hyperbolic function method [45], the modified Khater method [46], the Bernoulli sub-equation function method [47], and the exponential rational function method [48]. Figures 1, 2, and 3 are periodic wave solution; Figures 5, 7, and 9 are bright solitary wave solution; and Figures 4, 6, and 8 are dark solitary wave solution. The solutions show localized propagation of ions as shown in Figures 5, 7, and 9 or oscillated propagation of ions along the microtubules as shown in Figures 1, 2, and 3. The obtained solutions may depict the applications of microtubules of similar phenomena in natural science.

5 Stability analysis of the ionic currents along microtubule dynamical equations

In this section, we investigate the connection for the dispersion in equation (1). In order to determine whether the outcomes grow infinitely or disappear in given period, we investigate the real part of $\sigma(k)$ when the solution takes the form $e^{i\sigma t + ikx}$. If the real part of $\sigma(k)$ is negative for k , the solution may disappear subsequently and the form of

Figure 2: (a) 2D plot of the solution of $\phi_2(x, t)$ with $k = \pi$. (b) 3D plot of the solution of $\phi_2(x, t)$ with $k = \pi$.

Figure 3: (a) 2D plot of the solution of $\phi_3(x, t)$ with $k = -\frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_3(x, t)$ with $k = -\frac{\pi}{4}$.

Figure 4: (a) 2D plot of the solution of $\phi_4(x, t)$ with $k = -\frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_4(x, t)$ with $k = -\frac{\pi}{4}$.

the solution in stable case. If it is positive, the solution will lead to infinity and the form of the solution is unstable. Consider the solution of the form

$$f(x, t) = \varepsilon w(x, t) + p_0. \quad (12)$$

Substituting equation (12) into equation (1), we get the following results:

Figure 5: (a) 2D plot of the solution of $\phi_5(x, t)$ with $k = -\frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_5(x, t)$ with $k = -\frac{\pi}{4}$.

Figure 6: (a) 2D plot of the solution of $\phi_6(x, t)$ with $k = \frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_6(x, t)$ with $k = \frac{\pi}{4}$.

$$\begin{aligned}
 & -6bC_0p_0Z^{3/2}\varepsilon w^{(0,1)}(x, t) \\
 & -6bC_0Z^{3/2}\varepsilon^2w(x, t)w^{(0,1)}(x, t) \\
 & +3\beta G_0p_0Z^{3/2}\varepsilon w^{(0,1)}(x, t) \\
 & +3\beta G_0Z^{3/2}\varepsilon^2w(x, t)w^{(0,1)}(x, t) + L^3\varepsilon w^{(3,0)}(x, t) \\
 & +6L\varepsilon w^{(1,0)}(x, t) + 3ZC_0\varepsilon w^{(0,1)}(x, t) = 0,
 \end{aligned} \tag{13}$$

where p_0 is the constant. Using the linear term ε equation (13) yields

$$\begin{aligned}
 & -6bC_0p_0Z^{3/2}\varepsilon w^{(0,1)}(x, t) + 3\beta G_0p_0Z^{3/2}\varepsilon w^{(0,1)}(x, t) \\
 & + L^3\varepsilon w^{(3,0)}(x, t) + 6L\varepsilon w^{(1,0)}(x, t) \\
 & + 3ZC_0\varepsilon w^{(0,1)}(x, t) = 0.
 \end{aligned} \tag{14}$$

Figure 7: (a) 2D plot of the solution of $\phi_7(x, t)$ with $k = -\frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_7(x, t)$ with $k = -\frac{\pi}{4}$.

Figure 8: (a) 2D plot of the solution of $\phi_8(x, t)$ with $k = -\frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_8(x, t)$ with $k = -\frac{\pi}{4}$.

Figure 9: (a) 2D plot of the solution of $\phi_9(x, t)$ with $k = -\frac{\pi}{4}$. (b) 3D plot of the solution of $\phi_9(x, t)$ with $k = -\frac{\pi}{4}$.

Suppose that equation (14) has a solution of the form

$$w(x, t) = e^{i(k_1 x + \sigma t)}, \quad (15)$$

where k is the normalized wave number.

Substituting equation (15) into equation (14), we get the following results:

$$-6bC_0 p_0 \sigma Z^{3/2} + 3\beta G_0 p_0 \sigma Z^{3/2} - k_1^3 L^3 + 6k_1 L + 3\sigma ZC_0 = 0. \quad (16)$$

Solving for σ from equation (16) yields

$$\sigma(k_1) = \frac{6k_1 L - k_1^3 L^3}{3(2bC_0 p_0 Z^{3/2} - \beta G_0 p_0 Z^{3/2} - ZC_0)}. \quad (17)$$

The system is stable and $\sigma(k_1) < 0$ when $p_0 > 0$ and $k < 0$ or $p_0 < 0$ and $k > 0$.

6 Summary

In this article, the GERFM was successfully applied to find new solutions to the transmission line models of nano-ionic currents along microtubules. We find that the proposed

method is reliable and effective and gives analytical and exact solutions. In future work, GERFM will be applied to create new solutions for other types of nonlinear equations.

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