Research Article

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Fully Legendre spectral collocation technique for stochastic heat equations

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Abstract: For the stochastic heat equation (SHE), a very accurate spectral method is considered. To solve the SHE, we suggest using a shifted Legendre Gauss–Lobatto collocation approach in combination with a shifted Legendre Gauss–Radau collocation technique. A comprehensive theoretical formulation is offered, together with numerical examples, to demonstrate the technique’s performance and competency. The scheme’s superiority in tackling the SHE is demonstrated.

Keywords: quadrature, stochastic heat equation, collocation method

1 Introduction

For addressing differential equations, several numerical approaches, both global and local, were mentioned as approximation techniques. The numerical solutions for local techniques are supplied for specific places, whereas global methods can acquire solutions for the whole domain. Using finite difference techniques, numerical approximations for differential equations [1–4] are listed at specific points, whereas the finite element methods [5–7] divide the interval into sub-intervals and provide an approximate solution in each of them. Spectral techniques have recently

attracted much interest for dealing with many forms of differential and integral equations [8,9], owing to their ability to handle problems with infinite or finite domains [10,11]. The greatest benefits of spectral techniques are their rapid convergence (exponential rate) and high accuracy rate [12–15].

As a relatively young branch of research, stochastic differential equation (SDE) theory will play a significant role in the mathematical world. An SDE is a differential equation with one or more stochastic terms, and hence a stochastic solution. It discovered a wide range of applications in biology, chemistry, mechanics, economics, and other fields in the late 1950s [16]. Due to the fact that most SDE models do not have an explicit precise analytic solution, numerical approaches must be used to create an approximate solution to the problem at hand. The development of numerical techniques for approximating SDEs has become a subject of growing attention in recent years, see, for example, refs [17,18]. Several researchers looked at numerical solutions to SDEs. Refs [19–27] for spatial variable discretization using finite differences, finite elements, or spectral Galerkin methods, and then solving the resulting system of ordinary differential equations *via* Euler method.

The approximate solutions for the stochastic heat equations (SHEs) are found using the Legendre collocation technique based on Gauss–Lobatto and Gauss–Radau nodes. The spectral collocation technique was utilized for the temporal and spatial discretizations. For spatial discretization, the shifting Legendre Gauss–Lobatto collocation technique is used, along with proper boundary management. The temporal discretization was done using the Legendre Gauss–Radau technique with shifted Legendre. This change significantly improves accuracy. Following that, we estimate the aforementioned problem’s residuals at selected quadrature points. The shifted Legendre Gauss–Lobatto quadrature is used to numerically handle the integral terms after an appropriate mapping. Furthermore, the Brownian motion $W(x)$ is discretized using Lagrange interpolation for computing purposes. As a result, a system of algebraic equations is extracted. Various numerical problems confirm the accuracy of our method.

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The following is the format of this article. First, several noteworthy features are listed. In Section 3, the aforementioned technique is applied for SHEs with initial-boundary and stochastic process conditions. Various examples in Section 4 demonstrate the competency of our numerical method. Remarks are finally provided.

2 Preliminaries and notations

2.1 Properties of the shifted Legendre polynomials

The Legendre polynomials $\mathcal{J}_k(t)$ ($k = 0, 1, 2, \dots$) ($k = 0, 1, 2, \dots$) ($k = 0, 1, 2, \dots$) ($k = 0, 1, 2, \dots$) meet the Rodrigue

$$\mathcal{J}_k(t) = \frac{(-1)^k}{2^k k!} D^k((1-t)^k). \quad (2.1)$$

As a result, $\mathcal{J}_k^{(p)}(t)$ (the p th derivative of $\mathcal{J}_k(t)$) is

$$\mathcal{J}_k^{(p)}(t) = \sum_{i=0(t+k=\text{even})}^{k-p} C_p(k, t) P_i(t), \quad (2.2)$$

where

$$C_p(k, t) = \frac{2^{p-1}(2t+1)\Gamma\left(\frac{p+k-t}{2}\right)\Gamma\left(\frac{p+k+t+1}{2}\right)}{\Gamma(p)\Gamma\left(\frac{2-p+k-t}{2}\right)\Gamma\left(\frac{3-p+k+t}{2}\right)}.$$

The norm and inner product of space $L^2[-1, 1]$ are then denoted by $\|\chi\|$ and (χ, ν) .

In $L^2[-1, 1]$, the set of $\mathcal{J}_k(t)$ is a complete orthogonal system

$$(\mathcal{J}_j(t), \mathcal{J}_k(t)) = \int_{-1}^1 \mathcal{J}_j(t) \mathcal{J}_k(t) dt = h_k \Pi_{jk}, \quad (2.3)$$

where $h_i = \frac{2}{2i+1}$ and Π_{jk} is the Dirac function. Thus, for any $\nu \in L^2[-1, 1]$,

$$\nu(t) = \sum_{i=0}^{\infty} a_i \mathcal{J}_i(t), \quad a_i = \frac{1}{h_i} \int_{-1}^1 \nu(t) \mathcal{J}_i(t) dt. \quad (2.4)$$

Let $S_N[-1, 1]$ be the set of all polynomials with a maximum degree of N ($N \geq 0$). As a result, we get $\varphi \in S_{2N-1}[-1, 1]$ for any $\varphi \in S_{2N-1}[-1, 1]$.

$$\int_{-1}^1 \varphi(t) dt = \sum_{i=0}^N \varpi_{N,i} \varphi(t_{N,i}), \quad (2.5)$$

Legendre Gauss–Lobatto (L–GL) interpolation on the classical interval $[-1, 1]$ is indicated by $t_{N,k}$ ($0 \leq k \leq N$) and $\varpi_{N,k}$ ($0 \leq k \leq N$), which are nodes and Christoffel numbers.

The discrete inner product and the norm are specified as

$$\|\chi\|_N = (\chi, \nu)_N^{\frac{1}{2}}, \quad (\chi, \nu)_N = \sum_{j=0}^N \chi(t_{N,j}) \nu(t_{N,j}) \varpi_{N,j}. \quad (2.6)$$

2.2 Brownian motion

A scalar Brownian motion, also known as a standard Wiener process, is a random variable $W(t)$ that flows continuously on $[0, \mathcal{L}]$ and meets the following conditions:

- (1) $W(t) = 0$, with probability 1.
- (2) For $0 \leq s < t \leq \mathcal{L}$ the random variable given by increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, i.e., $W(t) - W(s) \sim \sqrt{t - s} \mathcal{N}(0, 1)$, where $\mathcal{N}(0, 1)$ denotes a normally distributed random variable with zero mean and unit variance.
- (3) For $0 \leq s < t < u < v \leq \mathcal{L}$, $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

2.3 Stochastic integral

The integral of a function $\mu(t)$ over an interval $[0, \mathcal{L}]$, with respect to a Brownian motion $W(t)$ as $\int_0^{\mathcal{L}} \mu(t) d\omega_t$.

2.4 Some properties of the Itô integral

Definition 2.1. (The Itô integral) If $\mu \in \nu(s_1, s_2)$, the Itô integral is stated as

$$\int_{s_1}^{s_2} \mu(t, t) d\omega_t(t) = \lim_{n \rightarrow \infty} \int_{s_1}^{s_2} \mu_{\mathcal{M}}(t, t) d\omega_t(t), \quad (2.7)$$

where $\mu_{\mathcal{M}}$ is a series of basic functions that results in

$$E \left[\int_{s_1}^{s_2} (\mu(t, t) - \mu_{\mathcal{M}}(t, t))^2 dt \right] \rightarrow 0 \quad \text{as } \mathcal{M} \rightarrow \infty. \quad (2.8)$$

Property 2.1. (Integration by parts) If $\mu(s, t) = \mu(s)$ is exclusively dependent on s and μ is continuous and constrained in $[0, t]$, afterward

$$\int_0^t \mu(s) d\omega_s = \mu(t)\omega_t - \int_0^t \omega_s d\mu_s.$$

Property 2.2. If $\mu \in \mathcal{V}(s_1, s_2)$, then

$$E \left[\left(\int_{s_1}^{s_2} \mu(t, t) d\omega_t(t) \right)^2 \right] = E \left[\int_{s_1}^{s_2} \mu^2(t, t) dt \right].$$

3 Fully spectral collocation treatment

3.1 Initial-boundary conditions

To begin, we devised a numerical method to handle SHEs of the type

$$\begin{aligned} & \frac{\partial \mathcal{Y}(x, t)}{\partial t} \\ &= \left(\kappa_1 + \kappa_2 \frac{d\omega(t)}{dt} \right) \frac{\partial^2 \mathcal{Y}(x, t)}{\partial x^2} + \kappa_3 \frac{\partial \mathcal{Y}(x, t)}{\partial x} \\ &+ \Pi(x, t, \omega), \quad (x, t) \in \Xi^\bullet \times \Xi^\diamond, \end{aligned} \quad (3.1)$$

where $\Xi^\bullet \equiv [0, x_{\text{end}}]$, and $\Xi^\diamond \equiv [0, t_{\text{end}}]$, which is related to

$$\begin{aligned} \mathcal{Y}(x, 0) &= \Theta_1(x), \quad x \in \Xi^\bullet, \\ \mathcal{Y}(0, t) &= \Theta_2(t), \quad \mathcal{Y}(x_{\text{end}}, t) = \Theta_3(t), \quad t \in \Xi^\diamond. \end{aligned} \quad (3.2)$$

To convert SHEs into linear algebraic systems, the shifted Legendre Gauss-Lobatto collocation and shifted Legendre Gauss-Radau collocation methods are used. The truncated solution is written as:

$$\mathcal{Y}_{N, \mathcal{M}}(x, t) = \sum_{\substack{l, r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(x, t), \quad (3.3)$$

with $\mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(x, t) = \mathcal{G}_{x_{\text{end}}, r_1}(x) \mathcal{G}_{t_{\text{end}}, r_2}(t)$ and $\mathcal{G}_{\mathcal{L}, s}(x)$ is assigned for shifted Legendre polynomials on $[0, \mathcal{L}]$.

$\frac{\partial}{\partial x}(\mathcal{Y}_{N, \mathcal{M}}(x, t))$ is calculated as

$$\frac{\partial}{\partial x}(\mathcal{Y}_{N, \mathcal{M}}(x, t)) = \sum_{\substack{r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(x, t), \quad (3.4)$$

where $\mathcal{G}_{r_1, r_2, x}^{x_{\text{end}}, t_{\text{end}}}(x, t) = \frac{\partial}{\partial x}(\mathcal{G}_{x_{\text{end}}, r_1}(x)) \mathcal{G}_{t_{\text{end}}, r_2}(t)$. Similarly, we find

$$\frac{\partial^2}{\partial x^2}(\mathcal{Y}_{N, \mathcal{M}}(x, t)) = \sum_{\substack{r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(x, t), \quad (3.5)$$

where $\mathcal{G}_{r_1, r_2, x}^{x_{\text{end}}, t_{\text{end}}}(x, t) = \frac{\partial^2}{\partial x^2}(\mathcal{G}_{x_{\text{end}}, r_1}(x)) \mathcal{G}_{t_{\text{end}}, r_2}(t)$, whereas the temporal derivative of first order is computed as

$$\frac{\partial}{\partial t}(\mathcal{Y}_{N, \mathcal{M}}(x, t)) = \sum_{\substack{r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(x, t), \quad (3.6)$$

where $\mathcal{G}_{r_1, r_2, t}^{x_{\text{end}}, t_{\text{end}}}(x, t) = \frac{\partial}{\partial t}(\mathcal{G}_{t_{\text{end}}, r_2}(t)) \mathcal{G}_{x_{\text{end}}, r_1}(x)$.

At certain nodes, the preceding derivatives of spatial and temporal variables are calculated as follows:

$$\begin{aligned} & \left(\frac{\partial}{\partial t}(\mathcal{Y}_{N, \mathcal{M}}(x, t)) \right)_{\substack{x=x_{N, n}^{x_{\text{end}}}, \\ t=t_{M, m}^{t_{\text{end}}}}} \\ &= \sum_{\substack{r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2, t}^{x_{\text{end}}, t_{\text{end}}}(x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}), \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \left(\frac{\partial}{\partial x}(\mathcal{Y}_{N, \mathcal{M}}(x, t)) \right)_{\substack{x=x_{N, n}^{x_{\text{end}}}, \\ \eta=\eta_{M, m}^{t_{\text{end}}}}} \\ &= \sum_{\substack{r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2, x}^{x_{\text{end}}, t_{\text{end}}}(x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}), \end{aligned} \quad (3.8)$$

$$\begin{aligned} & \left(\frac{\partial^2}{\partial x^2}(\mathcal{Y}_{N, \mathcal{M}}(x, t)) \right)_{\substack{x=x_{N, n}^{x_{\text{end}}}, \\ \eta=\eta_{M, m}^{t_{\text{end}}}}} \\ &= \sum_{\substack{r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2, x, x}^{x_{\text{end}}, t_{\text{end}}}(x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}), \end{aligned} \quad (3.9)$$

where $n = 0, 1, \dots, N$, $m = 0, 1, \dots, \mathcal{M}$.

For computing tasks, discretized Brownian motion is helpful, where $\omega(t)$ is stated at t discrete values and $\omega(t)$ is evaluated using Lagrange interpolation. As a result, we get

$$\omega_i = \omega_{i-1} + \Delta\omega_i, \quad i = 1, 2, \dots, \mathcal{M}, \quad (3.10)$$

$$\left(\frac{d\omega(t)}{dt} \right)_{t=t_{M, i}^{t_{\text{end}}}} = \frac{\Delta\omega_i}{\Delta t_{M, i}^{t_{\text{end}}}}, \quad i = 1, 2, \dots, \mathcal{M}, \quad (3.11)$$

where $\omega_0 = 0$, with the probability 1, $\omega_i = \omega(t_{M, i}^{t_{\text{end}}})$, $\Delta t_{M, i}^{t_{\text{end}}} = t_{M, i}^{t_{\text{end}}} - t_{M, i-1}^{t_{\text{end}}}$ also $\Delta\omega_i$ is an independent random variable of the form $\sqrt{\Delta t_{M, i}^{t_{\text{end}}}} \mathcal{N}(0, 1)$.

Alternatively, you may get the initial boundary by

$$\begin{aligned} & \sum_{\substack{l, r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(x, 0) = \Theta_1(x), \\ & \sum_{\substack{l, r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(0, t) = \Theta_2(t), \\ & \sum_{\substack{l, r_1=0, \dots, N \\ r_2=0, \dots, \mathcal{M}}} \varsigma_{r_1, r_2} \mathcal{G}_{r_1, r_2}^{x_{\text{end}}, t_{\text{end}}}(x_{N, N}^{x_{\text{end}}}, t) = \Theta_3(t). \end{aligned} \quad (3.12)$$

Equation (3.1) is constrained to zero at the $(N-1) \times (M)$ locations in the suggested spectral collocation approach. As a result of modifying (3.1)–(3.12),

$$\begin{aligned} & \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2, t}^{x_{\text{end}}, t_{\text{end}}} (x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}) \\ &= \left(\kappa_1 + \kappa_2 \frac{\Delta \omega_i}{\Delta t_{M, m}^{t_{\text{end}}}} \right) \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2, x, x}^{x_{\text{end}}, t_{\text{end}}} \\ & \quad (x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}) \\ &+ \kappa_3 \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2, x}^{x_{\text{end}}, t_{\text{end}}} (x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}) \\ &+ \Pi(x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}, \omega(t_{M, m}^{t_{\text{end}}})), \end{aligned} \quad (3.13)$$

where, $n = 1, \dots, N-1$, $m = 1, \dots, M$, additionally

$$\begin{aligned} & \sum_{\substack{l, n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2}^{x_{\text{end}}, t_{\text{end}}} (x_{N, k}^{x_{\text{end}}}, 0) = \Theta_1(x_{N, k}^{x_{\text{end}}}), \\ & \quad k = 1, \dots, N-1, \\ & \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2}^{x_{\text{end}}, t_{\text{end}}} (0, t_{M, l}^{t_{\text{end}}}) = \Theta_2(t_{M, l}^{t_{\text{end}}}), \\ & \quad l = 0, \dots, M, \\ & \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2}^{x_{\text{end}}, t_{\text{end}}} (x_{N, N}^{x_{\text{end}}}, t_{M, l}^{t_{\text{end}}}) = \Theta_3(t_{M, l}^{t_{\text{end}}}), \\ & \quad l = 0, \dots, M. \end{aligned} \quad (3.14)$$

As equations (3.13) and (3.14) are combined, we have a linear system of algebraic equations that are easily solvable.

3.2 Conditions based on stochastic process

We devised a numerical approach to deal with SHEs of the form

$$\begin{aligned} \frac{\partial \mathcal{Y}(x, t)}{\partial t} &= \left(\kappa_1 + \kappa_2 \frac{d\omega(t)}{dt} \frac{\partial^2}{\partial x^2} (\mathcal{Y}(x, t)) \right) \\ &+ \kappa_3 \frac{\partial}{\partial x} (\mathcal{Y}(x, t)) + \Pi(x, t, \omega), \\ & \quad (x, t) \in \Xi^\bullet \times \Xi^\diamond. \end{aligned} \quad (3.15)$$

which is related to

$$\begin{aligned} \mathcal{Y}(x, 0) &= \Theta_1(x), \quad x \in \Xi^\bullet, \\ \mathcal{Y}(0, t) &= \Theta_2(t) + \int_0^t \omega(\tau) d\tau, \\ \mathcal{Y}(x_{\text{end}}, t) &= \Theta_3(t) + \int_0^t \omega(\tau) d\tau, \quad t \in \Xi^\diamond. \end{aligned} \quad (3.16)$$

Using the preceding subsection's similar analysis, we arrive to

$$\begin{aligned} & \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2, t}^{x_{\text{end}}, t_{\text{end}}} (x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}) \\ &= \left(\kappa_1 + \kappa_2 \frac{\Delta \omega m}{\Delta t_{M, m}^{t_{\text{end}}}} \right) \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2, x, x}^{x_{\text{end}}, t_{\text{end}}} \\ & \quad (x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}) \\ &+ \kappa_3 \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2, x}^{x_{\text{end}}, t_{\text{end}}} (x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}) \\ &+ \Pi(x_{N, n}^{x_{\text{end}}}, t_{M, m}^{t_{\text{end}}}, \omega(t_{M, m}^{t_{\text{end}}})), \end{aligned} \quad (3.17)$$

where, $n = 1, \dots, N-1$, $m = 1, \dots, M$. The integral terms in (3.16) will convert into new ones if $\tau = \frac{t}{t_{\text{end}}} \theta$ is applied

$$\int_0^t \omega(\tau) d\tau = \frac{t}{t_{\text{end}}} \int_0^{t_{\text{end}}} \omega\left(\frac{t}{t_{\text{end}}} \theta\right) d\theta. \quad (3.18)$$

Using the shifted Legendre Gauss–Lobatto quadrature, we have for every $\mathcal{G} \in S_{2M+1}[0, L]$

$$\int_0^L \mathcal{G}(\theta) d\theta = \sum_{k=0}^K \varpi_{K, k} \mathcal{G}(\theta_{K, k}^{t_{\text{end}}}), \quad (3.19)$$

in which $\theta_{K, k}^{t_{\text{end}}}$ are nodes of the shifted Legendre Gauss–Lobatto. As a result, the stochastic process conditions are addressed as:

$$\sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2}^{x_{\text{end}}, t_{\text{end}}} (0, t) \quad (3.20)$$

$$= \Theta_2(t_{M, l}^{t_{\text{end}}}) + \frac{t}{t_{\text{end}}} \sum_{k=0}^K \varpi_{K, k} \omega\left(\frac{t}{t_{\text{end}}} \theta_{K, k}^{t_{\text{end}}}\right),$$

$$\begin{aligned} & \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2}^{x_{\text{end}}, t_{\text{end}}} (x_{N, N}^{x_{\text{end}}}, t) \\ &= \Theta_3(t_{M, l}^{t_{\text{end}}}) + \frac{t}{t_{\text{end}}} \sum_{k=0}^K \varpi_{K, k} \omega\left(\frac{t}{t_{\text{end}}} \theta_{K, k}^{t_{\text{end}}}\right). \end{aligned} \quad (3.21)$$

So, we have

$$\begin{aligned} & \sum_{\substack{n_1=0, \dots, N \\ r_2=0, \dots, M}} \varsigma_{n_1, r_2} \mathcal{G}_{n_1, r_2}^{x_{\text{end}}, t_{\text{end}}} (0, t_{M, l}^{t_{\text{end}}}) \\ &= \Theta_2(t_{M, l}^{t_{\text{end}}}) + \frac{t}{t_{\text{end}}} \sum_{k=0}^K \varpi_{K, k} \omega\left(\frac{t_{M, l}^{t_{\text{end}}}}{t_{\text{end}}} \theta_{K, k}^{t_{\text{end}}}\right), \end{aligned} \quad (3.22)$$

$$\sum_{\substack{r_1=0,\dots,N \\ r_2=0,\dots,M}} \varsigma_{r_1,r_2} \mathcal{G}_{r_1,r_2}^{x_{\text{end}},t_{\text{end}}} \left(x_{N,N}^{x_{\text{end}},t_{\text{end}}}, t_{M,l}^{t_{\text{end}}} \right) \quad (3.23)$$

$$= \Theta_3 \left(t_{M,l}^{t_{\text{end}}} \right) + \frac{t}{t_{\text{end}}} \sum_{k=0}^K \varpi_{K,k} \omega \left(\frac{t_{M,l}^{t_{\text{end}}}}{t_{\text{end}}} \theta_{K,k}^{t_{\text{end}}} \right).$$

As equations (3.17), (3.22), and (3.23) are combined, we have a linear system of algebraic equations that are easily solvable.

4 Numerical results

We demonstrate the accuracy and robustness of the spectral collocation method in this part by applying the algorithm to four test problems.

Example 1. The SHEs [28] are introduced

$$\frac{\partial}{\partial t}(\mathcal{Y}(x, t)) = \left(\frac{1}{\pi^2} + \beta \frac{d\omega(t)}{dt} \frac{\partial^2}{\partial x^2}(\mathcal{Y}(x, t)) \right), \quad (4.1)$$

$$(x, t) \in [0, 1] \times [0, 1],$$

with

$$\begin{aligned} \mathcal{Y}(x, 0) &= \sin(\pi x), \quad x \in [0, 1], \\ \mathcal{Y}(0, t) &= 0, \quad \mathcal{Y}(1, t) = 0, \quad t \in [0, 1], \end{aligned} \quad (4.2)$$

taking $\beta = 0$, we obtain $\mathcal{Y}(x, t) = e^{-t} \sin(\pi x)$.

At different parameter levels, Table 1 shows a comparison between our results and those in ref. [28]. The suggested approach produces superior numerical results than those published in ref. [28], based on these findings. It is also noticed that excellent estimates may be produced with a small number of collocation sites. In addition, we presented the greatest absolute errors obtained by our technique in Table 2 for Examples 1, 2, and 3 when $\beta = 0$. While we presented the absolute errors obtained by our approach for Example 1 when $\beta = 0.1$ in Table 3. Table 4 also contains the captions for the figures (Figures 1, 2, 3, 4, 5).

Example 2. We list the SHEs [28]

$$\frac{\partial}{\partial t}(\mathcal{Y}(x, t)) = \left(\frac{1}{2\pi^2} + \beta \frac{d\omega(t)}{dt} \frac{\partial^2}{\partial x^2}(\mathcal{Y}(x, t)) \right), \quad (4.3)$$

$$(x, t) \in [0, 1] \times [0, 1],$$

with

$$\begin{aligned} \mathcal{Y}(x, 0) &= \sin(2\pi x), \quad x \in [0, 1], \\ \mathcal{Y}(0, t) &= 0, \quad \mathcal{Y}(1, t) = 0, \quad t \in [0, 1], \end{aligned} \quad (4.4)$$

taking $\beta = 0$, we obtain $\mathcal{Y}(x, t) = e^{-2t} \sin(2\pi x)$.

At different values of parameters, Table 5 compares our results to those in ref. [28]. The suggested approach appears to produce superior numerical results than those published in ref. [28] based on these findings. Excellent

Table 1: Absolute errors for problem (4.1)

Wavelets Galerkin method [28] where $\bar{m} = 20$, $k = 2$, $M = 5$, and $N = 250$.					
(x, t)	(0.2, 0.2)	(0.4, 0.4)	(0.6, 0.6)	(0.8, 0.8)	(1.0, 1.0)
$E(x, t)$	6.5079×10^{-5}	3.7885×10^{-4}	5.5048×10^{-4}	4.0292×10^{-4}	3.5321×10^{-9}
(N, M, K)	Our method				
(4,4,4)	2.73233×10^{-4}	8.21153×10^{-4}	7.74369×10^{-4}	1.94591×10^{-4}	0
(8,8,8)	4.37034×10^{-8}	2.08469×10^{-8}	1.72441×10^{-8}	2.45231×10^{-8}	0
(12,12,12)	1.09446×10^{-12}	1.75415×10^{-14}	1.35447×10^{-14}	6.00964×10^{-13}	0
(16,16,16)	1.66533×10^{-16}	1.11022×10^{-16}	2.22045×10^{-16}	2.77556×10^{-16}	0

Table 2: Maximum absolute errors for Examples 1, 2, and 3, where $\beta = 0$

(N, M, K)	Our method				
(N, M, K)	(2,2,2)	(4,4,4)	(8,8,8)	(12,12,12)	(16,16,16)
Example 1	7.92311×10^{-2}	8.30599×10^{-4}	8.83531×10^{-8}	8.51113×10^{-13}	7.42220×10^{-18}
(N, M, K)	(2,2,2)	(6,6,6)	(12,12,12)	(16,16,16)	(20,20,20)
Example 2	1.13570×10^{-1}	8.77104×10^{-3}	1.21998×10^{-7}	1.35609×10^{-11}	5.95014×10^{-14}
(N, M, K)	(2,2,2)	(4,4,4)	(6,6,6)	(8,8,8)	(12,12,12)
Example 3	3.49581×10^{-1}	5.57078×10^{-3}	3.91833×10^{-5}	1.54329×10^{-7}	6.75349×10^{-13}

Table 3: Absolute errors for problem (4.1) at $(x_{N,i}^{x_{\text{end}}}, t_{M,j}^{t_{\text{end}}})$, (N, M, K) , and $\beta = 0.1$

(i, j)	(4,4,4)	(8,8,8)	(12,12,12)	(16,16,16)
(1,1)	0	7.77156×10^{-16}	4.36068×10^{-13}	4.39542×10^{-11}
(2,2)	3.10862×10^{-15}	1.15463×10^{-14}	2.16382×10^{-13}	5.77427×10^{-12}
(3,3)	4.4964×10^{-15}	9.37028×10^{-14}	4.53096×10^{-13}	6.74127×10^{-13}
(4,4)	7.28482×10^{-2}	6.01741×10^{-13}	8.58424×10^{-13}	8.30153×10^{-12}
(5,5)		2.84686×10^{-12}	5.24758×10^{-12}	1.18792×10^{-10}
(6,6)		1.12261×10^{-11}	3.7951×10^{-11}	8.21927×10^{-10}
(7,7)		1.89988×10^{-11}	1.38504×10^{-10}	3.15292×10^{-9}
(8,8)		3.33751×10^{-5}	1.20133×10^{-9}	1.77439×10^{-8}
(9,9)			2.06785×10^{-9}	2.28779×10^{-7}
(10,10)			8.10143×10^{-9}	2.06846×10^{-6}
(11,11)			2.58686×10^{-8}	7.35157×10^{-6}
(12,12)			5.27995×10^{-8}	2.36665×10^{-5}
(13,13)				6.68812×10^{-5}
(14,14)				1.27114×10^{-4}
(15,15)				2.79002×10^{-4}
(16,16)				3.11601×10^{-4}

Table 4: Absolute errors for problem (4.1) at $(x_{N,i}^{x_{\text{end}}}, t_{M,j}^{t_{\text{end}}})$, (N, M, K) , and $\beta = 0.1$

Figure 1	$\mathcal{Y}_{N,M}(x, t)$ of problem (4.1), where $\beta = 0$ and $N = M = K = 16$.
Figure 2	$\mathcal{E}(x, t)$ of problem (4.1), where $\beta = 0$ and $N = M = K = 16$.
Figure 3	x -curves of $\mathcal{Y}(x, t)$ and $\mathcal{Y}_{N,M}(x, t)$ of problem (4.1), where $\beta = 0$ and $N = M = K = 16$.
Figure 4	$\mathcal{E}(x, 0.5)$ of (4.1), where $\beta = 0$ and $N = M = K = 16$.
Figure 5	M_E convergence of problem (4.1), for $\beta = 0$.
Figure 6	$\mathcal{Y}_{N,M}(x, t)$ of problem (4.5), where $\beta = 0$ and $N = M = K = 20$.
Figure 7	$\mathcal{E}(x, t)$ of problem (4.5), where $\beta = 0$ and $N = M = K = 20$.
Figure 8	x -curves of $\mathcal{Y}(x, t)$ and $\mathcal{Y}_{N,M}(x, t)$ of problem (4.5), where $\beta = 0$ and $N = M = K = 20$.
Figure 9	$\mathcal{E}(x, 0.5)$ of problem (4.5), where $\beta = 0$ and $N = M = K = 20$.
Figure 10	$\mathcal{Y}_{N,M}(x, t)$ of problem (4.5), where $\beta = 0.2$ and $N = M = K = 16$.
Figure 11	$\mathcal{Y}_{N,M}(x, t)$ of problem (4.9), where $\beta = 0.1$ and $N = M = K = 12$.

estimates can also be produced with a small number of collocation sites. We also reported the absolute errors obtained by our approach for Example 2 when $\beta = 0.2$ in Table 6 (Figures 6, 7, 8, 9, and 10).

Example 3. We introduce the SHEs [28]

$$\frac{\partial}{\partial t}(\mathcal{Y}(x, t)) = \left(\frac{1}{2\pi^2} + \beta \frac{d\omega(t)}{dt} \frac{\partial^2}{\partial x^2}(\mathcal{Y}(x, t)) \right), \quad (4.5)$$

$$(x, t) \in [0, 1] \times [0, 1],$$

$$\begin{aligned} \mathcal{Y}(x, 0) &= \sin(2\pi x), \quad x \in [0, 1], \\ \mathcal{Y}(0, t) &= 0, \quad \mathcal{Y}(1, t) = 0, \quad t \in [0, 1], \end{aligned} \quad (4.6)$$

taking $\beta = 0$, we obtain $\mathcal{Y}(x, t) = e^{-2t} \sin(2\pi x)$.

At different values of parameters, Table 5 compares our results to those in ref. [28]. The suggested approach appears to produce superior numerical results than those published in ref. [28] based on these findings. Excellent

Figure 1: $\mathcal{Y}_{N,M}(x, t)$ of problem (4.1), where $\beta = 0$ and $N = M = K = 16$.

Figure 4: $\mathcal{E}(x, 0.5)$ of problem (4.1), where $\beta = 0$ and $N = M = K = 16$.

Figure 2: $\mathcal{E}(x, t)$ of problem (4.1), where $\beta = 0$ and $N = M = K = 16$.

Figure 5: M_E convergence of problem (4.1), for $\beta = 0$.

Figure 3: x -curves of $\mathcal{Y}(x, t)$ and $\mathcal{Y}_{N,M}(x, t)$ of problem (4.1), where $\beta = 0$ and $N = M = K = 16$.

estimates can also be produced with a small number of collocation sites. We also reported the absolute errors

obtained by our approach for Example 2 when $\beta = 0.2$ in Table 6.

Example 4. We introduce the SHEs [28]

$$\begin{aligned} \frac{\partial}{\partial t}(\mathcal{Y}(x, t)) &= \left(1 + \beta \frac{d\omega(t)}{dt} \frac{\partial^2}{\partial x^2}(\mathcal{Y}(x, t)) \right) \\ &\quad - 3 \frac{\partial}{\partial x}(\mathcal{Y}(x, t)), \\ (x, t) &\in [0, 1] \times [0, 1], \end{aligned} \quad (4.7)$$

with

$$\begin{aligned} \mathcal{Y}(x, 0) &= e^{2x}, \quad x \in [0, 1], \\ \mathcal{Y}(0, t) &= e^{2t}, \quad \mathcal{Y}(1, t) = e^{2-2t}, \quad t \in [0, 1], \end{aligned} \quad (4.8)$$

taking $\beta = 0$, we obtain $\mathcal{Y}(x, t) = e^{2x-2t}$.

At different parameter levels, Table 7 shows a comparison between our results and those in ref. [28]. The suggested approach produces superior numerical results than those published in ref. [28], based on these findings.

Table 5: Absolute errors for problem (4.5)

Wavelets Galerkin method [28] where $\bar{m} = 20$, $k = 2$, $M = 5$ and $N = 250$					
(x, t)	(0.2, 0.2)	(0.4, 0.4)	(0.6, 0.6)	(0.8, 0.8)	(1.0, 1.0)
$E(x, t)$	1.4636×10^{-4}	1.5676×10^{-3}	1.7341×10^{-3}	2.6439×10^{-4}	2.5840×10^{-9}
$(N, M, \mathcal{K})$	Our method				
(4,4,4)	4.36423×10^{-2}	7.02733×10^{-2}	5.02683×10^{-2}	2.63084×10^{-2}	0
(8,8,8)	1.73631×10^{-4}	1.75319×10^{-4}	1.17774×10^{-4}	5.24123×10^{-5}	0
(12,12,12)	8.26674×10^{-8}	3.77041×10^{-8}	2.52744×10^{-8}	2.48881×10^{-8}	0
(16,16,16)	8.46956×10^{-12}	1.19738×10^{-12}	8.02219×10^{-13}	2.54918×10^{-12}	0
(20,20,20)	2.22045×10^{-16}	6.66134×10^{-16}	1.11022×10^{-16}	1.91513×10^{-15}	0

Table 6: Absolute errors for problem (4.5) at $(x_{N,i}^{\text{end}}, t_{M,j}^{\text{end}})$, $(N, M, \mathcal{K})$ and $\beta = 0.2$

(i, j)	(6,6,6)	(8,8,8)	(12,12,12)	(16,16,16)
(1,1)	1.11022×10^{-14}	3.59268×10^{-13}	1.90914×10^{-12}	1.04627×10^{-11}
(2,2)	2.64733×10^{-13}	5.37348×10^{-14}	2.57394×10^{-12}	2.42544×10^{-11}
(3,3)	2.47735×10^{-12}	1.63114×10^{-12}	2.63203×10^{-11}	6.30784×10^{-12}
(4,4)	3.53273×10^{-12}	4.13689×10^{-11}	4.22672×10^{-11}	1.22569×10^{-12}
(5,5)	4.33525×10^{-11}	7.05007×10^{-11}	6.03086×10^{-10}	9.12692×10^{-11}
(6,6)	1.0197×10^{-2}	1.18168×10^{-9}	2.05257×10^{-8}	2.67056×10^{-9}
(7,7)		1.50789×10^{-9}	1.1151×10^{-7}	2.17597×10^{-8}
(8,8)		5.1252×10^{-4}	6.06801×10^{-7}	2.48228×10^{-7}
(9,9)			2.56535×10^{-6}	6.24128×10^{-6}
(10,10)			3.39759×10^{-6}	6.57906×10^{-6}
(11,11)			1.13805×10^{-5}	2.70452×10^{-4}
(12,12)			6.48755×10^{-5}	7.09638×10^{-4}
(13,13)				6.88505×10^{-4}
(14,14)				1.14674×10^{-2}
(15,15)				1.86774×10^{-2}
(16,16)				2.01006×10^{-3}

Figure 6: $\mathcal{Y}_{N,M}(x, t)$ of problem (4.5), where $\beta = 0$ and $N = M = \mathcal{K} = 20$. **Figure 7:** $\mathcal{E}(x, t)$ of problem (4.5), where $\beta = 0$ and $N = M = \mathcal{K} = 20$.

Figure 8: x -curves of $\mathcal{Y}(x, t)$ and $\mathcal{Y}_{N,M}(x, t)$ of problem (4.5), where $\beta = 0$ and $N = M = K = 20$.

Figure 9: $\mathcal{E}(x, 0.5)$ of problem (4.5), where $\beta = 0$ and $N = M = K = 20$.

It is also noticed that excellent estimates may be produced with a small number of collocation sites. In

Figure 10: $\mathcal{Y}_{N,M}(x, t)$ of problem (4.5), where $\beta = 0.2$ and $N = M = K = 16$.

addition, we provided the absolute errors obtained by our approach for Example 3 when $\beta = 0.1$ in Table 8 (Figure 11).

Example 5. Finally, we solve the SHEs [28]

$$\frac{\partial}{\partial t}(\mathcal{Y}(x, t)) = \left(1 + \beta \frac{d\omega(t)}{dt} \frac{\partial^2}{\partial x^2}(\mathcal{Y}(x, t))\right) + 2xt + \omega(t), \quad (x, t) \in [0, 1] \times [0, 1], \quad (4.9)$$

with

$$\mathcal{Y}(x, 0) = 0, \quad x \in [0, 1],$$

$$\mathcal{Y}(0, t) = \int_0^t \omega(\tau) d\tau, \quad \mathcal{Y}(1, t) = t^2 + \int_0^t \omega(\tau) d\tau, \quad (4.10)$$

$$t \in [0, 1],$$

the exact solution is $\mathcal{Y}(x, t) = xt^2 + \int_0^t \omega(\tau) d\tau$.

Table 7: Absolute errors for problem (4.7)

Wavelets Galerkin method [28] where $\bar{m} = 20$, $k = 2$, $M = 5$, and $N = 250$					
(x, t)	(0.2, 0.2)	(0.4, 0.4)	(0.6, 0.6)	(0.8, 0.8)	(1.0, 1.0)
$E(x, t)$	1.5377×10^{-5}	3.7708×10^{-4}	4.4558×10^{-4}	3.0031×10^{-4}	1.9104×10^{-6}
(N, M, K)	Our method				
(4,4,4)	1.74537×10^{-4}	6.3192×10^{-4}	2.37922×10^{-4}	6.45234×10^{-4}	5.57078×10^{-4}
(6,6,6)	7.02415×10^{-6}	3.1971×10^{-6}	5.63059×10^{-7}	4.57881×10^{-6}	3.91833×10^{-5}
(8,8,8)	1.49513×10^{-8}	1.14847×10^{-8}	6.8988×10^{-10}	2.88771×10^{-9}	1.54329×10^{-7}
(12,12,12)	8.99281×10^{-14}	3.38618×10^{-14}	1.34337×10^{-14}	4.55191×10^{-14}	6.75349×10^{-13}

Table 8: Absolute errors for problem (4.7) at $(x_{N,i}^{x_{\text{end}}}, t_{M,j}^{t_{\text{end}}})$, (N, M, K) , and $\beta = 0.1$

(i, j)	(4,4,4)	(6,6,6)	(8,8,8)	(12,12,12)
(1,1)	8.88178×10^{-16}	9.61453×10^{-14}	1.13753×10^{-12}	2.59797×10^{-11}
(2,2)	9.32587×10^{-15}	1.74971×10^{-13}	2.03837×10^{-13}	1.30873×10^{-12}
(3,3)	2.04281×10^{-14}	1.5854×10^{-13}	9.74776×10^{-14}	1.65556×10^{-12}
(4,4)	4.41567×10^{-1}	2.63567×10^{-13}	1.1886×10^{-11}	1.84912×10^{-11}
(5,5)		9.85878×10^{-14}	5.73759×10^{-11}	1.51753×10^{-10}
(6,6)		9.48059×10^{-3}	1.46754×10^{-10}	2.82679×10^{-9}
(7,7)			1.76612×10^{-10}	2.23406×10^{-8}
(8,8)			1.03497×10^{-2}	1.38597×10^{-7}
(9,9)				5.89702×10^{-7}
(10,10)				4.54438×10^{-6}
(11,11)				1.69804×10^{-5}
(12,12)				6.22283×10^{-4}

Figure 11: $\mathcal{Y}_{N,M}(x, t)$ of problem (4.9), where $\beta = 0.1$ and $N = M = K = 12$.

Finally, we reported the absolute errors obtained by our approach for Example 4 in Table 9, where $\beta = 0.1$.

5 Conclusion

To investigate SHEs, this work used the completely shifted Legendre collocation approach. The algorithm's strong numerical scheme yielded a slew of outstanding numerical results, demonstrating the algorithm's great efficiency. The research was conducted using both initial-boundary conditions and stochastic processes. The algorithm's results open the way for more study in this subject to be conducted in the future, with more results to be displayed.

Table 9: Absolute errors for problem (4.9) at $(x_{N,i}^{x_{\text{end}}}, t_{M,j}^{t_{\text{end}}})$, (N, M, K) , and $\beta = 0.1$

(i, j)	(4,4,4)	(6,6,6)	(8,8,8)	(12,12,12)
(1,1)	4.59071×10^{-3}	1.14365×10^{-2}	1.4868×10^{-2}	1.47869×10^{-3}
(2,2)	5.65413×10^{-2}	1.32981×10^{-2}	9.82781×10^{-2}	1.61673×10^{-4}
(3,3)	5.58898×10^{-2}	3.03539×10^{-2}	6.30974×10^{-2}	8.62726×10^{-3}
(4,4)	1.44329×10^{-14}	3.13059×10^{-2}	8.85331×10^{-2}	1.46635×10^{-2}
(5,5)		1.57833×10^{-2}	5.0286×10^{-2}	2.48492×10^{-4}
(6,6)		3.01981×10^{-14}	2.48021×10^{-2}	5.26329×10^{-2}
(7,7)			1.20099×10^{-2}	8.702×10^{-2}
(8,8)			1.8016×10^{-10}	5.90496×10^{-2}
(9,9)				5.90724×10^{-2}
(10,10)				3.35535×10^{-2}
(11,11)				1.03262×10^{-2}
(12,12)				1.01667×10^{-7}

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