Research Article

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New trigonometric B-spline approximation for numerical investigation of the regularized long-wave equation

<https://doi.org/10.1515/phys-2021-0087>

received September 02, 2021; accepted November 08, 2021

Abstract: The objective of this work is to propose a collocation technique based on new cubic trigonometric B-spline (NCTB-spline) functions to approximate the regularized long-wave (RLW) equation. This equation is used for modelling numerous problems occurring in applied sciences. The NCTB-spline collocation method is used to integrate the spatial derivatives. We use the Rubin–Graves linearization technique to linearize the non-linear term. The accuracy and efficiency of the technique are examined by employing it on three important numerical examples which have three invariants of motion *viz.* mass, momentum, and energy. It is observed that the error norms of the present method are less than the error norms of the methods available in the literature. The numerical values of these invariants have also been approximated, which remain conserved during the program run which shows that the propagation of the solitary wave is represented perfectly. The propagation of one and two solitary waves and undulations of waves are depicted graphically. The stability analysis shows that the method is unconditionally stable.

Keywords: RLW equation, NCTB-spline functions, Rubin–Graves-type linearization technique, collocation technique, stability analysis

1 Introduction

The regularized long-wave (RLW) equation is used as a model to describe long-wave behavior. Peregrine [1] introduced this equation to demonstrate the development of undular bore. Benjamin [2] discussed the properties of this equation. The aforesaid equation has been used to model various phenomena in science such as hydromagnetic waves, longitudinal dispersive waves, shallow water waves, anharmonic lattice, and rotating flow down a tube. Olver [3] proved that the aforesaid equation has only three conservation laws. The fixed conservation laws show that this equation is non-integrable. In fact, there are various initial and boundary conditions for that the general solution has not been obtained. Hence, there are only a few analytical solutions with a limited set of these conditions. So, there is a need for numerical methods for the RLW equation with different sets of initial and boundary conditions.

Various approximations have been found in the literature for the RLW equation with several sets of initial and boundary conditions. For instance, Gardner *et al.* [4] solved it numerically by the least square method based on linear space-time finite elements, whereas Dogan [5] used linear finite element-based Galerkin method. The authors of Ref. [6] used the splitting method in combination with the CB-spline method for solving non-linear RLW equation. The authors of Refs. [7,8] proposed linearized implicit and three-level finite difference methods, respectively, while a new finite difference method based on the quintic spline and splitting method has been proposed by Raslan [9]. Petrov-Galerkin finite element method, least-square quadratic finite element method and least-square cubic B-spline finite element method have been proposed by authors of Refs. [10–12], respectively, for solving the aforesaid equation. Saka *et al.* [13] and Zaki [14] used splitting methods in the combination of cubic and quadratic B-spline finite elements, respectively. The collocation method based on both quadratic and quintic B-splines has been proposed by Dağ *et al.* [15]. Mei and Chen [16] presented

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explicit multistep Galerkin method to solve this equation, whereas Dag and Derehi [17] investigated it numerically using radial basis-based meshless method. Eilbeck and McGuire [18] studied the solitary wave solutions numerically. Additionally, Cimpoiasu [19] investigated some traveling wave solutions for the long-short wave resonance model.

In Ref. [20], the authors presented a differential quadrature method based on cubic B-spline basis functions, whereas Saka and Dag [21] presented a collocation method based on quartic B-spline basis functions for the numerical solution of the aforesaid equation. The authors of Refs. [22–25] used Galerkin's method using linear finite elements, quadratic B-spline finite element-based lumped Galerkin method, B-spline finite element method, and quadratic B-spline Galerkin finite element method in combination with space-splitting technique, respectively, to investigate the motion of a single solitary wave, development of two solitary wave interaction and an undular bore, numerically. The authors of Ref. [26] proposed a fourth-order collocation method based on cubic B-splines to solve RLW and modified RLW equations. Guo *et al.* [27] applied multiple integral FVM based on Lagrange interpolation for the Rosenau-RLW equation. Recently, Dhiman and Tamsir [28] applied a trigonometric cubic B-spline collocation technique for Fisher's reaction-diffusion equation.

In this article, we consider the RLW equation as follows:

$$(u - \mu u_{xx})_t + (\varepsilon u + \alpha)u_x = 0, \quad x \in [a, b], \quad t \in (0, T], \quad (1)$$

subject to

$$u(x, 0) = \varphi(x), \quad x \in [a, b], \quad (2)$$

and the boundary conditions

$$\begin{aligned} u(a, t) = \beta_1, \quad u(b, t) = \beta_2, \quad u_x(a, t) = 0, \\ u_x(b, t) = 0, \quad t \in (0, T], \end{aligned} \quad (3)$$

where μ , ε , α are known positive parameters and the wave amplitude $u = u(x, t)$ is to be obtained. The terms β_1 and β_2 are given constants. Bona and Byrant [29] already proved that there exists a unique solution for the problem (1)–(3). The problem (1)–(3) conserve mass, momentum, and energy. These terms are physical quantities of motion, which are given by

$$\begin{aligned} I_1 &= \int_a^b u dx, \quad I_2 = \int_a^b (u^2 + \mu u_x^2) dx, \\ I_3 &= \int_a^b (u^3 + 3\mu u_x^2) dx. \end{aligned} \quad (4)$$

The rest of the article is organized as follows. The procedure of the new trigonometric B-spline collocation technique

is given in Section 2. In Section 3, a brief discretization of the problems using the proposed technique is given. The von Neumann stability is discussed in Section 4. In Section 5, numerical results are given. Finally, Section 6 focuses on the conclusions.

2 New cubic trigonometric B-spline (NCTB-spline) collocation technique

In this section, we introduce the collocation method based on NCTB-spline functions. First, we partition the problem domain $x \in [a, b]$ into mesh of equal length $h = x_{i+1} - x_i$ for $i = 0, 1, \dots, M$. For discrete form, we use $u(x_i, t_j) = u_i^j$ for $i = 0, 1, \dots, M$ and $j = 0, 1, \dots, N$. The NCTB-spline functions $NB_i(x)$ for $i = -1, 0, \dots, M, M+1$ are given by Ref. [30]

$$NB_i(x) = \begin{cases} d_{i-2}(g_3(z_{i-2})), & x \in [x_{i-2}, x_{i-1}), \\ \sum_{k=0}^3 c_{i-1,k} g_k(z_{i-1}), & x \in [x_{i-1}, x_i), \\ \sum_{k=0}^3 b_{i,k} g_k(z_{i-1}), & x \in [x_i, x_{i+1}), \\ a_{i+1} g_0(z_{i-1}), & x \in [x_{i+1}, x_{i+2}), \\ 0, & \text{else,} \end{cases} \quad (5)$$

where

$$\begin{aligned} z_i(x) &= \frac{\pi}{2} \left(\frac{x - x_i}{x_{i+1} - x_i} \right), \quad i = 1, 2, \dots, \\ a_i &= b_{i,3} = c_{i,0} = d_i = \frac{vl_i^2}{\mu_i \psi_i}, \\ b_{i,0} &= \frac{v\psi_i(\alpha_i l_i + \beta_i l_{i-1})}{\mu_i \psi_i^2}, \quad b_{i,1} = \frac{v\beta_i}{\psi_i}, \\ b_{i,2} &= \frac{vl_{i+1}}{\psi_i}, \quad c_{i,1} = \frac{vl_{i-1}}{\psi_i}, \\ c_{i,2} &= \frac{v\alpha_i}{\psi_i}, \quad c_{i,3} = \frac{v\psi_i(\alpha_i l_{i+1} + \beta_i l_i)}{\mu_i \psi_i^2}, \end{aligned}$$

where

$$\begin{aligned} l_i &= x_{i+1} - x_i, \quad \mu_i = l_{i-1} + l_i, \quad \alpha_i = l_{i-1} + vl_i, \\ \beta_i &= vl_{i-1} + l_i, \quad v = 1 + 2\eta, \quad \psi_i = vl_{i-2} + v^2 l_{i-1} + vl_i. \end{aligned}$$

The new trigonometric functions are defined as:

$$\begin{cases} g_0(z) = (1 - \sin z)^2 (1 + (1 - \eta) \sin z), \\ g_1(z) = \sin z (1 - \sin z) (2 - (1 - \eta)(1 - \sin z)), \\ g_2(z) = \cos z (1 - \cos z) (2 - (1 - \eta)(1 - \cos z)), \\ g_3(z) = (1 - \cos z)^2 (1 + (1 - \eta) \cos z), \end{cases} \quad (6)$$

Table 1: Values of $NB_i(x)$, $NB'_i(x)$, and $NB''_i(x)$ at the knots

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$NB_i(x)$	0	ξ_1	ξ_2	ξ_3	0
$NB'_i(x)$	0	ξ_4	ξ_5	ξ_6	0
$NB''_i(x)$	0	ξ_7	ξ_8	ξ_9	0

where $\eta \in (\frac{1}{2}, 2)$ is the shape parameter. The set of NCTB-spline basis functions $\{NB_i(x) : i = -1, 0, \dots, M, M+1\}$ forms a basis over the problem domain. Table 1 gives the values of $NB_i(x)$ and its two derivatives at the knots.

We assume that the approximation u_M at (x_i, t_j) is written as a linear combination of the NCTB-spline functions and unknown time-dependent coefficients as:

$$u_M(x, t) = \sum_{m=-1}^{M+1} C_m(t) NB_m(x). \quad (7)$$

The NCTB-spline and its four principle derivatives vanish outside of the region $[x_{i-2}, x_{i+2}]$. Since each NCTB-spline covers four elements, each element is covered by four NCTB-splines. Hence, principle four NCTB-splines cover the interval $[x_i, x_{i+1}]$. So, the variation of the $U_M(x, t)$, over the element, can be expressed as

$$u_M(x, t) = \sum_{m=i-1}^{i+1} C_m(t) NB_m(x). \quad (8)$$

From Eq. (8) and Table 1, we get the approximated $u(x, t)$, $u_x(x, t)$, and $u_{xx}(x, t)$ as follows:

$$u_i^j = \xi_1 C_{i-1}^j + \xi_2 C_i^j + \xi_3 C_{i+1}^j, \quad (9)$$

$$(u_x)_i^j = \xi_4 C_{i-1}^j + \xi_5 C_i^j + \xi_6 C_{i+1}^j, \quad (10)$$

$$(u_{xx})_i^j = \xi_7 C_{i-1}^j + \xi_8 C_i^j + \xi_9 C_{i+1}^j, \quad (11)$$

where $C_i^j = C_i(t_j)$.

3 Discretization of the RLW equation

This section considers the problem (1)–(3) when μ, ε, α are given. First, we discretize Eq. (1) as follows:

$$\frac{(u - \mu(u_{xx}))_i^{j+1} - (u - \mu(u_{xx}))_i^j}{\Delta t} + \frac{\varepsilon}{2}((uu_x)_i^{j+1} + (uu_x)_i^j) + \frac{\alpha}{2}((u_x)_i^{j+1} + (u_x)_i^j) = 0. \quad (12)$$

We linearize the non-linear term by Rubin–Graves technique [31] as follows:

$$u^{j+1}(u_x)^{j+1} = u^j(u_x)^{j+1} + u^{j+1}(u_x)^j - u^j(u_x)^j. \quad (13)$$

Using Eq. (13) in Eq. (12), we get

$$(u - \mu(u_{xx}))_i^{j+1} + \frac{\Delta t \varepsilon}{2} (u_i^j (u_x)_i^{j+1} + u_i^{j+1} (u_x)_i^j) + \frac{\Delta t \alpha}{2} (u_x)_i^{j+1} = (u - \mu(u_{xx}))_i^j - \frac{\Delta t \alpha}{2} (u_x)_i^j. \quad (14)$$

Simplifying the above equation, we get

$$P_i^j u_i^{j+1} + Q_i^j (u_x)_i^{j+1} - \mu(u_{xx})_i^{j+1} = u_i^j - \frac{\Delta t \alpha}{2} (u_x)_i^j - \mu(u_{xx})_i^j, \quad (15)$$

where

$$P_i^j = 1 + \frac{\Delta t \varepsilon}{2} (u_x)_i^j, \quad Q_i^j = \frac{\Delta t \varepsilon}{2} u_i^j + \frac{\Delta t \alpha}{2}.$$

Now, using equations (9)–(11), we get

$$\begin{aligned} & (P_i^j \xi_1 + Q_i^j \xi_4 - \mu \xi_7) C_{i-1}^{j+1} + (P_i^j \xi_2 + Q_i^j \xi_5 - \mu \xi_8) C_i^{j+1} \\ & + (P_i^j \xi_3 + Q_i^j \xi_6 - \mu \xi_9) C_{i+1}^{j+1} \\ & = \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) C_{i-1}^j + \left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 \right) C_i^j \\ & + \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) C_{i+1}^j, \\ & i = 1, 2, \dots, M-1, \quad j = 0, 1, \dots, N. \end{aligned} \quad (16)$$

Now, we discretize the boundary conditions as follows:

$$\xi_1 C_{-1}^j + \xi_2 C_0^j + \xi_3 C_1^j = \beta_1, \quad j = 1, 2, \dots, N, \quad (17)$$

$$\xi_1 C_{M-1}^{j+1} + \xi_2 C_M^{j+1} + \xi_3 C_{M+1}^{j+1} = \beta_2, \quad j = 1, 2, \dots, N. \quad (18)$$

From equations (17) and (18), we get

$$C_{-1}^j = \frac{\beta_1}{\xi_1} - \frac{\xi_2}{\xi_1} C_0^j - \frac{\xi_3}{\xi_1} C_1^j, \quad j = 1, 2, \dots, N, \quad (19)$$

$$C_{M+1}^j = \frac{\beta_2}{\xi_3} - \frac{\xi_1}{\xi_3} C_{M-1}^j - \frac{\xi_2}{\xi_3} C_M^j, \quad j = 1, 2, \dots, N. \quad (20)$$

For $i = 0$, using (19) in (16), we get

$$\begin{aligned} & \left(P_0^j \xi_2 + Q_0^j \xi_5 - \mu \xi_8 - \frac{\xi_2}{\xi_1} (P_0^j \xi_1 + Q_0^j \xi_4 - \mu \xi_7) \right) C_0^{j+1} \\ & + (P_0^j \xi_3 + Q_0^j \xi_6 - \mu \xi_9 - \frac{\xi_3}{\xi_1} (P_0^j \xi_1 + Q_0^j \xi_4 - \mu \xi_7)) C_1^{j+1} \\ & = \left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 - \frac{\xi_2}{\xi_1} \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) \right) C_0^j \\ & + \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 - \frac{\xi_3}{\xi_1} \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) \right) C_1^j \\ & + \frac{\alpha_1}{\xi_1} \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) - \frac{\alpha_1}{\xi_1} (P_0^j \xi_1 + Q_0^j \xi_4 - \mu \xi_7), \quad j = 0, 1, \dots, N. \end{aligned} \quad (21)$$

For $i = M$, using (20) in (16), we get

$$\begin{aligned}
& \left((P_M^j \xi_1 + Q_M^j \xi_4 - \mu \xi_7) - \frac{\xi_1}{\xi_3} (P_M^j \xi_3 + Q_M^j \xi_6 - \mu \xi_9) \right) C_{M-1}^{j+1} + \left((P_M^j \xi_2 + Q_M^j \xi_5 - \mu \xi_8) - \frac{\xi_2}{\xi_3} (P_M^j \xi_3 + Q_M^j \xi_6 - \mu \xi_9) \right) C_M^{j+1} \\
&= \left(\left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) - \frac{\xi_1}{\xi_3} \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) \right) C_{M-1}^j + \left(\left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 \right) - \frac{\xi_2}{\xi_3} \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) \right) C_M^j \\
&+ \frac{\alpha_2}{\xi_3} \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 - (P_M^j \xi_3 + Q_M^j \xi_6 - \mu \xi_9) \right), \quad j = 0, 1, \dots, N.
\end{aligned} \quad (22)$$

At t_{j+1} , $j = 0, 1, \dots, N$, Eqs. (21), (16), and (22) can be written into the form of a linear system of order $(M+1) \times (M+1)$ as follows:

$$\begin{bmatrix}
\hat{q}_0^j - \frac{\xi_2}{\xi_1} \hat{p}_0^j & \hat{r}_0^j - \frac{\xi_3}{\xi_1} \hat{p}_0^j & & & & \\
\hat{p}_1^j & \hat{q}_1^j & \hat{r}_1^j & & & \\
& \hat{p}_2^j & \hat{q}_2^j & \hat{r}_2^j & & \\
& & \ddots & \ddots & \ddots & \\
& & & \hat{p}_{M-2}^j & \hat{q}_{M-2}^j & \hat{r}_{M-2}^j \\
& & & & \hat{p}_M^j - \frac{\xi_1}{\xi_3} \hat{r}_M^j & \hat{q}_M^j - \frac{\xi_2}{\xi_3} \hat{r}_M^j
\end{bmatrix}
\begin{bmatrix}
C_0^{j+1} \\
C_1^{j+1} \\
C_2^{j+1} \\
\vdots \\
C_{M-2}^{j+1} \\
C_{M-1}^{j+1} \\
C_M^{j+1}
\end{bmatrix}
=
\begin{bmatrix}
R_0^j \\
R_1^j \\
R_2^j \\
\vdots \\
R_{M-2}^j \\
R_{M-1}^j \\
R_M^j
\end{bmatrix},$$

where

$$\begin{aligned}
& \hat{p}_i^j = (P_i^j \xi_1 + Q_i^j \xi_4 - \mu \xi_7), \quad \hat{q}_i^j = (P_i^j \xi_2 + Q_i^j \xi_5 - \mu \xi_8), \quad \hat{r}_i^j = (P_i^j \xi_3 + Q_i^j \xi_6 - \mu \xi_9), \\
& R_0^j = \left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 - \frac{\xi_2}{\xi_1} \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) \right) C_0^j + \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 - \frac{\xi_3}{\xi_1} \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) \right) C_1^j \\
&+ \frac{\alpha_1}{\xi_1} \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 - (P_0^j \xi_1 + Q_0^j \xi_4 - \mu \xi_7) \right), \quad j = 0, 2, \dots, N, \\
& R_i^j \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) C_{i-1}^j + \left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 \right) C_i^j + \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) C_{i+1}^j, \\
& \quad i = 1, 2, \dots, M-1, \quad j = 0, 1, \dots, N, \\
& R_M^j = \left(\left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) - \frac{\xi_1}{\xi_3} \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) \right) C_{M-1}^j + \left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 - \frac{\xi_2}{\xi_3} \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) \right) C_M^j \\
&+ \frac{\alpha_2}{\xi_3} \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 - (P_M^j \xi_3 + Q_M^j \xi_6 - \mu \xi_9) \right), \quad j = 0, 1, \dots, N.
\end{aligned}$$

Now, we find the vector $(C_{-1}^0, C_0^0, \dots, C_M^0, C_{M+1}^0)$ from the initial condition which gives $M+1$ equation in $N+3$ unknowns. To eliminate the unknowns C_{-1}^0 and C_{M+1}^0 , we use $u_x(a, t) = 0$ and $u_x(b, t) = 0$, which gives

$$C_{-1}^0 = -\frac{\xi_5}{\xi_4} C_0^0 - \frac{\xi_6}{\xi_4} C_1^0, \quad (23)$$

$$C_{M+1}^0 = -\frac{\xi_4}{\xi_6} C_{M-1}^0 - \frac{\xi_5}{\xi_6} C_M^0. \quad (24)$$

Removing C_{-1}^0 and C_{M+1}^0 from the system, we have a tridiagonal system of order $(M+1) \times (M+1)$ as follows:

$$\begin{bmatrix} \xi_2 - \frac{\xi_1 \xi_5}{\xi_4} & \xi_3 - \frac{\xi_1 \xi_6}{\xi_4} & & & \\ & \xi_1 & \xi_2 & \xi_3 & \\ & & \xi_1 & \xi_2 & \xi_3 \\ & & & \ddots & \ddots \\ & & & & \xi_1 & \xi_2 & \xi_3 \\ & & & & & \xi_1 - \frac{\xi_1 \xi_4}{\xi_6} & \xi_2 - \frac{\xi_1 \xi_5}{\xi_6} \end{bmatrix} \begin{bmatrix} C_0^0 \\ C_1^0 \\ C_2^0 \\ \vdots \\ C_{M-1}^0 \\ C_M^0 \end{bmatrix} = \begin{bmatrix} \varphi(x_0) - \frac{\xi_1}{\xi_4} \varphi'(x_0) \\ \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \varphi(x_{M-1}) \\ \varphi(x_M) - \frac{\xi_1}{\xi_6} \varphi'(x_M) \end{bmatrix}.$$

4 Stability analysis

Now, the von-Neumann stability analysis [32–36] is carried out for the RLW equation. We consider the discretized system (12) as

$$(u - \mu(u_{xx}))_i^{j+1} + \frac{\Delta t \varepsilon}{2} (u_i^j (u_x)_i^{j+1} + u_i^{j+1} (u_x)_i^j) + \frac{\Delta t \alpha}{2} (u_x)_i^{j+1} = (u - \mu(u_{xx}))_i^j - \frac{\Delta t \alpha}{2} (u_x)_i^j. \quad (25)$$

Using equations (9)–(11), we get

$$\begin{aligned} & \left[\left(1 + \frac{\Delta t \varepsilon}{2} (u_x)_i^j \right) \xi_1 + \left(\frac{\Delta t \varepsilon}{2} u_i^j + \frac{\Delta t \alpha}{2} \right) \xi_4 - \mu \xi_7 \right] C_{i-1}^{j+1} \\ & + \left[\left(1 + \frac{\Delta t \varepsilon}{2} (u_x)_i^j \right) \xi_2 + \left(\frac{\Delta t \varepsilon}{2} u_i^j + \frac{\Delta t \alpha}{2} \right) \xi_5 - \mu \xi_8 \right] C_i^{j+1} + \left[\left(1 + \frac{\Delta t \varepsilon}{2} (u_x)_i^j \right) \xi_3 + \left(\frac{\Delta t \varepsilon}{2} u_i^j + \frac{\Delta t \alpha}{2} \right) \xi_6 - \mu \xi_9 \right] C_{i+1}^{j+1} \\ & = \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) C_{i-1}^j + \left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 \right) C_i^j + \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) C_{i+1}^j. \end{aligned} \quad (26)$$

Now, we assume $u = \bar{k}_1$ and $u_x = \bar{k}_2$ as local constants for known level to get

$$\begin{aligned} & (\hat{A} \xi_1 + \hat{B} \xi_4 - \mu \xi_7) C_{i-1}^{j+1} + (\hat{A} \xi_2 + \hat{B} \xi_5 - \mu \xi_8) C_i^{j+1} + (\hat{A} \xi_3 + \hat{B} \xi_6 - \mu \xi_9) C_{i+1}^{j+1} \\ & = \left(\xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu \xi_7 \right) C_{i-1}^j + \left(\xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu \xi_8 \right) C_i^j + \left(\xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu \xi_9 \right) C_{i+1}^j, \end{aligned} \quad (27)$$

where

$$\hat{A} = \left(1 + \frac{\Delta t \varepsilon}{2} k_2 \right), \quad \hat{B} = \left(\frac{\Delta t \varepsilon}{2} k_1 + \frac{\Delta t \alpha}{2} \right).$$

Next, we consider $C_i^j = \delta^j e^{ki\phi}$ as trial solutions at a given point x_i , where $\phi = \theta h$ and $k = \sqrt{-1}$. The θ is the mode number and h is the element size. Substituting the trial solution in the above equation, we get

$$\begin{aligned}
& (\hat{A}_1 \cos(k\phi) + \hat{A}_2 + \hat{A}_3 \cos(k\phi) \\
& - i(\hat{A}_1 + \hat{A}_3) \sin(k\phi)) \delta \hat{B} \\
& = \hat{B}_1 \cos(k\phi) + \hat{B}_2 + \hat{B}_3 \cos(k\phi) \\
& - i(\hat{B}_1 + \hat{B}_3) \sin(k\phi),
\end{aligned} \quad (28)$$

where

$$\begin{aligned}
\hat{A}_1 &= \hat{A}\xi_1 + \hat{B}\xi_4 - \mu\xi_7, \quad \hat{A}_2 = \hat{A}\xi_2 + \hat{B}\xi_5 - \mu\xi_8, \\
\hat{A}_3 &= \hat{A}\xi_3 + \hat{B}\xi_6 - \mu\xi_9, \\
\hat{B}_1 &= \xi_1 - \frac{\Delta t \alpha}{2} \xi_4 - \mu\xi_7, \quad \hat{B}_2 = \xi_2 - \frac{\Delta t \alpha}{2} \xi_5 - \mu\xi_8, \\
\hat{B}_3 &= \xi_3 - \frac{\Delta t \alpha}{2} \xi_6 - \mu\xi_9.
\end{aligned}$$

Now, simplifying the terms in equation (28), we get

$$|\delta|^2 = \frac{(\hat{B}_1 - \hat{B}_2 + \hat{B}_3)^2 + 4(\hat{B}_1 + \hat{B}_3)\hat{B}_2 \cos^2(k\phi)}{(\hat{A}_1 - \hat{A}_2 + \hat{A}_3)^2 + 4(\hat{A}_1 + \hat{A}_3)\hat{A}_2 \cos^2(k\phi)}, \quad (29)$$

where

$$\begin{aligned}
\hat{A}_1 - \hat{A}_2 + \hat{A}_3 &= (\xi_1 - \xi_2 + \xi_3) - \frac{\Delta t \alpha}{2} (\xi_4 - \xi_5 + \xi_6) \\
&\quad - \mu(\xi_7 - \xi_8 + \xi_9) + \frac{\Delta t \alpha}{2} (k_2(\xi_1 - \xi_2 + \xi_3) \\
&\quad + \alpha(k_1 + 2)(\xi_4 - \xi_5 + \xi_6)), \\
\hat{B}_1 - \hat{B}_2 + \hat{B}_3 &= (\xi_1 - \xi_2 + \xi_3) - \frac{\Delta t \alpha}{2} (\xi_4 - \xi_5 + \xi_6) \\
&\quad - \mu(\xi_7 - \xi_8 + \xi_9).
\end{aligned}$$

Since $\xi_1 - \xi_2 + \xi_3 > 0$ and $\xi_4 - \xi_5 + \xi_6 > 0$, we get $(\hat{A}_1 - \hat{A}_2 + \hat{A}_3)^2 \geq (\hat{B}_1 - \hat{B}_2 + \hat{B}_3)^2$. Also, $\hat{A}_1 \geq \hat{B}_1$, $\hat{A}_2 \geq \hat{B}_2$, and $\hat{A}_3 \geq \hat{B}_3$. Therefore, we get $|\delta| \leq 1$, and hence the method is unconditionally stable for the discretized system of the RLW equation.

5 Results and discussion

In this section, we consider three numerical examples in order to check the accuracy and efficiency of the proposed method. The L_2 and L_∞ errors are given by

$$L_2 = \left(h \sum_{i=0}^M |u_i - u_i^e|^2 \right)^{1/2}, \quad L_\infty = \max_{0 \leq i \leq M} |u_i - u_i^e|, \quad (30)$$

where u_i and u^e are numerical and analytical solutions at x_i .

5.1 Example 1

First, we consider the RLW equation (1) in $x \in [-40, 60]$ with analytical solution [8,26,37]

Table 2: Comparison of the present solutions with the solutions obtained by Mittal and Rohila [26], in terms of L_∞ , for Example 1

t	$c = 0.1$		$c = 0.3$	
	Mittal and Rohila [26]	Present	Mittal and Rohila [26]	Present
1	4.57×10^7	2.01×10^7	1.00×10^5	5.40×10^6
2	3.79×10^5	1.83×10^7	1.84×10^5	5.00×10^6
3	3.80×10^5	1.01×10^7	6.09×10^5	5.50×10^6
4	3.80×10^5	1.06×10^7	6.08×10^5	5.25×10^6
5	2.23×10^6	1.90×10^7	6.08×10^5	4.57×10^6
6	2.75×10^6	1.07×10^7	6.08×10^5	3.10×10^6
7	3.14×10^6	1.05×10^7	6.09×10^5	2.25×10^6
8	3.00×10^6	1.54×10^7	6.09×10^5	2.12×10^6
9	3.82×10^6	1.50×10^7	6.08×10^5	1.80×10^6
10	3.78×10^6	1.46×10^7	6.09×10^5	1.35×10^6

$$u(x, t) = 3c \operatorname{sech}^2(kx - kvt - kx_0), \quad (31)$$

and the boundary conditions (3), where $\beta_1 = \beta_2 = 0$. This exact solution corresponds to the single solitary wave

Table 3: Comparison of the present method and existing method results with $c = 0.03$, for Example 1

Methods	L_∞	I_1	I_2	I_3
Analytical	—	3.97992	0.81046	2.57900
Present method	1.12×10^6	3.97993	0.81046	2.57901
Dag and Ozer [12]	1.50×10^3	3.96466	0.80461	2.56971
Jain et al. [6]	6.70×10^2	4.41219	0.89734	2.85361
Kutluay and Esen [8]	2.10×10^4	3.97997	0.81045	2.57901
Mittal and Rohila [26]	7.81×10^5	3.97993	0.81046	2.57901
Raslan [9]	2.27×10^4	3.97995	0.80972	2.57607
QBCM1 [21]	8.30×10^5	3.97995	0.81046	2.57901
QBCM2 [21]	1.29×10^4	3.97995	0.81046	2.57901
Zaki [14]	1.81×10^4	3.97781	0.80963	2.5762

Table 4: L_2 and L_∞ error norms with $c = 0.1$, $\Delta t = 0.1$, and $\Delta x = 0.125$, for Example 1

t	QBSCM $L_2 \times 10^4$	QBSCM $L_\infty \times 10^4$	Present method $L_2 \times 10^4$	Present method $L_\infty \times 10^4$
0	0	0	0	0
4	0.1757	0.0693	0.1258	0.0548
8	0.2249	0.0887	0.1565	0.0652
12	0.3355	0.1072	0.2547	0.0923
16	0.4075	0.1224	0.3582	0.1567
20	0.4315	0.1321	0.3649	0.1825

Figure 1: Numerical and analytical solutions together with absolute error norms at: (a) $t = 4$, (b) $t = 8$, and (c) $t = 16$, for Example 1.

centrally located at x_0 initially with amplitude $3c$ and with k , where $k = \frac{1}{2} \left(\frac{c\varepsilon}{\mu\nu} \right)^{\frac{1}{2}}$. The term $\nu = 1 + c\varepsilon$ stands for the wave velocity. The exact values of mass, momentum, and energy are given by

$$I_1 = \frac{6c}{k}, \quad I_2 = \frac{12c^2}{k} + \frac{48kc^2\mu}{5}, \quad I_3 = \frac{36c^2}{k} + \frac{144c^3}{5k}. \quad (32)$$

We fix $\varepsilon = 1$, $\mu = 1$, $\alpha = 1$, and $x_0 = 0$ to approximate Example 1. First, we approximate the error norms for Example 1 with parameters $M = 200$, $\Delta t = 0.001$ for $x = c = 0.1$ and 0.3 at different t , which are presented in Table 2. It is clear from this table that the obtained results are better than the results presented by Mittal and Rohila [26]. Next, we computed the L_∞ error norms

Figure 2: Single solitary wave profiles with $M = 200$ for: (a) $c = 0.1$ and (b) $c = 0.3$, for Example 1.

with $M = 200$, $\Delta t = 0.1$, $t = 20$, which are presented in Tables 3 and 4 for $c = 0.1$ and 0.3 , respectively, together with the exact values of mass, energy, and momentum. We found from these tables that the present results are better than the results obtained by Dag and Ozer [12], Jain *et al.* [6], Kutluay and Esen [8], Mittal and Rohila [26], Raslan [9], two methods of Saka and Dag [21] viz. QBCM1 and QBCM2, and Zaki [14]. Moreover, during the program run, we observe that the approximated values of mass, energy, and momentum remain almost constants, i.e., propagation of the solitary wave represented accurately. The comparison between numerical and analytical solutions together with absolute error norms for $M = 200$, $\Delta t = 0.1$ for $c = 0.1$ at $t = 4, 8$, and 16 are depicted in Figure 1. One can see from these figures that there is an excellent agreement between approximate and analytical solutions. Figure 2 shows the profiles of single solitary wave at various time with $M = 200$, $\Delta t = 0.1$ for $c = 0.1$ and 0.3 , respectively, at $t = 1, 5, 10, 15$, and 20 .

5.2 Example 2

Now, we consider the RLW (1) in interval $x \in [0, 120]$ with the initial condition [8,9,26,37]

$$u(x, 0) = \sum_{j=1}^2 3C_j \operatorname{sech}^2(k_j(x - x_j)), \quad (33)$$

where $C_j = \frac{4k_j^2}{1 - 4k_j^2}$, $j = 1, 2$. The RLW Eq. (1) with this initial condition represents two positive solitary wave equations. For the numerical computation of this example, we choose $k_1 = 0.4$, $k_2 = 0.3$, $x_1 = 15$, $x_2 = 35$,

$\varepsilon = 1$, $\alpha = 1$, and $\mu = 1$. Figure 3 depicts the propagation of two positive solitary waves with $N = 300$ for $t = 0, 5, 10, 15, 20$, and 25 , respectively. We also check the efficiency of the proposed method by computing the invariant mass, momentum, and energy. Tables 5 and 6 show the comparison of the computed invariants with the invariants presented in Refs. [8,26]. It is clear from these figures and tables that an excellent agreement is found between present and existing results. Also, it is noticed that the exact values of mass, energy, and momentum remain conserved during the program run which shows that the propagation of the solitary wave represented perfectly.

5.3 Example 3

Finally, we consider the RLW Eq. (1) with initial condition [8,26,37–39]

$$u(x, 0) = 0.5\beta_1 \left[1 - \tanh\left(\frac{x - x_c}{d}\right) \right], \quad (34)$$

to study the development of the undular bore together with the boundary conditions

$$u(a, t) = \beta_1, \quad u(b, t) = 0. \quad (35)$$

The term d denotes the slope between the shallow and deeper water, whereas $u(x, 0) = 0$ represents the elevation of the water above the equilibrium surface at initial time. The change in water level of magnitude $u(x, 0)$ is centered at $x = x_c$. We choose the parameters $\beta_1 = 0.1$, $\varepsilon = 1.5$, $\mu = 1$, $\alpha = 1$, $x_c = 0$, $a = -36$, $b = 300$, $\Delta t = 0.1$,

Figure 3: Plot of the numerical solutions for $M = 300$ at different time levels: (a) $t = 0$, (b) $t = 5$, (c) $t = 10$, (d) $t = 15$, (e) $t = 20$, and (f) $t = 25$, for Example 2.

Table 5: Numerical values of mass and momentum at different time levels, for Example 2

t	I_1 (Mass) Present	I_2 (Momentum) Present	I_1 (Mass) [26]	I_2 (Momentum) [26]	I_1 (Mass) [8]	I_2 (Momentum) [8]
0	37.9165	120.4126	37.9165	120.5220	37.9165	120.3515
2	37.9168	120.4115	37.9169	120.5230	37.9168	120.3571
4	37.9170	120.4356	37.9170	120.5240	37.9170	120.3584
6	37.9171	120.4360	37.9171	120.5320	37.9170	120.3586
8	37.9173	120.4275	37.9173	120.5560	37.9172	120.3583
10	37.9174	120.4596	37.9174	120.5950	37.9173	120.3570
12	37.9174	120.4675	37.9174	120.5340	37.9173	120.3915
14	37.9174	120.3570	37.9174	120.2570	37.9174	120.4156
16	37.9175	120.3045	37.9175	120.1940	37.9174	120.3886
18	37.9175	120.3495	37.9175	120.3570	37.9174	120.3653
20	37.9174	120.4169	37.9175	120.4620	37.9174	120.3599
22	37.9174	120.4168	37.9175	120.4620	37.9174	120.3599
24	37.9175	120.4215	37.9175	120.5160	37.9175	120.3595

Table 6: Numerical values of energy, for Example 2

t	I_3 (Energy) Present	I_3 (Energy) [26]	I_3 (Energy) [8]
0	744.0811	744.0810	744.0814
2	744.0760	744.0770	744.0387
4	744.0560	744.0660	744.0110
6	744.0201	744.0320	744.9796
8	743.8815	743.8910	743.8679
10	743.4185	743.4130	743.4202
12	742.3056	742.2550	742.3387
14	741.5525	741.5420	741.5781
16	742.5314	742.5810	742.4890
18	743.5232	743.5710	743.4752
20	743.9081	743.9290	743.8638
22	743.9081	743.9290	743.8638
24	743.8676	743.9175	744.0037

$\Delta x = 0.24$, $d = 2$, and 5 and we run the program until $t = 250$. The initial functions at $t = 0$ are depicted in Figure 4 for $d = 5, 2$ while the undulation profiles at $t = 250$ are shown in Figure 5 for $d = 5$ and 2, respectively. Graphs in Figure 5 represent the undular bore profiles when the gentle slope $d = 5$ and steep slope $d = 2$, respectively, are used. A few more undulations can be seen when a steep slope, i.e., $d = 0$ is used as the initial condition. It is noticed that an increase in number of undulations is because of the form of the initial function. Also, we see that the magnitudes of leading undulations for the $d = 5$ and 2 become closer and closer for the longer run. An excellent agreement between the present and existing results [1,4,15] is found.

Figure 4: Initial profile at $t = 0$ for: (a) $d = 5$ and (b) $d = 2$, for Example 3.

Figure 5: Undulation profiles at time $t = 250$ for: (a) $d = 5$ and (b) $d = 2$, for Example 3.

6 Conclusion

An NCTB-spline collocation technique has been proposed for the numerical computation of the RLW equation. The proposed collocation technique is used to integrate the spatial derivatives, whereas the discretization of the time derivative is done by the usual finite difference method. The numerical results have been presented in tabular form as well as graphically at different time levels. The present solutions have been compared with the solutions obtained by methods of Dag and Ozer [12], Jain *et al.* [6], Kutluay and Esen [8], Mittal and Rohila [26], Raslan [9], two methods of Saka and Dag [21] viz. QBCM1 and QBCM2, and Zaki [14] as well as with the exact solutions. It has been observed that the present error norms are less than the error norms presented in Refs. [6,8,9,12,14,21,26]. We noticed that the physical quantities of motion remain conserved during the program run, which shows that the propagation of the solitary wave is represented perfectly. The stability analysis of the discretized system of the RLW equation shows that the method is unconditionally stable.

Funding information: The authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

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