Research Article

Aly R. Seadawy*, Shafiq U. Rehman, Muhammad Younis, Syed T. R. Rizvi, and Ali Althobaiti

On solitons: Propagation of shallow water waves for the fifth-order KdV hierarchy integrable equation

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Abstract: This article studies the fifth-order KdV (5KdV) hierarchy integrable equation, which arises naturally in the modeling of numerous wave phenomena such as the propagation of shallow water waves over a flat surface, gravity–capillary waves, and magneto-sound propagation in plasma. Two innovative integration norms, namely, the $\left(\frac{G'}{G^2}\right)$ -expansion and ansatz approaches, are used to secure the exact soliton solutions of the 5KdV type equations in the shapes of hyperbolic, singular, singular periodic, shock, shock-singular, solitary wave, and rational solutions. The constraint conditions of the achieved solutions are also presented. Besides, by selecting appropriate criteria, the actual portrayal of certain obtained results is sorted out graphically in three-dimensional, two-dimensional, and contour graphs. The results suggest that the procedures used are concise, direct, and efficient, and that they can be applied to more complex nonlinear phenomena.

Keywords: solitons, 5KdV equation, ansatz approach, $\left(\frac{G'}{G^2}\right)$ -expansion function method

1 Introduction

The study of nonlinear phenomena is a glorious field for the researchers due to its significant applications in

this advanced era of technology. The analytic solutions of nonlinear partial differential equations (NLPDEs) are the attractive and testing field of examination for mathematicians and researchers [1–6]. For the investigation of exact wave structures, NLPDEs have a huge impact in diverse areas of nonlinear sciences like mathematical physics, mechanics, chemistry, thermodynamic, physical science, electromagnetism, bio-math, and vice versa. To extract and study the exact and solitary waves of complicated nonlinear problems apart from the mathematical perspective can be analyzed using different techniques. Therefore, the new techniques are developed and modified in recent times [7–14, 20–25].

Moreover, solitons models are widely useful in the mechanism of solitary wave-based communications links, optical pulse compressors, fiber-optic amplifiers, and several others. The solitons can propagate in nonlinear dispersive media. Remarkably, the interest is growing steadily to the study of shock and solitary solitons in recent years. To extract the various type of solutions like soliton solutions, traveling wave solutions, cnoidal and snoidal waves, trigonometric wave solutions, and many more have been the challenging task for the researchers; for details, see also refs. [26–34]. In this article, our main purpose is to obtain diverse nonlinear dynamical wave structures such as hyperbolic, solitary, singular, shock, shock-singular soliton, singular periodic, and rational function solutions for the 5KdV hierarchy integrable equation with Lax operator.

This equation can be reduced to other equations, namely, Lax's fifth-order KdV equation, Sawada–Kotera equation, and Kaup–Kupershmidt equation for different values of constants. It is the first equation in a hierarchy of integrable equations with Lax operator [35–37].

Consider the fifth-order KdV equation with constant-coefficients in the following form:

$$u_t + \alpha u u_{3x} + \beta u_x u_{2x} + \gamma u^2 u_x + u_{5x} = 0, \quad (1)$$

where α, β, γ are nonzero constants. With the choice of $\alpha = 10, \beta = 20$, and $\gamma = 30$, Eq. (1) reduces to Lax's fifth-order KdV equation [38]:

* **Corresponding author: Aly R. Seadawy**, Mathematics Department, Faculty of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia, e-mail: aly742001@yahoo.com

Shafiq U. Rehman: Department of Computer Science, University of the Punjab, Lahore 54000, Pakistan

Muhammad Younis: PUCIT, University of the Punjab, Lahore 54000, Pakistan

Syed T. R. Rizvi: Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan

Ali Althobaiti: Department of Mathematics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

$$u_t + 10uu_{3x} + 20u_xu_{2x} + 30u^2u_x + u_{5x} = 0. \quad (2)$$

If we take $\alpha = 15$, $\beta = 15$, and $\gamma = 45$, Eq. (1) reduces to the Sawada–Kotera Eq. [39] (equivalent to the Caudrey–Dodd–Gibbon equation [40]):

$$u_t + 15uu_{3x} + 15u_xu_{2x} + 45u^2u_x + u_{5x} = 0. \quad (3)$$

When $\alpha = 10$, $\beta = 25$, and $\gamma = 20$, Eq. (1) reduces to Kaup–Kupershmidt (KK) equation:

$$u_t + 10uu_{3x} + 25u_xu_{2x} + 20u^2u_x + u_{5x} = 0. \quad (4)$$

The 5KdV type equations is modeled to discuss the non-linear phenomenon in the propagation of shallow water waves over a flat surface, gravity–capillary waves, and magneto-sound propagation in plasmas [40,41].

The enduring article is organized as follows: Section 2 presents application of $\left(\frac{G'}{G^2}\right)$ -expansion function method. In Section 3, anstaz approaches are presented. In Section 4, concluding remarks is revealed.

2 Application of $\left(\frac{G'}{G^2}\right)$ -expansion function method

To solve the proposed system, the following wave hypothesis is used as follows: $u(x, t) = Q(\vartheta)$, where $\vartheta = \rho x - ct$,

and $c \neq 0$, where c and ρ are constants. Putting the transformation into Eq. (1), we get,

$$\rho^5 Q^{(5)}(\vartheta) + \alpha \rho^3 Q(\vartheta) Q^{(3)}(\vartheta) + \gamma \rho Q(\vartheta)^2 Q'(\vartheta) - c Q'(\vartheta) + \beta \rho^3 Q'(\vartheta) Q''(\vartheta) = 0. \quad (5)$$

On integrating Eq. (5) one time with respect to ϑ , we get

$$c_1 + Q(\vartheta)(\alpha \rho^3 Q''(\vartheta) - c) + \rho^5 Q^{(4)}(\vartheta) + \frac{1}{2} \rho^3 (\beta - \alpha) Q'(\vartheta)^2 + \frac{1}{3} \gamma \rho Q(\vartheta)^3 = 0, \quad (6)$$

where c_1 is constant of integration. Applying the balance principle in Eq. (6) provides $n = 2$. So, the nontrivial solution of Eq. (6) turns into following

$$Q(\vartheta) = \beta_0 + \sum_{j=1}^n \left(\beta_j \left(\frac{G'}{G^2} \right)^j + \delta_j \left(\frac{G'}{G^2} \right)^{-j} \right),$$

$$Q(\vartheta) = \beta_0 + \beta_1 \left(\frac{G'}{G^2} \right) + \delta_1 \left(\frac{G'}{G^2} \right)^{-1} + \beta_2 \left(\frac{G'}{G^2} \right)^2 + \delta_2 \left(\frac{G'}{G^2} \right)^{-2}, \quad (7)$$

where β_i , $i = 0, 1, 2$ and δ_j , $j = 1, 2$ are unknowns and are calculated later. On solving Eq. (7) and (6) together, we get nonlinear sets of algebraic expression. Tackling the computational software Mathematica gives us following sets.

Set-1:

$$\beta_0 = -\frac{2\Gamma(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})}{\gamma\Omega}, \quad \beta_1 = 0, \quad \beta_2 = -\frac{3(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})}{\gamma}, \quad \delta_1 = 0, \quad \delta_2 = 0,$$

$$c_1 = -\frac{8\Gamma^3\rho^5\Omega(2\alpha^2\sqrt{\Theta\rho^4\Omega^4} + \rho^2\Omega^2((\alpha - \beta)(2\alpha + \beta)^2 - 12\gamma(\alpha - 2\beta)) - \alpha\beta\sqrt{\Theta\rho^4\Omega^4} - \beta^2\sqrt{\Theta\rho^4\Omega^4} + 4\gamma\sqrt{\Theta\rho^4\Omega^4})}{3\gamma^2},$$

$$c = \frac{2\Gamma^2\rho^3(\rho^2\Omega^2(2\alpha\beta + \beta^2 - 12\gamma) + \beta\sqrt{\Theta\rho^4\Omega^4})}{\gamma}.$$

Set-2:

$$\beta_0 = -\frac{2\Omega(\Gamma^2\rho^2(2\alpha + \beta) + \sqrt{\Gamma^4\Theta\rho^4})}{\gamma\Gamma}, \quad \beta_1 = 0, \quad \beta_2 = 0, \quad \delta_1 = 0, \quad \delta_2 = -\frac{3(\Gamma^2\rho^2(2\alpha + \beta) + \sqrt{\Gamma^4\Theta\rho^4})}{\gamma},$$

$$c_1 = -\frac{8\Gamma\rho^5\Omega^3(2\alpha^2\sqrt{\Gamma^4\Theta\rho^4} + \Gamma^2\rho^2((\alpha - \beta)(2\alpha + \beta)^2 - 12\gamma(\alpha - 2\beta)) - \alpha\beta\sqrt{\Gamma^4\Theta\rho^4} - \beta^2\sqrt{\Gamma^4\Theta\rho^4} + 4\gamma\sqrt{\Gamma^4\Theta\rho^4})}{3\gamma^2},$$

$$c = \frac{2\rho^3\Omega^2(\Gamma^2\rho^2(2\alpha\beta + \beta^2 - 12\gamma) + \beta\sqrt{\Gamma^4\Theta\rho^4})}{\gamma}.$$

Set-3:

$$\begin{aligned}\beta_0 &= -\frac{2\Gamma(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})}{\gamma\Omega}, \quad \beta_1 = 0, \quad \beta_2 = -\frac{3(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})}{\gamma}, \\ \delta_1 &= 0, \quad \delta_2 = -\frac{3\Gamma^2(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})}{\gamma\Omega^2}, \\ c_1 &= -\frac{512\Gamma^3\rho^5\Omega(2\alpha^2\sqrt{\Theta\rho^4\Omega^4} + \rho^2\Omega^2((\alpha - \beta)(2\alpha + \beta)^2 - 12\gamma(\alpha - 2\beta)) - \alpha\beta\sqrt{\Theta\rho^4\Omega^4} - \beta^2\sqrt{\Theta\rho^4\Omega^4} + 4\gamma\sqrt{\Theta\rho^4\Omega^4})}{3\gamma^2}, \\ c &= \frac{32\Gamma^2\rho^3(\rho^2\Omega^2(2\alpha\beta + \beta^2 - 12\gamma) + \beta\sqrt{\Theta\rho^4\Omega^4})}{\gamma},\end{aligned}$$

where $\Theta = (2\alpha + \beta)^2 - 40\gamma$ for all above cases.

For **Set-1**,

• **Trigonometric solutions:**

If $\Gamma\Omega > 0$,

$$u_{1,1}(x, t) = -\frac{\Gamma(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})(2\Lambda_1\Lambda_2\sin(2\vartheta\sqrt{\Gamma\Omega}) + (\Lambda_1^2 - \Lambda_2^2)\cos(2\vartheta\sqrt{\Gamma\Omega}) + 5(\Lambda_1^2 + \Lambda_2^2))}{2\gamma\Omega(\Lambda_1\sin(\vartheta\sqrt{\Gamma\Omega}) - \Lambda_2\cos(\vartheta\sqrt{\Gamma\Omega}))^2}, \quad (8)$$

• **Hyperbolic solution:**

If $\Gamma\Omega < 0$,

$$u_{1,2}(x, t) = \frac{(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))\left(-\frac{3|\Gamma\Omega|(\Lambda_1e^{2\vartheta\sqrt{|\Gamma\Omega|}} + \Lambda_2)^2}{(\Lambda_2 - \Lambda_1e^{2\vartheta\sqrt{|\Gamma\Omega|}})^2} - 2\Gamma\Omega\right)}{\gamma\Omega^2}, \quad (9)$$

Setting, $\Lambda_1 = \Lambda_2$, we achieve singular wave solution as:

$$u_{1,2}(x, t) = \frac{(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))(-3|\Gamma\Omega|\coth^2(\vartheta\sqrt{|\Gamma\Omega|}) - 2\Gamma\Omega)}{\gamma\Omega^2}. \quad (10)$$

• **Rational solution:**

If $\Gamma = 0$, $\Omega \neq 0$,

$$u_{1,3}(x, t) = -\frac{3\Lambda_1^2(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))}{\gamma\Omega^2(\vartheta\Lambda_1 + \Lambda_2)^2}. \quad (11)$$

If we take $\Lambda_1 = \Lambda_2$, a plane wave solution is found as follows (Figures 1–3):

$$u_{1,3}(x, t) = \frac{3(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))}{\gamma(\vartheta + 1)^2\Omega^2}. \quad (12)$$

For **Set-2**,

• **Trigonometric solutions:**

If $\Gamma\Omega > 0$,

$$u_{2,1}(x, t) = \frac{\Omega(\Gamma^2\rho^2(2\alpha + \beta) + \sqrt{\Gamma^4\Theta\rho^4})(2\Lambda_1\Lambda_2\sin(2\vartheta\sqrt{\Gamma\Omega}) + (\Lambda_1^2 - \Lambda_2^2)\cos(2\vartheta\sqrt{\Gamma\Omega}) - 5(\Lambda_1^2 + \Lambda_2^2))}{2\gamma\Gamma(\Lambda_2\sin(\vartheta\sqrt{\Gamma\Omega}) + \Lambda_1\cos(\vartheta\sqrt{\Gamma\Omega}))^2}, \quad (13)$$

Figure 1: Simulations of Eq. (8).

Figure 2: Simulations of Eq. (10).

Figure 3: Simulations of Eq. (12).

Figure 4: Simulations of Eq. (13).

• **Hyperbolic solution:**

If $\Gamma\Omega < 0$,

$$u_{2,2}(x, t) = \frac{\Omega(\Gamma^2\rho^2(2\alpha + \beta) + \sqrt{\Gamma^4\Theta\rho^4})\left(-\frac{3\Omega(\Lambda_2 - \Lambda_1 e^{2\vartheta\sqrt{|\Gamma\Omega|}})^2}{|\Gamma\Omega|(\Lambda_1 e^{2\vartheta\sqrt{|\Gamma\Omega|}} + \Lambda_2)^2} - \frac{2}{\Gamma}\right)}{\gamma}, \quad (14)$$

Setting, $\Lambda_1 = \Lambda_2$, we achieve shock wave solution as follows (Figures 4 and 5):

$$u_{2,2}(x, t) = \frac{\Omega(\Gamma^2\rho^2(2\alpha + \beta) + \sqrt{\Gamma^4\Theta\rho^4})\left(-\frac{3\Omega \tanh^2(\vartheta\sqrt{|\Gamma\Omega|})}{|\Gamma\Omega|} - \frac{2}{\Gamma}\right)}{\gamma}. \quad (15)$$

For **Set-3**,

• **Trigonometric solutions:**

If $\Gamma\Omega > 0$,

$$u_{3,1}(x, t) = -2\Gamma(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4}) \times \frac{4\Lambda_1\Lambda_2(\Lambda_1^2 - \Lambda_2^2)\sin(4\vartheta\sqrt{\Gamma\Omega}) + (\Lambda_1^4 - 6\Lambda_2^2\Lambda_1^2 + \Lambda_2^4)\cos(4\vartheta\sqrt{\Gamma\Omega}) + 5(\Lambda_1^2 + \Lambda_2^2)^2}{\gamma\Omega((\Lambda_2^2 - \Lambda_1^2)\sin(2\vartheta\sqrt{\Gamma\Omega}) + 2\Lambda_1\Lambda_2\cos(2\vartheta\sqrt{\Gamma\Omega}))^2}, \quad (16)$$

For soliton solution, take $\Lambda_1 = \Lambda_2$, we get periodic wave solution as follows:

$$u_{3,1}(x, t) = \frac{2\Gamma(\cos(4\vartheta\sqrt{\Gamma\Omega}) - 5)\sec^2(2\vartheta\sqrt{\Gamma\Omega})(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))}{\gamma\Omega}. \quad (17)$$

• **Hyperbolic solution:**

If $\Gamma\Omega < 0$,

$$u_{3,2}(x, t) = -\frac{3(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})}{\gamma} \times \left(-\frac{\sqrt{|\Gamma\Omega|}}{\Omega} \left(\frac{\Lambda_1 \sinh(2\sqrt{|\Gamma\Omega|}\vartheta) + \Lambda_1 \cosh(2\sqrt{|\Gamma\Omega|}\vartheta) + \Lambda_2}{\Lambda_1 \sinh(2\sqrt{|\Gamma\Omega|}\vartheta) + \Lambda_1 \cosh(2\sqrt{|\Gamma\Omega|}\vartheta) - \Lambda_2}\right)\right)^2 \\ - \frac{3\Gamma^2(\rho^2\Omega^2(2\alpha + \beta) + \sqrt{\Theta\rho^4\Omega^4})}{\gamma\Omega^2} \times \left(-\frac{\sqrt{|\Gamma\Omega|}}{\Omega} \left(\frac{\Lambda_1 \sinh(2\sqrt{|\Gamma\Omega|}\vartheta) + \Lambda_1 \cosh(2\sqrt{|\Gamma\Omega|}\vartheta) + \Lambda_2}{\Lambda_1 \sinh(2\sqrt{|\Gamma\Omega|}\vartheta) + \Lambda_1 \cosh(2\sqrt{|\Gamma\Omega|}\vartheta) - \Lambda_2}\right)\right)^{-2}, \quad (18)$$

Setting, $\Lambda_1 = \Lambda_2$, we achieve combo shock-singular wave solution as follows:

$$u_{3,2}(x, t) = \frac{(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))\left(-\frac{3\Gamma^2 \tanh^2(\vartheta\sqrt{|\Gamma\Omega|})}{|\Gamma\Omega|} - \frac{3|\Gamma\Omega| \coth^2(\vartheta\sqrt{|\Gamma\Omega|})}{\Omega^2} - \frac{2\Gamma}{\Omega}\right)}{\gamma}. \quad (19)$$

• **Rational solution:**

If $\Gamma = 0$, $\Omega \neq 0$,

$$u_{3,3}(x, t) = -\frac{3\Lambda_1^2(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))}{\gamma\Omega^2(\vartheta\Lambda_1 + \Lambda_2)^2}. \quad (20)$$

If we take $\Lambda_1 = \Lambda_2$, a plane wave solution is obtained as follows:

$$u_{3,3}(x, t) = \frac{3(\sqrt{\rho^4\Omega^4\Theta} + \rho^2\Omega^2(2\alpha + \beta))}{\gamma(\vartheta + 1)^2\Omega^2}. \quad (21)$$

For all solutions $\vartheta = \rho x - ct$ and $\Theta = (2\alpha + \beta)^2 - 40\gamma$ (Figures 6–8).

Figure 5: Simulations of Eq. (15).

Figure 6: Simulations of Eq. (17).

Figure 7: Simulations of Eq. (19).

Figure 8: Simulations of Eq. (21).

3 The solitary, singular, and shock wave solutions

To construct the solutions, the solitary wave ansatz method is used in the following sections.

3.1 Solitary wave

For solitary or nontopological wave solutions, we have

$$u(x, t) = \frac{A}{\cosh^r \vartheta} \text{ and } \vartheta = B(x - vt), \quad (22)$$

with the condition $r > 0$.

Where the parameters A , B , v denote the amplitude, inverse width, and velocity of the solitary wave, respectively. From Eq. (22), one can obtain:

$$u_t = \frac{ABvr \tanh \vartheta}{\cosh^r \vartheta}, \quad (23)$$

$$uu_{xxx} = \frac{-A^2 B^3 r^3 \tanh \vartheta}{\cosh^{2r} \vartheta} + \frac{A^2 B^3 r(r+1)(r+2) \tanh \vartheta}{\cosh^{2r+2} \vartheta}, \quad (24)$$

$$u_x u_{xx} = \frac{-A^2 B^3 r^3 \tanh \vartheta}{\cosh^{2r} \vartheta} + \frac{A^2 B^3 r^2(r+1) \tanh \vartheta}{\cosh^{2r+2} \vartheta}, \quad (25)$$

$$u^2 u_x = \frac{-A^3 Br \tanh \vartheta}{\cosh^{3r} \vartheta}, \quad (26)$$

$$u_{xxxxx} = \frac{-AB^5 r^5 \tanh \vartheta}{\cosh^r \vartheta} + \frac{AB^5 r(r+1)(r+2)(2r^2+4r+4) \tanh \vartheta}{\cosh^{r+2} \vartheta} - \frac{AB^5 r(r+1)(r+2)(r+3)(r+4) \tanh \vartheta}{\cosh^{r+4} \vartheta}. \quad (27)$$

After substituting Eq. (23)–(27) into the Eq. (1), we have

$$\begin{aligned} & \frac{ABvr \tanh \vartheta}{\cosh^r \vartheta} - \frac{\alpha A^2 B^3 r^3 \tanh \vartheta}{\cosh^{2r} \vartheta} + \frac{\alpha A^2 B^3 r(r+1)(r+2) \tanh \vartheta}{\cosh^{2r+2} \vartheta} - \frac{\beta A^2 B^3 r^3 \tanh \vartheta}{\cosh^{2r} \vartheta} + \frac{\beta A^2 B^3 r^2(r+1) \tanh \vartheta}{\cosh^{2r+2} \vartheta} \\ & - \frac{\gamma A^3 Br \tanh \vartheta}{\cosh^{3r} \vartheta} - \frac{AB^5 r^5 \tanh \vartheta}{\cosh^r \vartheta} + \frac{AB^5 r(r+1)(r+2)(2r^2+4r+4) \tanh \vartheta}{\cosh^{r+2} \vartheta} \\ & - \frac{AB^5 r(r+1)(r+2)(r+3)(r+4) \tanh \vartheta}{\cosh^{r+4} \vartheta} = 0. \end{aligned} \quad (28)$$

From Eq. (28), we get $r = 2$, by equating the exponents $r + 4 = 3r$. To solve Eq. (28), equate the coefficients of the linearly independent terms to 0. Thus, it turns

$$-AB^5 r(r+1)(r+2)(r+3)(r+4) + \alpha A^2 B^3 r(r+1)(r+2) + \beta A^2 B^3 r^2(r+1) - \gamma A^3 Br = 0,$$

$$ABvr - AB^5 r^5 = 0,$$

$$AB^5 r(r+1)(r+2)(2r^2+4r+4) - \alpha A^2 B^3 r^3 - \beta A^2 B^3 r^3 = 0.$$

On inserting $r = 2$ and solving the above algebraic equations, we secure:

$$B = \sqrt{\frac{A(\alpha + \beta)}{60}}, \quad v = 16B^4 \text{ and} \\ A = \frac{3(2\alpha + \beta \pm \sqrt{4\alpha^2 + 4\alpha\beta + \beta^2 - 40\gamma B^2})}{\gamma}.$$

Figure 9: Simulations of Eq. (29).

Moreover, it is observed that the soliton width B , given in above case, provokes a constraint condition that is presented by

$$A(\alpha + \beta) > 0.$$

Hence, the solitary wave solution of the 5KdV equation is written as follows (Figure 9):

$$u(x, t) = \frac{3(2\alpha + \beta \pm \sqrt{4\alpha^2 + 4\alpha\beta + \beta^2 - 40\gamma B^2})}{\gamma \cosh^2\{B(x - 16B^4t)\}}. \quad (29)$$

3.2 Shock wave

For shock or topological wave solutions, we have

$$u(x, t) = A \tanh^r \vartheta, \quad \text{where } \vartheta = B(x - vt), \quad (30)$$

where the parameters A , B , v represent the amplitude, inverse width, and velocity of the solitary wave, respectively. From Eq. (30), one can obtain:

$$u_t = -ABvr \tanh^{r-1} \vartheta + ABvr \tanh^{r+1} \vartheta, \quad (31)$$

$$\begin{aligned} uu_{xxx} = & A^2 B^3 r(r-1)(r-2) \tanh^{2r-3} \vartheta \\ & - A^2 B^3 r(3r^2 - 3r + 2) \tanh^{2r-1} \vartheta \\ & + A^2 B^3 r(3r^2 + 3r + 2) \tanh^{2r+1} \vartheta \\ & - A^2 B^3 r(r+1)(r+2) \tanh^{2r+3} \vartheta, \end{aligned} \quad (32)$$

$$\begin{aligned} u_x u_{xx} = & A^2 B^3 r^2(r-1) \tanh^{2r-3} \vartheta \\ & - A^2 B^3 r^2(3r-1) \tanh^{2r-1} \vartheta \\ & - A^2 B^3 vr^2(3r+1) \tanh^{2r+1} \vartheta \\ & + A^2 B^3 vr^2(r+1) \tanh^{2r+3} \vartheta, \end{aligned} \quad (33)$$

$$u^2 u_x = A^3 Br \{\tanh^{3r-1} \vartheta - \tanh^{3r+1} \vartheta\}, \quad (34)$$

$$\begin{aligned}
u_{xxxxx} = & AB^5r(r-1)(r-2)(r-3)(r-4) \tanh^{r-5}\vartheta \\
& - 5AB^5r(r^4 - 6r^3 + 15r^2 - 18r + 8) \tanh^{r-3}\vartheta \\
& + AB^5r(10r^4 - 20r^3 + 50r^2 - 40r + 16) \tanh^{r-1}\vartheta \\
& - AB^5r(10r^4 + 20r^3 + 50r^2 + 40r + 16) \tanh^{r+1}\vartheta \\
& + 5AB^5r(r^4 + 6r^3 + 15r^2 + 18r + 8) \tanh^{r+3}\vartheta \\
& - AB^5r(r+1)(r+2)(r+3)(r+4) \tanh^{r+5}\vartheta.
\end{aligned} \quad (35)$$

After substituting Eq. (31)–(35) into Eq. (1), the following Eq. can be obtained:

$$\begin{aligned}
& -ABvr \tanh^{r-1}\vartheta + ABvr \tanh^{r+1}\vartheta \\
& + AB^5r(r-1)(r-2)(r-3)(r-4) \tanh^{r-5}\vartheta \\
& + \alpha A^2B^3r(r-1)(r-2) \tanh^{2r-3}\vartheta \\
& - \alpha A^2B^3r(3r^2 - 3r + 2) \tanh^{2r-1}\vartheta \\
& + \alpha A^2B^3r(3r^2 + 3r + 2) \tanh^{2r+1}\vartheta \\
& - \alpha A^2B^3r(r+1)(r+2) \tanh^{2r+3}\vartheta \\
& + \beta A^2B^3r^2(r-1) \tanh^{2r-3}\vartheta \\
& - \beta A^2B^3r^2(3r-1) \tanh^{2r-1}\vartheta \\
& - \beta A^2B^3r^2(3r+1) \tanh^{2r+1}\vartheta \\
& + \beta A^2B^3vr^2(r+1) \tanh^{2r+3}\vartheta \\
& + \gamma A^3Br\{\tanh^{3r-1}\vartheta - \tanh^{3r+1}\vartheta\} \\
& + AB^5r(r-1)(r-2)(r-3)(r-4) \tanh^{r-5}\vartheta \\
& - 5AB^5r(r^4 - 6r^3 + 15r^2 - 18r + 8) \tanh^{r-3}\vartheta \\
& + AB^5r(10r^4 - 20r^3 + 50r^2 - 40r + 16) \tanh^{r-1}\vartheta \\
& - AB^5r(10r^4 + 20r^3 + 50r^2 + 40r + 16) \tanh^{r+1}\vartheta \\
& + 5AB^5r(r^4 + 6r^3 + 15r^2 + 18r + 8) \tanh^{r+3}\vartheta \\
& - AB^5r(r+1)(r+2)(r+3)(r+4) \tanh^{r+5}\vartheta = 0.
\end{aligned} \quad (36)$$

From Eq. (36), we get $r = 2$, by equating the exponents $r + 5 = 3r + 1$. Moreover, to solve Eq. (36), equate the coefficients of the linearly independent terms to 0. Thus, it turns

$$\begin{aligned}
& -\alpha A^2B^3r(r+1)(r+2) + \beta A^2B^3r^2(r+1) - \gamma A^3Br \\
& - AB^5r(r+1)(r+2)(r+3)(r+4) = 0, \\
& \alpha A^2B^3r(3r^2 + 3r + 2) - \beta A^2B^3vr^2(3r+1) + \gamma A^3Br \\
& + 5AB^5r(r^4 + 6r^3 + 15r^2 + 18r + 8) = 0, \\
& ABvr - \alpha A^2B^3r(3r^2 - 3r + 2) - \beta A^2B^3r^2(3r-1) \\
& - AB^5r(10r^4 + 20r^3 + 50r^2 + 40r + 16) = 0, \\
& -ABr + \alpha A^2B^3r(r-1)(r-2) + \beta A^2B^3r^2(r-1) \\
& + AB^5r(10r^4 - 20r^3 + 50r^2 - 40r + 16) = 0.
\end{aligned}$$

On inserting $r = 2$ and solving the above algebraic equations, we secure:

$$\begin{aligned}
A &= \frac{1 + 136B^4}{2\beta B^2}, \quad B = B \text{ and} \\
v &= \frac{4\alpha + 544\alpha B^4 + 5\beta + 1296B^4\beta}{\beta}.
\end{aligned}$$

Hence, the shock wave solution of the 5KdV equation is given by the following equation: (Figure 10)

$$\begin{aligned}
u(x, t) &= \left(\frac{1 + 136B^4}{2\beta B^2} \right) \\
&\tanh^2 \left\{ B \left(x - \frac{4\alpha + 544\alpha B^4 + 5\beta + 1296B^4\beta}{\beta} t \right) \right\}. \quad (37)
\end{aligned}$$

3.3 Singular wave

For singular wave solution, we have

$$u(x, t) = \frac{A}{\sinh^r \vartheta} \text{ and } \vartheta = B(x - vt), \quad (38)$$

where the parameters A , B , v represent the amplitude, the inverse width, and the velocity of the solitary wave, respectively. From Eq. (38), one can obtain:

$$u_t = \frac{ABrv}{\sinh^r \vartheta \tanh \vartheta}, \quad (39)$$

$$uu_{xxx} = -\frac{A^2B^3r^3}{\sinh^{2r} \vartheta \tanh \vartheta} - \frac{A^2B^3r(r+1)(r+2)}{\sinh^{2r+2} \vartheta \tanh \vartheta}, \quad (40)$$

$$u_x u_{xx} = -\frac{A^2B^3r^3}{\sinh^{2r} \vartheta \tanh \vartheta} - \frac{A^2B^3r^2(r+1)}{\sinh^{2r+2} \vartheta \tanh \vartheta}, \quad (41)$$

$$u^2 u_x = -\frac{A^3Br}{\sinh^{3r} \vartheta \tanh \vartheta}, \quad (42)$$

$$\begin{aligned}
u_{xxxxx} &= \frac{-AB^5r^5}{\sinh^r \vartheta \tanh \vartheta} \\
&- \frac{AB^5r(r+1)(r+2)(2r^2 + 4r + 4)}{\sinh^{r+2} \vartheta \tanh \vartheta} \\
&- \frac{AB^5r(r+1)(r+2)(r+3)(r+4)}{\sinh^{r+4} \vartheta \tanh \vartheta}. \quad (43)
\end{aligned}$$

After inserting Eq. (39)–(43) into Eq. (1), the following equation is achieved:

Figure 10: Simulations of Eq. (37).

Figure 11: Simulations of Eq. (45).

$$\begin{aligned}
& \frac{ABrv}{\sinh^r \vartheta \tanh \vartheta} - \frac{\alpha A^2 B^3 r^3}{\sinh^{2r} \vartheta \tanh \vartheta} \\
& - \frac{\alpha A^2 B^3 r(r+1)(r+2)}{\sinh^{2r+2} \vartheta \tanh \vartheta} - \frac{\beta A^2 B^3 r^3}{\sinh^{2r} \vartheta \tanh \vartheta} \\
& - \frac{\beta A^2 B^3 r^2(r+1)}{\sinh^{2r+2} \vartheta \tanh \vartheta} - \frac{\gamma A^3 Br}{\sinh^{3r} \vartheta \tanh \vartheta} \\
& - \frac{AB^5 r(r+1)(r+2)(2r^2+4r+4)}{\sinh^{r+2} \vartheta \tanh \vartheta} \\
& - \frac{AB^5 r(r+1)(r+2)(r+3)(r+4)}{\sinh^{r+4} \vartheta \tanh \vartheta} \\
& - \frac{AB^5 r^5}{\sinh^r \vartheta \tanh \vartheta} = 0.
\end{aligned} \quad (44)$$

From Eq. (44), we get $r = 2$, by equating the exponents $r + 4 = 3r$. Moreover, to solve Eq. (44), equate the coefficients of the linearly independent terms to 0. Thus, it turns

$$\begin{aligned}
& -AB^5 r(r+1)(r+2)(r+3)(r+4) - \alpha A^2 B^3 r(r+1)(r+2) \\
& - \beta A^2 B^3 r^2(r+1) - \gamma A^3 Br = 0, \\
& ABvr - AB^5 r^5 = 0, \\
& -AB^5 r(r+1)(r+2)(2r^2+4r+4) - \alpha A^2 B^3 r^3 - \beta A^2 B^3 r^3 = 0.
\end{aligned}$$

On inserting $r = 2$ and solving the above algebraic equations, we secure:

$$\begin{aligned}
B &= \sqrt{-\frac{A(\alpha + \beta)}{60}}, \quad v = 16B^4 \text{ and} \\
A &= \frac{-3(2\alpha + \beta \pm \sqrt{4\alpha^2 + 4\alpha\beta + \beta^2 - 40\gamma B^2})}{\gamma}.
\end{aligned}$$

Moreover, it may be noted that the soliton width B , given in the above case, provokes a constraint condition that is given by

$$A(\alpha + \beta) < 0.$$

Hence, the singular wave solution of the 5KdV equation is given by (Figure 11)

$$u(x, t) = \frac{-3(2\alpha + \beta \pm \sqrt{4\alpha^2 + 4\alpha\beta + \beta^2 - 40\gamma B^2})}{\gamma \sinh^2\{B(x - 16B^4 t)\}}. \quad (45)$$

4 Concluding remarks

We retrieved the novel hyperbolic, trigonometric, shock, solitary, singular, shock-singular, singular periodic, and rational function solutions to the 5KdV equation. It is also noted that this equation can be reduced to other equations, namely, Lax's fifth-order KdV equation, Sawada–Kotera equation, and Kaup–Kupershmidt equation for different values of parameters. Moreover the solitary wave ansatz

method forms noteworthy highlights that make it reasonable for the acquiring of single soliton solutions for a wide class of NLPDEs and we obviously see that the consistency, which has been applied effectively. We also have depicted the graphical view of some wave structures of the studied system in this article. By implementing the proposed methodologies, the diverse exact wave structures are extracted and some of represented physically with different parameters. Compared with the solutions obtained by using other methods, our reported solutions are new ones. The results of this work will be a source of inspiration and motivation for future discussion in nonlinear physical sciences. The calculations also reveals us the importance of these methods to find the exact wave solutions in a more general way. The results obtained in this article enrich the analysis of 5KdV equations.

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