Research Article

Mashael M. AlBaidani, Abdul Hamid Ganie*, and Asia Fahd Mohammad Almuteb

Generalized notion of integral inequalities of variables

https://doi.org/10.1515/phys-2022-0070
received February 18, 2022; accepted July 10, 2022

Abstract: The fractional structures of variables using Riemann–Liouville notion have been analyzed by various authors. The novel idea of this article is to introduce the new notion of weighted behavior on random variables using integral inequalities. In view of these, we obtain some new generalized fractional inequalities by using this new fractional integration of continuous random variables.

Keywords: random variables, fractional integral inequalities, Riemann–Liouville integral

1 Introduction

In recent times, the integral inequalities play a vital role in science, engineering, and technology. The authors are much attracted to study and analyze the structure in the fields like physics, statistics, biology, chemistry, and engineering as witnessed in refs. [1–3] and many others. The basic concern of fractional calculus has a straight knock on the solution of various problems of fast growing sciences to stimulate much interest in such field and to show its visibility in them. The different types of procedures and applications of fractional derivatives have been developed as can be found in refs. [4–10]. In this direction, Riemann–Liouville (RL) and Grunwald–Letnikov are most notable authors. It was Leonhard Euler in around 1720s gave the concept of how to extend the factorial to non-integer values. This developed a rich theory utilizing over the scientific world.

For a real number $\mho$, the Euler gamma function $\Gamma(\mho)$ for $\mho \notin \{0, -1, -2, -3, \ldots\}$ can be expressed as an improper integral:

$$\Gamma(\mho) = \int_0^\infty e^{-t} t^{\mho-1} \mathrm{d}t.$$

It is observed that
(i) $\Gamma(\mho + 1) = \mho!$ for $\mho \in \mathbb{N}$,
(ii) $\Gamma(\mho + 1) = \mho\Gamma(\mho)$ for $\mho \in \mathbb{R} \setminus \{0, -1, -2, -3, \ldots\}$.

It was Grunwald–Letnikow who forwarded the definition of fractional derivative order η as follows:

$$\mathcal{I}^{\eta}[\xi(t)] = \lim_{h\to 0}\left[\frac{1}{h^{\eta}}\sum_{r=0}^{\infty}\gamma(\eta, r)\xi(t + (h - r)h)\right], \quad (1)$$

$$\gamma(\eta, r) = (-1)^r \frac{\Gamma(\eta + 1)}{r!\Gamma(\eta - r + 1)}, \quad (2)$$

where Γ is the gamma function and h is the time increment as can be found in refs. [11–14], and many others.

The Riemann–Liouville fractional integral of order $\eta \geq 0$ as can be seen in refs. [15–17] is defined as follows:

$$\mathcal{I}_0^{\eta} = \int_0^t g(\tau)\mathrm{d}\tau \quad (3)$$

and

$$g(\tau) = \frac{1}{\Gamma(\eta + 1)}[f(\tau)(t - \tau)^{\eta}]. \quad (4)$$

For $0 < \eta < 1$, we see from (2) that

$$\gamma(\eta, 0) = 1 \quad (5)$$

and

$$-\sum_{r=1}^{\infty}\gamma(\eta, r) = 1. \quad (6)$$

* **Corresponding author: Abdul Hamid Ganie,** Basic Science Department, College of Science and Theoretical Studies, Saudi Electronic University-Abha Male, Saudi Arabia; Department of Applied Sciences and Humanities, SSM College of Engineering Parihaspora, Pattan-193121, Jammu and Kashmir, India, e-mail: a.ganie@seu.edu.sa
Mashael M. AlBaidani: Department of Mathematics, College of Science and Humanities, Prince Sattam Bin Abdulaziz University, Al Kharj, 11942, Saudi Arabia, e-mail: m.albaidani@psau.edu.sa
Asia Fahd Mohammad Almuteb: Department of Mathematics, College of Science and Humanities, Prince Sattam Bin Abdulaziz University, Al Kharj, 11942, Saudi Arabia, e-mail: Asia-almoteab@outlook.sa

Using the notion of theory of probability concept, we have following:

Using (5) the "present" (*i.e.*, $\xi(0)$) is noticed in (1) having one as probability.

The classical approach of the Riemann–Liouville fractional derivative was reformulated by Caputo and he then gave the solution of fractional differential equations under given initial conditions. It was Grunwald–Letnikov who later put forward the fractional calculus given by Leibnitz in a new way. Importance of fractional calculus by using its execution to real-world issues in applied analyses, statistics, physics, and fluid mechanics eta cetera can be witnessed in refs. [18–29] and many others and the classical approach was generalized.

Now as in refs. [30–39], we have following definitions in an extended sense:

Definition 1.1. The left and right Riemann–Liouville fractional integral of order $\kappa \geq 0$, respectively, are given by

$$ {}_{+}\mathfrak{I}^{\kappa}_{\alpha}=\frac{1}{\Gamma(\kappa)}\int\limits_{\alpha}^{u}(u-t)^{\kappa-1}\mathfrak{F}(t)\mathrm{d}t \quad \text{for } u>\alpha, \tag{7}$$

and

$$ {}_{-}\mathfrak{I}^{\kappa}_{\beta}=\frac{1}{\Gamma(\kappa)}\int\limits_{u}^{\beta}(u-t)^{\kappa-1}\mathfrak{F}(t)\mathrm{d}t \quad \text{for } u<\beta, \tag{8}$$

where $\Gamma(\kappa)=\int_0^{\infty}e^{-v}v^{\kappa-1}\mathrm{d}v$ is called as the Gamma function and ${}_{+}\mathfrak{I}^{0}_{\alpha}={}_{+}\mathfrak{I}^{0}_{\alpha}\mathfrak{F}(u)=\mathfrak{F}(u)$.

For $\kappa \geq 0$ and $\mho \geq 0$, we have following well-known results:

$$\begin{aligned} {}_{+}\mathfrak{I}^{\kappa}_{\alpha+}\mathfrak{I}^{\mho}_{\alpha} &= {}_{+}\mathfrak{I}^{\kappa+\mho}_{\alpha}\\ {}_{+}\mathfrak{I}^{\kappa}_{\alpha+}\mathfrak{I}^{\mho}_{\alpha} &= {}_{+}\mathfrak{I}^{\mho}_{\alpha+}\mathfrak{I}^{\kappa}_{\alpha}. \end{aligned} \tag{9}$$

Also, as in refs. [30,35], and many others, we have following definitions:

Definition 1.2. For $1 \leq p < \infty$, the space $\mathfrak{L}_{p,k}(\alpha,\beta)$ of real-valued Lebesgue measurable functions $\mathfrak{F}$ on $[\alpha,\beta]$ such that

$$\|\mathfrak{F}\|_{\mathfrak{L}_{p,k}(\alpha,\beta)}=\left[\int\limits_{\alpha}^{\beta}|\mathfrak{F}(v)|^{p}v^{k}\mathrm{d}v\right]^{\frac{1}{p}}<\infty, \tag{10}$$

for $\kappa \geq 0$.

Definition 1.3. For $c \in \mathbb{R}$ and $1 \leq p < \infty$, we consider the space $\mathfrak{X}^{p}_{c}(\alpha,\beta)$ of all real-valued Lebesgue measurable functions $\mathfrak{F}$ on $[\alpha,\beta]$ such that

$$\|\mathfrak{F}\|_{\mathfrak{X}^{p}_{c}}=\left[\int\limits_{\alpha}^{\beta}|v^{c}\mathfrak{F}(v)|^{p}\frac{\mathrm{d}v}{v}\right]^{\frac{1}{p}}<\infty, \tag{11}$$

and choosing $p=\infty$, we define it as follows:

$$\|\mathfrak{F}\|_{\mathfrak{X}^{\infty}_{c}}=\operatorname*{ess\,sup}_{\alpha\leq v\leq\beta}|v^{c}\mathfrak{F}(v)|. \tag{12}$$

Remark 1.4. By choosing $c=\frac{(k+1)}{p}$ with $1 \leq p < \infty$, $k \geq 0$, then $\mathfrak{X}^{p}_{c}(\alpha,\beta)$ gets coincide to $\mathfrak{L}_{p,k}(\alpha,\beta)$-space, and when we assume $c=\left(\frac{1}{p}\right)$ for $1 \leq p < \infty$ then space $\mathfrak{X}^{p}_{c}(\alpha,\beta)$ reduced to the classical space $\mathfrak{L}^{p}(\alpha,\beta)$.

Definition 1.5. Let $\mathfrak{F} \in \mathfrak{L}_{1,s}$ and $k \geq 0$, the generalized Riemann–Liouville fractional integrals ${}_{+}\mathfrak{I}^{\kappa,s}_{\alpha}$ and ${}_{-}\mathfrak{I}^{\kappa,s}_{\alpha}$ with order $\kappa \geq 0$ are given by

$$ {}_{+}\mathfrak{I}^{\kappa,s}_{\alpha}=\frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)}\int\limits_{\alpha}^{v}(v^{s+1}-t^{s+1})^{\kappa-1}t^{s}\mathfrak{F}(t)\mathrm{d}t \quad \text{for } v>a. \tag{13}$$

$$ {}_{-}\mathfrak{I}^{\kappa,s}_{\beta}=\frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)}\int\limits_{v}^{\beta}(v^{s+1}-t^{s+1})^{\kappa-1}t^{s}\mathfrak{F}(t)\mathrm{d}t \quad \text{for } \beta>v. \tag{14}$$

It is observed that the integral formulas given by (13) and (14) are, respectively, known as right generalized Riemann–Liouville integral and left generalized Riemann–Liouville fractional integral.

Throughout the article, random variable of $\mathcal{X}$ will be abbreviated by r.v. and $\mho:[\alpha,\beta]\to\mathbb{R}^{+}$ as a positive continuous function.

Following the authors [10–16,36–40], we have following generalized definitions:

Definition 1.6. Let $\mathcal{X}$ be a r.v. having a positive p.d.f. $\mathfrak{F}$ given on $[\alpha,\beta]$. Then, for $s \geq 0$ and $\alpha < \varsigma \leq \beta$, the $\mho$-weighted fractional expectation function of order κ is defined as follows:

$$\begin{aligned} E_{\mathcal{X},\kappa,\mho}(\varsigma) &= {}_{+}\mathfrak{I}^{\kappa,s}_{\alpha}[\varsigma\mho\mathfrak{F}(\varsigma)]\\ &= \frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)}\int\limits_{\alpha}^{\varsigma}(\varsigma^{s+1}-t^{s+1})^{\kappa-1}t^{s+1}\mho(t)\mathfrak{F}(t)\mathrm{d}t. \end{aligned} \tag{15}$$

Definition 1.7. We define the $\mho$-weighted fractional expectation for a r.v. $\mathcal{X}-E(\mathcal{X})$ of order κ as follows:

$$\begin{aligned} E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma) = \frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)}\int\limits_{\alpha}^{\varsigma}&(\varsigma^{s+1}-t^{s+1})^{\kappa-1}\\ &\times(t-E(\mathcal{X}))\mho(t)t^{s}\mathfrak{F}(t)\mathrm{d}t. \end{aligned} \tag{16}$$

Note by choosing $\varsigma = \beta$, the aforementioned definitions reduces to following:

Definition 1.8. Let $\mathcal{X}$ be a r.v. with a positive p.d.f. $\mathfrak{F}$ defined on $[\alpha, \beta]$. Then, for $s \geq 0$, we define the $\mho$-weighted fractional expectation function of order κ as follows:

$$E_{\mathcal{X},\kappa,\mho,\mho}(\varsigma) = \frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)} \int_{\alpha}^{\beta} (\beta^{s+1} - t^{s+1})^{\kappa-1} \times t^{s+1} \mho(t) \mathfrak{F}(t) \mathrm{d}t. \tag{17}$$

Definition 1.9. For mathematical expectation $E(\mathcal{X})$ of r.v. $\mathcal{X}$ having a positive p.d.f. $\mathfrak{F}$. Then, on $[\alpha, \beta]$, $s \geq 0$ and $\alpha < \varsigma \leq \beta$, the $\mho$-weighted fractional variance function of order κ is defined as follows:

$$\begin{aligned}\sigma^2_{\mathcal{X},\kappa,\mho}(\varsigma) &= {}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[(\varsigma - E(\mathcal{X}))^2 \mho \mathfrak{F}(\varsigma)] \\ &= \frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)} \int_{\alpha}^{\varsigma} (\varsigma^{s+1} - t^{s+1})^{\kappa-1} (t - E(\mathcal{X}))^2 t^s \times \mho(t) \mathfrak{F}(t) \mathrm{d}t.\end{aligned} \tag{18}$$

Definition 1.10. If $\varsigma = \beta$, then $\mho$-weighted fractional variance function of order κ is given as follows:

$$\sigma^2_{\mathcal{X},\kappa,\mho}(\varsigma) = \frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)} \int_{\alpha}^{\beta} (\beta^{s+1} - t^{s+1})^{\kappa-1} \times (t - E(\mathcal{X}))^2 t^s \mho(t) \mathfrak{F}(t) \mathrm{d}t. \tag{19}$$

We have following important points:

Remark 1.11. (D1): Choosing $s = 0$ and $\kappa = 1$ and $\mho(t) = 1$ for every $t \in [\alpha, \beta]$ in Definition 1.6, we get $E_{\mathcal{X},1,1} = = E(\mathcal{X})$ as the classical expectation of r.v. $\mathcal{X}$.

(D2): Setting $s = 0$ and $\kappa = 1$ and $\mho(t) = 1$ for every $t \in [\alpha, \beta]$ in Definition 1.8, we get $\sigma^2_{\mathcal{X},1,1} = \sigma^2(\mathcal{X}) = \int_{\alpha}^{\beta} (t - E(X))^2 \mathfrak{F}(t) \mathrm{d}t$ as classical variance of r.v. $\mathcal{X}$.

(D3): Setting $\kappa = 1$ and $\mho(t) = 1$ for every $t \in [\alpha, \beta]$, we get the well-known result $I^{\kappa}[\mathfrak{F}(\beta)] = 1$.

2 Main results

This section will be concerning the new generalization of outcomes of continuous r.v. having fractional integral order.

Theorem 2.1. *Let $\mathcal{X}$ be a r.v. with p.d.f. $\mathfrak{F} : [\alpha, \beta] \to \mathbb{R}^+$. Then for each $\alpha < \varsigma \leq \beta$, $\kappa \geq 0$ and $s \geq 0$, we see*

(*i*) *the inequality*

$$\begin{aligned}&{}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\sigma^2_{\mathcal{X},\kappa,\mho}(\varsigma) - (E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))^2 \\ &\leq \|\mathfrak{F}\|^2_{\infty} \left(\frac{(s+1)^{1-\kappa}(\varsigma^{s+1} - \alpha^{s+1})^{\kappa}}{\Gamma(\kappa+1)} {}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho(\varsigma)\varsigma^{2s+2}] - ({}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho(\varsigma)\varsigma])^2 \right)\end{aligned} \tag{20}$$

holds provided $\mathfrak{F} \in L_{\infty}[\alpha, \beta]$ and

(*ii*) *the inequality*

$$\begin{aligned}&{}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\sigma^2_{\mathcal{X},\kappa,\mho}(\varsigma) - (E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))^2 \\ &\leq \frac{1}{2}(\varsigma^{s+1} - \alpha^{s+1})^2 ({}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho(\varsigma)\varsigma])^2\end{aligned} \tag{21}$$

holds.

Proof. Define a function $\mathfrak{Q}$ for $t, m \in (\alpha, \varsigma)$, $\alpha < \varsigma \leq \beta$ as follows:

$$\mathfrak{Q}(t, m) = (\mathfrak{Q}_1(t) - \mathfrak{Q}_1(m))(\mathfrak{Q}_2(t) - \mathfrak{Q}_2(m)), \tag{22}$$

where $\kappa \geq 0$.

Now multiplying (22) by $\frac{(\varsigma^{s+1} - t^{s+1})^{\kappa-1}}{\Gamma(\kappa)} t^s \mathfrak{p}(t)$ both sides, where the $\mathfrak{p}$ is a function $\mathfrak{p} : [\alpha, \beta] \to \mathbb{R}^+$ with $t \in (\alpha, \varsigma)$, and then integrating the resulting identity from α to ς and have

$$\begin{aligned}&\frac{(s+1)^{1-\kappa}}{\Gamma(\kappa)} \int_{\alpha}^{\varsigma} (\varsigma^{s+1} - t^{s+1})^{\kappa-1} \mathfrak{p}(t) \mathfrak{Q}(t, m) t^s \mathrm{d}t \\ &= {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1\mathfrak{Q}_2(\varsigma)] - \mathfrak{Q}_2(m) {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)] \\ &\quad - \mathfrak{Q}_1(m) {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_2(\varsigma)] \\ &\quad + \mathfrak{Q}_1(m)\mathfrak{Q}_2(m) {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}(\varsigma)].\end{aligned} \tag{23}$$

Now multiplying (23) by $\frac{(\varsigma^{s+1} - m^{s+1})^{\kappa-1}}{\Gamma(\kappa)} \mathfrak{p}(m) m^s$ for $m \in (\alpha, \varsigma)$, and then integrating the resulting identity over (α, ς) with respect to m, we see

$$\begin{aligned}&\frac{(s+1)^{2-2\kappa}}{\Gamma^2(\kappa)} \int_{\alpha}^{\varsigma}\int_{\alpha}^{\varsigma} (\varsigma^{s+1} - t^{s+1})^{\kappa-1} \\ &\quad \times (\varsigma^{s+1} - m^{s+1})^{\kappa-1} \mathfrak{p}(t)\mathfrak{p}(m)\mathfrak{Q}(t, m) t^s m^s \mathrm{d}t\mathrm{d}m \\ &= 2 {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}(\varsigma)] {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1\mathfrak{Q}_2(\varsigma)] \\ &\quad - 2 {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)] {}_{+}\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_2(\varsigma)].\end{aligned} \tag{24}$$

Now in (24), choosing $\mathfrak{p}(\varsigma) = \mho(\varsigma)\mathfrak{F}(\varsigma)$ and $\mathfrak{Q}_1(\varsigma) = \mathfrak{Q}_2(\varsigma) = \varsigma^{s+1} - E(\mathcal{X})$, $\varsigma \in (\alpha, \beta)$, we see

$$\begin{aligned}&\frac{(s+1)^{2-2\kappa}}{\Gamma^2(\kappa)}\int_\alpha^\varsigma\int_\alpha^\varsigma(\varsigma^{s+1}-t^{s+1})^{\kappa-1}(\varsigma^{s+1}-m^{s+1})^{\kappa-1}\\&\times \mho(t)\mathfrak{F}(t)\mho(m)\mathfrak{F}(m)(t^{s+1}-m^{s+1})^2t^sm^s\mathrm{d}t\mathrm{d}m\\&=2{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&-2[{}_+\mathfrak{I}_\alpha^{\kappa,s}\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2].\end{aligned}\tag{25}$$

But, on the other hand, we see

$$\begin{aligned}&\frac{(s+1)^{2-2\kappa}}{\Gamma^2(\kappa)}\int_\alpha^\varsigma\int_\alpha^\varsigma(\varsigma^{s+1}-t^{s+1})^{\kappa-1}(\varsigma^{s+1}-m^{s+1})^{\kappa-1}\\&\times \mathfrak{F}(t)\mho(t)\mathfrak{F}(m)\mho(m)(t^{s+1}-m^{s+1})^2t^sm^s\mathrm{d}t\mathrm{d}m\\&\le\|\mathfrak{F}\|_\infty^2\left(2\frac{(s+1)^{1-\kappa}(\varsigma^{s+1}-\alpha^{s+1})^\kappa}{\Gamma(\kappa+1)}{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho(\varsigma)\varsigma^{2s+2}]\right.\\&\left.-2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho(\varsigma)\varsigma])^2\right).\end{aligned}\tag{26}$$

Consequently, using definitions 1.6, 1.7, and 1.9, the part (i) of the result follows from (25) and (26).

(ii) We have

$$\begin{aligned}&\frac{(s+1)^{2-2\kappa}}{\Gamma^2(\kappa)}\int_\alpha^\varsigma\int_\alpha^\varsigma(\varsigma^{s+1}-t^{s+1})^{\kappa-1}(\varsigma^{s+1}-m^{s+1})^{\kappa-1}\\&\times \mathfrak{F}(t)\mathfrak{F}(m)\varsigma(t)\varsigma(m)(t^{s+1}-m^{s+1})^2t^sm^s\mathrm{d}t\mathrm{d}m\\&\le\sup_{t,m\in[\alpha,t]}|(t^{s+1}-m^{s+1})|^2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)])^2\\&=(t^{s+1}-\alpha^{s+1})^2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)])^2.\end{aligned}\tag{27}$$

Consequently, from (25) and (27), we get (21) as desired. □

Corollary 2.2. *If* $\mho(t)=1$ *for every* $t\in[\alpha,\beta]$ *and* $\kappa\ge0$, *the continuous r.v.* $\mathcal{X}$ *having p.d.f.* $\mathfrak{F}$ *in* $[\alpha,\beta]$, *then,*

(*i*) *the inequality*

$$\begin{aligned}&\frac{\beta^{s+1}-\alpha^{s+1}}{\Gamma(\kappa)}\sigma_{\mathcal{X},\kappa,\mho}^2-E_{\mathcal{X},\kappa}^2\\&\le\|\mathfrak{F}\|_\infty^2\left(\frac{(\beta^{s+1}-\alpha^{s+1})^{2\kappa+2}}{\Gamma(\kappa+1)\Gamma(\kappa+3)}-\left[\frac{(\beta^{s+1}-\alpha^{s+1})^{\kappa+1}}{\Gamma(\kappa+1)}\right]^2\right)\end{aligned}\tag{28}$$

gets satisfied if $s\ge0$ *and* $\mathfrak{F}\in L_\infty[\alpha,\beta]$;

(*ii*) *the inequality*

$$\frac{\beta^{s+1}-\alpha^{s+1}}{\Gamma(\kappa)}\sigma_{\mathcal{X},\kappa,\mho}^2-E_{\mathcal{X},\kappa}^2\le\frac12\left(\frac{(\beta^{s+1}-\alpha^{s+1})^{2\kappa}}{\Gamma^2(\kappa)}\right)\tag{29}$$

gets satisfied for any $s\ge0$.

Deduction 2.3. The first part of Theorem 1 in ref. [20] will be deduced if we set $\kappa=1$, $\mho(t)=1$ for every $t\in[\alpha,\beta]$ and $s=0$ in (i) of Corollary 2.2.

Deduction 2.4. The last part of Theorem 1 in ref. [20] will be deduced if we set $\kappa=1$, $\mho(t)=1$ for every $t\in[\alpha,\beta]$ and $s=0$ in (ii) of Corollary 2.2.

Theorem 2.5. *For a r.v.* $\mathcal{X}$ *having a p.d.f.* $\mathfrak{F}:[\alpha,\beta]\to\mathbb{R}^+$. *Then*

(*i*) *for all* $\alpha<\varsigma\le\beta$, $\kappa\ge0$, $\mho\ge0$ *and* $s\ge0$, *we have*

$$\begin{aligned}&{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\sigma_{\mathcal{X},\mho}^2(\varsigma)+{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]\sigma_{\mathcal{X},\kappa,\mho}^2(\varsigma)\\&-2(E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))(E_{\mathcal{X}-E(\mathcal{X}),\mho}(\varsigma))\\&\le\|\mathfrak{F}\|_\infty^2\left(\frac{(s+1)^{1-\kappa}(\varsigma^{s+1}-\alpha^{s+1})^\kappa}{\Gamma(\kappa+1)}{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\varsigma^{2s+2}]\right)\\&+\|\mathfrak{F}\|_\infty^2\left(\frac{(s+1)^{1-\mho}(\varsigma^{s+1}-\alpha^{s+1})^\mho}{\Gamma(\mho+1)}{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma^{2s+2}]\right.\\&\left.-2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma])({}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\varsigma])\right)\end{aligned}\tag{30}$$

holds for $\mathfrak{F}\in L_\infty[\alpha,\beta]$ *and*

(*ii*) *the inequality*

$$\begin{aligned}&{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\sigma_{\mathcal{X},\mho}^2(\varsigma)+{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]\sigma_{\mathcal{X},\kappa,\mho}^2(\varsigma)\\&-2(E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))(E_{\mathcal{X}-E(\mathcal{X}),\mho}(\varsigma))\\&\le(\varsigma^{s+1}-\alpha^{s+1})({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma])({}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\varsigma])\end{aligned}\tag{31}$$

holds for any $\alpha<\varsigma\le\beta$, $\kappa\ge0$, $\mho\ge0$ *and* $s\ge0$.

Proof. From (22), we can write

$$\begin{aligned}&\frac{(s+1)^{2-\kappa-\mho}}{\Gamma(\kappa)\Gamma(\mho)}\int_\alpha^\varsigma\int_\alpha^\varsigma(\varsigma^{s+1}-t^{s+1})^{\kappa-1}(\varsigma^{s+1}-m^{s+1})^{\kappa-1}\\&\times\mathfrak{p}(t)\mathfrak{p}(m)\mathfrak{Q}(t,m)t^sm^s\mathrm{d}t\mathrm{d}m\\&={}_+\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}(\varsigma)]{}_+\mathfrak{I}_{\alpha,\mho}^{\mho,s}[\mathfrak{p}\mathfrak{Q}_1\mathfrak{H}_2(\varsigma)]\\&+{}_+\mathfrak{I}_{\alpha,\mho}^{\mho,s}[\mathfrak{p}(\varsigma)]{}_+\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1\mathfrak{H}_2(\varsigma)]\\&-{}_+\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{H}_2(\varsigma)]{}_+\mathfrak{I}_{\alpha,\mho}^{\mho,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)]\\&-{}_+\mathfrak{I}_{\alpha,\mho}^{\mho,s}[\mathfrak{p}\mathfrak{H}_2(\varsigma)]{}_+\mathfrak{I}_{\alpha,\mho}^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)].\end{aligned}\tag{32}$$

For $\varsigma\in(\alpha,\beta)$, put $\mathfrak{p}(\varsigma)=\mho(\varsigma)\mathfrak{F}(\varsigma)$, $\mathfrak{Q}_1(\varsigma)=\mathfrak{Q}_2(\varsigma)=\varsigma^{s+1}-E(\mathcal{X})$ in (32), we see

$$\begin{aligned}&\frac{(s+1)^{2-\kappa-\mho}}{\Gamma(\kappa)\Gamma(\mho)}\int_\alpha^\varsigma\int_\alpha^\varsigma(\varsigma^{s+1}-t^{s+1})^{\kappa-1}(\varsigma^{s+1}-m^{s+1})^{\kappa-1}\\&\times\mho(t)\mathfrak{F}(t)\mho(m)\mathfrak{F}(m)(t^{s+1}-m^{s+1})^2t^sm^s\mathrm{d}t\mathrm{d}m\\&={}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&+{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&-2{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(t^{s+1}\\&-E(\mathcal{X}))].\end{aligned}\tag{33}$$

But we also see

$$\frac{(s+1)^{2-\kappa-\mho}}{\Gamma(\kappa)\Gamma(\mho)}\int_{\alpha}^{\varsigma}\int_{\alpha}^{\varsigma}(\varsigma^{s+1}-t^{s+1})^{\kappa-1}(\varsigma^{s+1}-m^{s+1})^{\kappa-1}$$
$$\times\mho(t)\mathfrak{F}(t)\mho(m)\mathfrak{F}(m)(t^{s+1}-m^{s+1})^2t^sm^s\mathrm{d}t\mathrm{d}m$$
$$\le\|\mathfrak{F}\|_\infty^2\left[\frac{(s+1)^{1-\kappa}(\varsigma^{s+1}-\alpha^{s+1})^{\kappa}}{\Gamma(\kappa+1)}{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\varsigma^{2s+2}]\right.\tag{34}$$
$$+\frac{(s+1)^{1-\mho}(\varsigma^{s+1}-\alpha^{s+1})^{\mho}}{\Gamma(\mho+1)}{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma^{2s+2}]$$
$$\left.-2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma])({}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\varsigma])\right].$$

Consequently, using this and by involving (33) will establish part (i) of the result.

Now to establish (ii), we shall make use of following

$$\sup_{t,m\in[\alpha,\varsigma]}|(t^{s+1}-m^{s+1})|^2=(\varsigma^{s+1}-\alpha^{s+1})^2$$

and get

$$\frac{(s+1)^{2-\kappa-\mho}}{\Gamma(\kappa)\Gamma(\mho)}\int_{\alpha}^{\varsigma}\int_{\alpha}^{\varsigma}(\varsigma^{s+1}-t^{s+1})^{\kappa-1}$$
$$\times\mho(t)g(t)\mho(m)g(m)(t^{s+1}-m^{s+1})^2t^sm^s\mathrm{d}t\mathrm{d}m$$
$$\le(\varsigma^{s+1}-\alpha^{s+1})^2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma])({}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\varsigma]).$$

Thus, using this equation and applying (33) will prove part (ii), *i.e.*, (31) is proved. □

Theorem 2.6. *Let $\mathcal{X}$ be a r.v. having p.d.f. $g:[\alpha,\beta]\to\mathbb{R}^+$. Then*

$$\begin{aligned}&{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\sigma^2_{\mathcal{X},\kappa,\mho}(\varsigma)-(E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))^2\\&\quad\le\frac{1}{4}(\beta^{s+1}-\alpha^{s+1})^2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma])^2\end{aligned}\tag{35}$$

for every $\alpha<\varsigma\le\beta$, $\kappa\ge0$ and $s\ge0$.

Proof. From Theorem 3.1 of ref. [40], one can write

$$\begin{aligned}0&\le|{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1^2(\varsigma)]-({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)])^2|\\&\le\frac{1}{4}({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}(\varsigma)])^2(\mathcal{M}-\mathfrak{m})^2.\end{aligned}\tag{36}$$

For $\varsigma\in[\alpha,\beta]$, we choose $\mathfrak{p}(\varsigma)=\mho\mathfrak{F}(\varsigma)$ and $\mathfrak{Q}_1(\varsigma)=\varsigma^{s+1}-E(\mathcal{X})$, then $\mathcal{M}=\beta^{s+1}-E(\mathcal{X})$ and $\mathfrak{m}=\alpha^{s+1}-E(\mathcal{X})$. Thus, from (36), we can have

$$\begin{aligned}0&\le{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&\quad-({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2])^2\\&\le\frac{1}{4}({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)])^2(\beta^{s+1}-\alpha^{s+1})^2.\end{aligned}\tag{37}$$

This yields that

$$\begin{aligned}&{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\sigma^2_{\mathcal{X},\kappa,\mho}(\varsigma)-(E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))^2\\&\quad\le\frac{1}{4}(\beta^{s+1}-\alpha^{s+1})^2({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\varsigma])^2.\end{aligned}\tag{38}$$

This completes the proof. □

Now choosing $\varsigma=\beta$ and $\mho=1$, we have following corollary:

Corollary 2.7. *For a p.d.f. g of r.v. $\mathcal{X}$, we have*

$$\begin{aligned}&\frac{(\beta^{s+1}-\alpha^{s+1})^{\kappa-1}}{\Gamma(\kappa)}\sigma^2_{\mathcal{X},\kappa}(\varsigma)-(E_{\mathcal{X}-E(\mathcal{X}),\kappa}(\varsigma))^2\\&\quad\le\frac{1}{4\Gamma^2(\kappa)}(\beta^{s+1}-\alpha^{s+1})^{2\kappa}\end{aligned}$$

for $\kappa\ge0$ and $s\ge0$.

Deduction 2.8. The Theorem 3.7 of ref. [22] will be deduced if we set $s=1$ and $\mho(t)=1$ for every $t\in[\alpha,\beta]$ in Corollary 2.7.

Theorem 2.9. *For a r.v. $\mathcal{X}$ with p.d.f. $g:[\alpha,\beta]\to\mathbb{R}^+$. Then*

$$\begin{aligned}&{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\sigma^2_{\mathcal{X},\mho}(\varsigma)+{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]\sigma^2_{\mathcal{X},\kappa,\mho}(\varsigma)\\&\quad+2(\alpha^{s+1}-E(\mathcal{X}))(\beta^{s+1}-E(\mathcal{X}))\\&\quad\times{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]\\&\le(\alpha^{s+1}+\beta^{s+1}-2E(\mathcal{X}))({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]\\&\quad(E_{\mathcal{X}-E(\mathcal{X}),\mho}(\varsigma))+{}_+\mathfrak{I}_\alpha^{\beta,s}[\mho\mathfrak{F}(\varsigma)](E_{\mathcal{X}-E(\mathcal{X}),\alpha}(\varsigma)))\end{aligned}\tag{39}$$

for every $\alpha<\varsigma\le\beta$, $\kappa\ge0$, $\mho\ge0$ and $s\ge0$.

Proof. From Theorem 3.4 of ref. [40], we can write

$$\begin{aligned}&[{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}\mathfrak{Q}_1^2(\varsigma)]+[{}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1^2(\varsigma)]\\&\quad-2{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)]]^2\\&\le[(\mathcal{M}{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}(\varsigma)]-{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)])({}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)]\\&\quad-\mathfrak{m}{}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}(\varsigma)])+({}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)]\\&\quad-\mathfrak{m}{}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}(\varsigma)])(\mathcal{M}{}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}(\varsigma)]-{}_+\mathfrak{I}_\alpha^{\mho,s}[\mathfrak{p}\mathfrak{Q}_1(\varsigma)])]^2.\end{aligned}\tag{40}$$

Choosing $\mathfrak{p}(\varsigma)=\mho\mathfrak{F}(\varsigma)$ and $\mathfrak{Q}_1(\varsigma)=\varsigma^{s+1}-E(\mathcal{X})$ in (40) yields

$$\begin{aligned}&[{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&\quad+{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&\quad-2{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]]^2\\&\le[M{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]-{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]\\&\quad\times({}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]-m{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)])\\&\quad\times({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]-m{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)])\\&\quad\times(M{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]-{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))])]^2.\end{aligned}\tag{41}$$

Now from (33) and (41) and using the fact that the left hand side of (33) is positive, we see

$$\begin{aligned}&{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&\quad+{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2]\\&\quad-2{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]\\&\le M{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]-{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]\\&\quad\times({}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]-m{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)])\\&\quad\times({}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]-m{}_+\mathfrak{I}_\alpha^{\kappa,s}[\mho\mathfrak{F}(\varsigma)])\\&\quad\times(M{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)]-{}_+\mathfrak{I}_\alpha^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))]).\end{aligned}\tag{42}$$

This yields us

$$\begin{aligned}
&{}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho\mathfrak{F}(\varsigma)]_{+}\mathfrak{I}_{\alpha}^{\mho,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2] \\
&\quad + {}_{+}\mathfrak{I}_{\alpha}^{\mho,s}[\mathfrak{F}(\varsigma)]_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho\mathfrak{F}(\varsigma)(\varsigma^{s+1}-E(\mathcal{X}))^2] \\
&\leq \mathcal{M}({}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho\mathfrak{F}(\varsigma)](E_{\mathcal{X}-E(\mathcal{X}),\mho}(\varsigma)) \\
&\quad + {}_{+}\mathfrak{I}_{\alpha}^{\mho,s}[\mho\mathfrak{F}(\varsigma)(E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))]) \\
&\quad + \mathfrak{m}({}_{+}\mathfrak{I}_{\alpha}^{\kappa,s}[\mho\mathfrak{F}(\varsigma)](E_{\mathcal{X}-E(\mathcal{X}),\mho}(\varsigma)) \\
&\quad + {}_{+}\mathfrak{I}_{\alpha}^{\mho,s}[\mho\mathfrak{F}(\varsigma)(E_{\mathcal{X}-E(\mathcal{X}),\kappa,\mho}(\varsigma))]).
\end{aligned}$$

Consequently, the result is established by simple calculation with utilizing the values of $\mathcal{M}$ and $\mathfrak{m}$ from Theorem 2.6. □

3 Conclusion

In this paper, we have presented various concepts of fractional calculus. Also, some new generalizations of outcomes for continuous random variables having fractional order have been given. Furthermore, certain definitions like $\mho$-weighted fractional expectation and variance, have been introduced and their various properties have been studied. Moreover, new bounds and inequalities have been established. The consequences of the results obtained in this manuscript are more general and extensive than the pre-existing known results.

Acknowledgments: The authors are thankful to the anonymous reviewers for their valuable comments and suggestions towards the improvement of the article.

Funding information: The authors state no funding involved.

Author contributions: The author have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

References

[1] Barnett NS, Cerone P, Dragomir SS, Roumeliotis J. Some inequalities for the expectation and variance of a random variable whose PDF is n-time differentiable. J Inequalit Pure Appl Math (JIPAM). 2000;1(2):1–13.
[2] Qi F, Li AJ, Zhao WZ, Niu DW, Cao J. Extensions of several integral inequalities. J Inequalit Pure Appl Math. 2006;7(3):1–6.
[3] Qi F. Several integral inequalities. J Inequalit Pure Appl Math. 2000;1(2):1–9.
[4] Agarwal P, Dragomir SS, Jleli M, Samet B. Advances in mathematical inequalities and applications. 1st ed. Birkhäuser, Singapore: Springer Nature Singapore Pte Ltd.; 2018. 978-981-13-3012-4, 978-981-13-3013-1.
[5] Agarwal P, Jleli M, Tomar M. Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. J Inequal Appl. 2017;2017:55.
[6] Agarwal P. Some inequalities involving Hadamard-type k-fractional integral operators. Math Methods Appl Sci. 2017;40(11):3882–91. doi: 10.1002/mma.4270.
[7] Ali MA, Abbas M, Budak H, Agarwal P, Murtaza G, Chu Yu-M. New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. Adv Differ Equ. 2021;2021:64.
[8] Ganie AH. New bounds for random variables of fractional order. Pak J Stat. 2022;38(2):211–8.
[9] Higazy M, Hijaz H, Ganie AH, Botmart T, El-Mesady A. Theoretical analysis and computational modeling of nonlinear fractional-order-two predators mode. Results Phys. 2021;17:1–26.
[10] Agarwal P, Restrepo JE. An extension by means of ω-weighted classes of the generalized Riemann–Liouville k-fractional integral inequalities. J Math Inequalities. 2020;14(1):35–46.
[11] Anastassiou GA, Hooshamandasl MR, Ghasemi A, Moftakharzadeh F. Montgomery identities for fractional integrals and related fractional inequalities. J Ineq Pure Appl Math. 2009;10(4):1–6.
[12] Ganie AH, Saeed AM, Sadia S, Ali U. The Rayleigh-Stokes problem for a heated generalized second-graded fluid with fractional derivative: an implicit scheme via Riemann–Liouville integral. Math Probl Eng. 2022;2022. Article ID 6948461, 13 pp.
[13] Amin R, Shah K, Ahmad H, Abdul HG, Haleem A, Botmart T. Haar wavelet method for solution of variable order linear fractional integro-differential equations. AIMS Math. 2022;7(4):5431–43.
[14] Machado JAT. A probabilistic interpretation of the fractional-order differentiation. Fract Calc Appl Anal. 2003;6(1):73–80.
[15] Podlubny I. Geometri and physical interpretation of fractional integration and fractional differentiation. Fract Calc Appl Anal. 2002;5(4):367–86.
[16] Das S, Pan I, Halder K, Gupta A. LQR based improved discrete PID controller design via optimum selection of weighting matrices using fractional order integral performance index. Appl Mat Model. 2013;37:4253–368.
[17] Ganie AH. Some new approach of spaces of non-integral order. Nonlinear Sci Appl. 2021;14(2):89–96.
[18] Ganie AH. New approach for structural behaviour of variables. J Nonlinear Sci Appl. 2021;14:351–8.
[19] Mehrez K, Agarwal P. New Hermite-Hadamard type integral inequalities for convex functions and their applications. J Comput Appl Math. 2019;350:274–85.
[20] Barnett NS, Cerone P, Dragomir SS, Roumeliotis J. Some inequalities for the dispersion of a random variable whose PDF is defined on a finite interval. J Inequalities Pure Appl Math. 2001;9(1):1–18.
[21] Belarbi S, Dahmani Z. On some new fractional integral inequalities. J Inequalities Pure Appl Math. 2009;10(3):1–12.

[22] Dahmani Z. New applications of fractional calculus on probabilistic random variables. Acta Math Univ Comen. 2017;86(2):299–307.

[23] Dahmani Z. New inequalities in fractional integrals. Int J Nonlinear Sci. 2010;9(4):493–7.

[24] Kumar P. Moment inequalities of a r.v. defined over a finite interval. J Inequalities Pure Appl Math. 2002;3(3):1–24.

[25] Dahmani Z. Fractional integral inequalities for continuous r.v.s. Malaya J Math. 2014;2(2):172–9.

[26] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. North-Holland Mathematics Studies. vol. 204. Amsterdam: Elsevier; 2006.

[27] Blutzer RL, Torvik PJ. On the fractional calculus model of viscoelastic behaviour. J Rheol. 1996;30:133–5.

[28] Jonsson T, Yngvason J. Waves and distributions. River Edge, NJ: World Scientific Publishing Co., Inc.; 1995.

[29] Khan Z, Ahmad H. Qualitative properties of solutions of fractional differential and difference equations arising in physical models. Fractals. 2021;29(5):2140024.

[30] Houas M. Certain weighted integral inequalities involving the fractional hypergeometric operators. Sci A Math Sci. 2016;27:87–97.

[31] Liu W, Ngo QA, Huy VN. Several interesting integral inequalities. J Math Inequal. 2009;3(2):201–12.

[32] Sharma R, Devi S, Kapoor G, Ram S, Barnett NS. A brief note on some bounds connecting lower order moments for r.v.s defined on a finite interval. Int J Theoret Appl Sci (IJTAS). 2009;1(2):83–5.

[33] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives, theory and applications. Yverdon, Switzerland: Gordon and Breach; 1993.

[34] Troparevsky MI, Seminara SA, Fabio MA. A review on fractional differential equations and a numerical method to solve some boundary value problems. In: Nonlinear systems-theoretical aspects and recent applications. United Kingdom: IntechOpen Limited; 2019. doi: 10.5772/intechopen.86273.

[35] Yıldırım H, Kırtay Z. Ostrowski inequality for generalized fractional integral and related inequalities. Malaya J Mat. 2014;2(3):322–9.

[36] Nale AB, Panchal SK, Chinchane VL. On weighted fractional inequalities using Hadmard fractiona integral operator. Palestine J Math. 2021;10(2):614–24.

[37] Naik PA, Owolabi K, Zu J, Naik MD. Modeling the transmission dynamics of COVID-19 pandemic in Caputo type fractional derivative. J Multiscale Model. 2021;12(3):2150006. doi: 10.1142/S1756973721500062.

[38] Singh A, Ganie AH, Albaidan MM. Some new inequalities using nonintegral notion of variables. Adv Math Phys. 2021;2021. Article ID 8045406, 6pp.

[39] Naik PA, Zu J, Owolabi K. Modeling the mechanics of viral kinetics under immune control during primary infection of HIV-1 with treatment in fractional order. Physics A. 2020;545:123816.

[40] Dahmani Z, Khameli A, Bezziou M, Sarikaya MZ. Some estimations on continuous random variables involving fractional calculus. Int J Anal Appl. 2017;1(15):8–17.