Research Article

Qian Xiang, Yi Zhong, Qun-Ying Xie, and Li Zhao*

Flat and bent branes with inner structure in two-field mimetic gravity

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Abstract: Inspired by the work Zhong et al. (2018), we study the linear tensor perturbation of both the flat and bent thick branes with inner structure in two-field mimetic gravity. The master equations for the linear tensor perturbations are derived by taking the transverse and traceless gauges. For the Minkowski and Anti-de-Sitter brane, the brane systems are stable against the tensor perturbation. The effective potentials of the tensor perturbations of both the flat and bent thick branes are volcano-like, and this structure may potentially lead to the zero-mode and the resonant modes of the tensor perturbation. We further illustrate the results of massive resonant modes.

Keywords: flat brane, bent brane, tensor perturbation, two-field mimetic gravity, resonant mode

1 Introduction

The braneworld theory has received much attention over the past years since it proposes a new route to possibly solve the gauge hierarchy problem and the cosmological constant problem [1–4]. The Randall–Sundrum (RS) models [3,4] are typical examples of them with a non-factorizable metric and a warped extra dimension. It is shown that RS models give rise to thin brane profiles because the warp factor has cusp singularities at the brane positions. Several proposals for generalizing thin branes into thick branes have been presented in the literature. The thick branes are obtained by introducing one or more bulk scalar fields coupled to gravity [5–15] or realized by pure geometric frameworks without background matter fields considered in refs [16–19]. Most of the models study Minkowski branes, and a few of them consider the curvature of the embedded brane, which includes de Sitter (dS) or anti-de Sitter (AdS) geometry.

On the other hand, mimetic gravity is proposed by Chamseddine and Mukhanov [20] as one of the extensions of general relativity (GR). In the original setup, a physical metric $g_{\mu\nu}$ is defined in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field $\phi$ with the relation $g_{\mu\nu} = -\tilde{g}_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$. This model regards the scalar field as the conformal degree of freedom to mimic cold dark matter. The mimetic model is then investigated by adding a potential $V(\phi)$ of the scalar field to explain the cosmological issues [21,22]. With the appropriate choice of potential, it is possible to provide an inflationary mechanism and a bouncing universe within this framework [21]. The cosmological behavior of mimetic $F(R)$ gravity is investigated [23]. The current values of dark energy and dark matter densities are of the same order of magnitude seems to indicate that we are currently living at a special time in the history of the universe, which gives rise to the cosmic coincidence problem. The de Sitter solution of the dark matter and dark energy parameter can be approached in cosmology, which would try to explain or alleviate the cosmological coincidence problem [24,25]. Therefore, this stimulates some interest in phenomenology and its observational viability [26–33] and leads to the Hamiltonian analyses of different mimetic models [34–38]. Mimetic gravity has also been widely discussed in a variety of modified gravity theories [39–50]. Another interest in mimetic gravity is to investigate the mimetic gravity in braneworld scenarios. The late-time cosmic expansion and inflation are investigated in the mimetic RS braneworld [27]. Following this work, the late-time acceleration and
perturbation behavior are studied for the brane–anti-brane system [51]. Later, the tensor and scalar perturbations on several thick branes are investigated in mimetic gravity [52]. For a review of the topics related to cosmology and astrophysics in mimetic theory, see ref. [53].

As we know, the scalar fields are usually introduced to generate the topological defects, such as kinks and domain walls, for realizing the thick branes, so it is natural to generate the domain walls by the mimetic scalar fields. However, suffering from the ghost and gradient instabilities, the original mimetic theory with a single field could not suffice [54]. For the single-field mimetic scenario, one can go to a ghost-free theory by adding higher derivative terms to the original theory [55]. For the two-field extension of the mimetic gravity put forward in ref. [56], the double scalar fields version not only avoids the above problem but also allows us to construct the thick branes with a complicated inner structure. It is common to find the background solution of the thick branes via the first-order formalism [11] or the extension method [57]. However, in the two-field mimetic theory, the gravity and two mimetic scalars are coupled. It is relatively harder to find the domain wall solutions if the four-dimensional geometry is either flat or curved. In this work, we apply the reconstruction technique [58] to seek the background solutions of three cases of thick branes. The approach gives the form of the warp factor and scalar fields, and provides a direct way to investigate the other variables. We consider that the thick domain walls possess four-dimensional dS and AdS symmetries as well as the Poincaré one. Because dS and AdS branes have different inner structures from flat brane, it is of interest to consider the tensor perturbation of gravity on the bent branes.

This article is organized as follows. In Section 2, we introduce the mimetic thick brane model and consider three cases of thick branes. In Section 3, we analyze the stability of the model under the linear tensor perturbation and study the localization of gravity zero-mode. Finally, a brief conclusion is presented in Section 5. Throughout this article, capital Latin letters $M$, $N$, ... represent the five-dimensional coordinate indices running over 0, 1, 2, 3, and 5 and lowercase Greek letters $\mu$, $\nu$, ... represent the four-dimensional coordinate indices running over 0, 1, 2, and 3.

2 Brane setup and field equations

We consider the five-dimensional, two-field mimetic gravity where the action is the Einstein–Hilbert action constructed in terms of the physical metric $g_{\mu\nu}$. For our model, in the natural unit, the action can be written as a Lagrange multiplier formulation as follows:

$$S = \int d^4 x dy \sqrt{-g} \left( \frac{R}{2} + L_m \right),$$

(1)

with the Lagrangian of the two interacting mimetic scalar fields is generalized as follows:

$$L_m = \lambda \left[ g^{MN} \partial_M \phi_1 \partial_N \phi_1 + g^{MN} \partial_M \phi_2 \partial_N \phi_2 - U(\phi_1, \phi_2) \right] - V(\phi_1, \phi_2).$$

(2)

In the original mimetic model, $U(\phi) = -1$, and here, the potential is extended into the form of $U(\phi_1, \phi_2)$ with double mimetic fields. The Lagrange multiplier $\lambda$ enforces the mimetic constraint as follows:

$$g^{MN} \partial_M \phi_1 \partial_N \phi_1 + g^{MN} \partial_M \phi_2 \partial_N \phi_2 - U(\phi_1, \phi_2) = 0.$$  

(3)

The variation of the action (1) concerning the metric $g_{MN}$ and the two scalar fields $(\phi_1, \phi_2)$ yields the following field equations, respectively:

$$g_{MN} + 2\lambda \partial_M \phi_1 \partial_N \phi_1 + 2\lambda \partial_M \phi_2 \partial_N \phi_2 - L_m g_{MN} = 0,$$

$$2\lambda \partial^2 \phi_1 + 2\lambda \partial^2 \phi_2 + \lambda \frac{\partial U(\phi_1, \phi_2)}{\partial \phi_1} \partial \phi_1 + \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_1} = 0,$$

$$2\lambda \partial^2 \phi_2 + 2\lambda \partial^2 \phi_2 + \lambda \frac{\partial U(\phi_1, \phi_2)}{\partial \phi_2} \partial \phi_2 + \frac{\partial V(\phi_1, \phi_2)}{\partial \phi_2} = 0.$$  

(4)

The line-element for a warped five-dimensional geometry is generally assumed as follows:

$$ds^2 = a^2(y) \hat{g}_{\mu\nu} dx^\mu dx^\nu + dy^2,$$

(5)

with $y$ the extra spatial coordinate. We deal with $a = a(y)$, $\phi_1 = \phi_1(y)$, and $\phi_2 = \phi_2(y)$ when considering the static brane. The metrics $\hat{g}_{\mu\nu}$ on the branes reads,

$$\hat{g}_{\mu\nu} = \left\{ \begin{array}{ll}
-dt^2 + (dx_1^2 + dx_2^2 + dx_3^2) & M_4 \text{ brane,} \\
-dt^2 + e^{2\Lambda_4} (dx_1^2 + dx_2^2 + dx_3^2) & \text{AdS}_4 \text{ brane,} \\
e^{-2\sqrt{-\Lambda_4}} (-dt^2 + dx_1^2 + dx_2^2) + dx_3^2 & \text{AdS}_4 \text{ brane,} \\
\end{array} \right.$$

(6)

where $\Lambda_4$ is related to the four-dimensional cosmological constant of dS$_4$ or AdS$_4$ brane [59,60].

We know that Eqs. (3) and (4) determine the solution of the brane system. There are three independent equations and six variables in this system. To solve this system, we should preset three variables. Based on the reconstruction technique, we will give the form of $a(y)$, $\phi_1(y)$, $\phi_2(y)$, and try to find the solution of $U(\phi_1, \phi_2)$,
V(ϕ₁, ϕ₂), and λ(ϕ₁, ϕ₂). Here, the warp factor a(y) and the scalar fields ϕ₁(y) and ϕ₂(y) are given as follows:

\[ a(y) = \operatorname{sech}(k(y - b)) + \operatorname{sech}(k(y)) + \operatorname{sech}(k(y + b)), \]

\[ \phi_1(y) = \tanh(k(y - b)) + \tanh(k(y + b)), \]

\[ \phi_2(y) = \tanh(k(y - b)) - \tanh(k(y + b)), \]

where k and b are parameters with dimension mass and length, respectively. To display the configuration of a(y), ϕ₁(y), and ϕ₂(y), we introduce the dimensionless quantities \( \bar{y} = ky \) and \( \bar{b} = kb \). The shapes of the warp factor and the two scalar fields are depicted in Figure 1. The above warp factor indicates that the bulk space-time is asymptotically AdS, which is essential for the localization of gravitation. The brane splits from a single brane into three sub-branes as the parameter \( \bar{b} \) increases. Thus, these branes have a rich inner structure. The scalar field \( \phi_1(\bar{y}) \) supports the topological solution, which changes from a single-kink to a double-kink configuration with the increasing of \( \bar{b} \). Without loss of generality, the other scalar field \( \phi_2(\bar{y}) \) is assumed as a non-topologically lump-like solution.

2.1 Flat brane

For Minkowski brane, we take \( \Lambda_4 = 0 \). With the ansatz (6), the Einstein tensors are expressed as follows:

\[ G_{\mu\nu} = 3\eta_{\mu
u}(a(y)a''(y) + a'(y)^2), \]
\[ G_{55} = 6a'(y)^2a(y)^2. \]

Eqs. (4) and (3) can be reduced to

\[ \frac{3\alpha'^2}{a^2} + \frac{3\alpha''}{a} + V(\phi_1, \phi_2) + \lambda(U(\phi_1, \phi_2) - \phi_1^{12} - \phi_2^{12}) = 0, \]
\[ \frac{6\alpha'^2}{a^2} + V(\phi_1, \phi_2) + \lambda(U(\phi_1, \phi_2) + \phi_1^{12} + \phi_2^{12}) = 0, \]

where the prime denotes the derivative with respect to \( y \). Plugging Eqs. (7)–(9) into (15), we obtain the following analytic solution:

\[ U(\phi_1(y), \phi_2(y)) = 2k^2[k(y - b)] + \tanh^2(k(y + b))], \]
2.2 Bent branes

We now turn attention to the case of $\Lambda_4 \neq 0$. The presence of $\Lambda_4$ makes the field equation and background solution complex. For the $dS_4$ geometry ($\Lambda_4 > 0$), with the ansatz (6), the Einstein tensors are expressed as follows:

$$ G_{\mu\nu} = -\frac{\mathcal{E}(\Lambda_4 - a'(y)^2 - a(y)a''(y))a^2(y)\tilde{g}_{\mu\nu}}{a^2(y)} , \quad G_{55} = -\frac{6(\Lambda_4 - a'(y)^2)}{a^2(y)}. $$

Eqs. (4) and (3) can be reduced to

$$ -\frac{3\Lambda_4 - (a'(y)^2 - a(y)a''(y))}{a^2(y)} + V(\phi_1, \phi_2) + \lambda(U(\phi_1, \phi_2) - \phi_{12} - \phi_{22}) = 0, $$

$$ -\frac{6(\Lambda_4 - a'(y)^2)}{a^2(y)} + V(\phi_1, \phi_2) + \lambda(U(\phi_1, \phi_2) + \phi_{12} + \phi_{22}) = 0, $$

$$ \frac{8\lambda a'(y)\phi_1'}{a(y)} + 2\lambda\phi_1'' + 2\lambda'\phi_1' + \lambda\frac{\partial U(\phi_1, \phi_2)}{\partial \phi_1} + \lambda\frac{\partial V(\phi_1, \phi_2)}{\partial \phi_1} = 0, $$

$$ \frac{8\lambda a'(y)\phi_2'}{a(y)} + 2\lambda\phi_2'' + 2\lambda'\phi_2' + \lambda\frac{\partial U(\phi_1, \phi_2)}{\partial \phi_2} + \lambda\frac{\partial V(\phi_1, \phi_2)}{\partial \phi_2} = 0, $$

$$ \phi_{12}^2 + \phi_{22}^2 = U(\phi_1, \phi_2). $$

Now the system can be solved as follows:

$$ U(\phi_1(y), \phi_2(y)) = 2k^2[\text{sech}^2(k(y-b)) + \text{sech}^2(k(y+b))], $$

$$ V(\phi_1(y), \phi_2(y)) = \frac{3}{[\text{sech}(k(y-b)) + \text{sech}(k(y+b)) + \text{sech}(ky)]^2} \times \left[ \Lambda_4 - k^2(\text{tanh}(k(y-b)) \text{sech}(k(y-b)) + \text{tanh}(k(y+b)) \text{sech}(k(y+b)) + \text{tanh}(ky) \text{sech}(ky) \right]^2 + k^2(\text{sech}(k(y-b)) + \text{sech}(k(y+b)) + \text{sech}(ky)) \times \left( \text{sech}^2(k(y-b)) - \frac{1}{2}(\cosh(2k(y+b)) - 3) \text{sech}^2(k(y+b)) - \text{tanh}^2(k(y-b)) \text{sech}(k(y-b)) + \text{sech}^2(k(y) \text{tanh}(ky) \text{sech}(ky)). $$
\(\lambda(\phi_1(y), \phi_2(y)) = \frac{-3}{8k^2[\text{sech}(k(y-b)) + \text{sech}(k(y+b)) + \text{sech}(ky)][\text{sech}(k(y-b)) + \text{sech}(k(y+b))]}\) 
\[\times[-2A_\lambda + 2k^2\text{sech}(k(y-b)) + 2k^2\text{sech}(k(y+b)) + 2k^2\text{sech}(ky)\text{sech}(k(y-b)) + k^2(\cosh(2k(y-b)) + \cosh(2k(y+b)) - \cosh(4bk) + 3)\text{sech}(k(y-b))\text{sech}(k(y+b)) + k^2\text{sech}(ky)(2\text{sech}(k(y-b)) + (\cosh(2k(y+b)) - \cosh(2bk) + \cosh(2ky) + 3)\text{sech}(k(y+b)) + 2k^2\text{sech}(k(y-b))].\]

For AdS case \((\Lambda_4 < 0)\), we change \(\Lambda_4 \rightarrow -\Lambda_4\), and the solution of dS\(_4\) brane is transformed into that of AdS\(_4\) brane. This result is interesting, since it simplifies the calculation, significantly.

So far, we have obtained the background solution of the three cases of thick branes. In more detail, the potential \(U(y)\) is the same; however, \(V(y)\) and \(\lambda(y)\) take different values for they are related to the parameter \(\Lambda_4\). For the sake of clarity, by performing the rescaling of the quantities,

\[\hat{y} = ky, \hat{\Lambda}_4 = \Lambda_4 k^2, \hat{\lambda}(\hat{y}) = V(\hat{y}) k^2, \hat{U}(\hat{y}) = U(\hat{y}) k^2, \hat{\lambda}(\hat{y}) = \lambda(\hat{y}),\]

we can obtain the dimensionless quantities. The profiles of the potential \(\hat{U}(\hat{y})\) with the increasing of the parameter \(\hat{b}\) are shown in Figure 2. In Figure 3, we compare the behaviors of \(\hat{V}(\hat{y})\) in the three cases of thick branes.

Figure 3 shows that the potential \(\hat{V}(\hat{y})\) in AdS\(_4\) brane opens downwards, different from dS\(_4\) brane where the curve opens upwards. When \(\Lambda_4 \rightarrow 0\), \(\hat{V}(\hat{y})\) in dS\(_4\) and AdS\(_4\) branes approach to the values in M\(_4\) brane. The shapes of the Lagrange multiplier \(\hat{\lambda}(\hat{y})\) in terms of \(\hat{y} = ky\) are plotted in Figure 4. When \(\Lambda_4 \rightarrow 0\), \(\hat{\lambda}(\hat{y})\) in dS\(_4\) and AdS\(_4\) branes also tend to the values in M\(_4\) brane.

### 3 Linear tensor perturbation

In the case of the tensor perturbation, we suppose that the space-time undergoes a small perturbation \(\delta g_{MN}^{(1)}\) on a fixed background \(g_{MN}\) as follows:

\[\bar{g}_{MN} = g_{MN} + \delta g_{MN}^{(1)},\]

where \(g_{MN}\) represents the five-dimensional Minkowski, dS, or AdS metric. The inverse of the perturbed metric will be

\[\bar{g}^{MN} = g^{MN} + \delta g^{(1)MN} + \cdots \]

with the first-order perturbed metric as \(\delta g^{(1)MN} = -g^{MP}g^{NQ}g_{PQ}^{(1)}\) and the \(n\)-order perturbed metric as follows: \(\delta g^{(n)MN} = (-1)^n\delta g_{P_1}^{(1)M}g_{P_1}^{(1)N}g_{P_1}^{(1)P_1}\cdots g_{P_1}^{(1)P_1}g_{P_1}^{(n)N}\). For the above metric perturbations, the first-order perturbations are expressed as follows:

\[\delta Y_{PQ}^{(1)} = \frac{1}{2}g^{LP}(\nabla_M\delta g_{NP}^{(1)} + \nabla_N\delta g_{MP}^{(1)} - \nabla_P\delta g_{MN}^{(1)})\]
\[ \delta R^{(i)}_{MN} = \nabla_k \delta R^{(i)}_{MN} - \nabla_N \delta R^{(i)}_{MK} \quad (29) \]
\[ \delta R_{MN} = \nabla_k \delta R^{K}_{MN} - \nabla_N \delta R^{K}_{MK}, \quad (30) \]

where \( \nabla_N \) denotes the covariant derivative corresponding to the five-dimensional metric \( g_{5 MN} \). In general, it is complicated to take into account a full set of fluctuations of the metric around the background where gravity is coupled to scalars. Fortunately, there is a sector where the metric fluctuations decouple from the scalars, which is the one associated with the transverse and traceless (TT) part of the metric fluctuation. Based on these relations, we will consider the linear tensor perturbation of flat and bent branes by taking the TT gauge condition.

### 3.1 Flat brane

For the tensor perturbation of flat brane, the perturbed metric is given as follows:

\[ ds^2 = a^2(y)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2, \quad (31) \]

where \( \eta_{\mu\nu} \) describes Minkowski geometry, \( h_{\mu\nu} \) represents the tensor perturbation and satisfies TT gauge condition \( \eta^{\mu\nu}\partial_\mu h_{\nu\nu} = 0 \) and \( \eta^{\mu\nu}h_{\mu\nu} = 0 \). The linear perturbations of the Ricci tensor and curvature scalar are obtained as follows:

\[
\begin{aligned}
\delta R^{(i)}_{\mu\nu} &= \frac{1}{2}(\partial_\mu \partial_\nu h_\sigma + \partial_\sigma \partial_\nu h_\mu - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h) \\
&\quad - (3a^2 + 2a' a'' + \frac{1}{2}a'' a^2 - \frac{1}{2}a a'' a') h', \\
\delta R^{(i)}_{\mu\nu} &= \frac{1}{2}(\partial_\mu h_\nu - \partial_\nu h_\mu), \\
\delta R^{(i)}_{\mu\nu} &= \frac{1}{2}(\partial_\mu h_\nu - \partial_\nu h_\mu), \\
\delta R^{(i)} &= \delta (g^{MN}R_{MN}) = a^{-2}(\partial_\mu \partial_\nu h_{\mu\nu} - \Box h) - 5a' a' - h', \\
\end{aligned}
\]

where \( \Box^{(4)} = \eta^{\mu\nu}\partial_\mu \partial_\nu \) is the four-dimensional d’Alembert operator, and \( h = \eta^{\mu\nu}h_{\mu\nu} \).

Under the TT condition, the perturbation of the \( \mu\nu \) components of the Einstein tensor reads

\[
\delta G^{(i)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + (3a^2 + 3a a')h_{\mu\nu} - 2a a' h_{\mu\nu}
\]

where the four-dimensional d’Alembertian is defined as \( \Box^{(4)} = \eta^{\mu\nu}\partial_\mu \partial_\nu \). Using Eqs. (19) and (33), the perturbation equation reads

\[
-\frac{1}{2}\Box h_{\mu\nu} - 2a a' h_{\mu\nu} - \frac{1}{2}a^2 h_{\mu\nu} = 0.
\]

By imposing a coordinate transformation, \( dz = \frac{1}{a(z)}dy \) and a rescaling, on \( h_{\mu\nu} = a(z)^{-2}h_{\mu\nu} \), the perturbation equation (34) can be calculated as follows:

\[
\Box h_{\mu\nu} + \frac{\partial^2 h_{\mu\nu}}{a^2(z)^2} = 0.
\]

Considering the Kaluza–Klein (KK) decomposition \( h_{\mu\nu} = e^{m^2y}e^{p_\nu x}H(z) \) with \( p^2 = -m^2 \), where the polarization tensor \( e_{\mu\nu} \) satisfies the TT condition \( \eta^{\mu\nu}\partial_\mu e_{\nu\nu} = 0 \) and \( \eta^{\mu\nu}e_{\mu\nu} = 0 \), we obtain the Schrödinger-like equation for \( H(z) \):

\[
-\partial_z^2 + V_{\text{eff}}(z)H(z) = m^2H(z),
\]

where \( m \) is the mass of the KK mode, and the effective potential \( V_{\text{eff}}(z) \) is given by [11] as follows:

\[
V_{\text{eff}}(z) = \frac{a^2z^2}{a^2(z)^2} = \left( \partial_z \ln a(z) \right)^2 + \partial_z \partial_z \left( \partial_z \ln a(z) \right).
\]

This equation can be factorized as

\[
-\left[ \partial_z + \partial_z \ln a(z) \right] \left[ \partial_z - \partial_z \ln a(z) \right] H(z) = m^2H(z),
\]

and this structure ensures that the eigenvalues are non-negative, which means that the brane is stable against the tensor perturbation. Since the potential vanishes for large \( z \), this is the only bound state, namely, the massless zero mode (\( m = 0 \)).
H_0(z(y)) \propto a^2(z(y))
\quad = [\text{sech}(k(y - b)) + \text{sech}(k(y) + \text{sech}(k(y + b)))]^{\frac{3}{2}}. \quad (39)

To localize the gravity zero-mode, H_0(z) should obey the normalization condition \( \int_{-\infty}^{\infty} |H_0(z)|^2 dz < \infty \). It can be normalized if
\[ \int_{-\infty}^{\infty} |H_0(z)|^2 dz = \int_{-\infty}^{\infty} |H_0(z(y))|^2 d\bar{y} 
\quad = \int \left( \text{sech}(k(y - b)) + \text{sech}(k(y) + \text{sech}(k(y + b)) \right) d\bar{y} < \infty, \quad (40)\]

which is finite when \( k > 0 \); in other words, the normalized zero-mode can be achieved for \( k > 0 \) such that the observable four-dimensional gravity is recovered on the brane. The behaviors of the dimensionless effective potential and zero-mode in terms of \( \bar{y} \) are shown in Figure 5. The effective potentials have a well with a negative minimum inside the brane and satisfy \( V_{\text{eff}}(\bar{y} \to \pm \infty) \to 0 \) when far from the brane. As the parameter \( b \) increases, the volcano-like potential gradually changes to a multi-well potential, and at last, splits into three-well potential; meanwhile, the wave function of the graviton zero-mode also splits.

### 3.2 Bent branes

We now turn to the case of \( \Lambda_4 \neq 0 \) and consider the following perturbed metric:
\[ ds^2 = a^2(z) \hat{g}_{\mu\nu} + h_{\mu\nu} dz^2, \quad (41)\]
where the four-dimensional metric is decomposed into a small perturbation \( h_{\mu\nu} \) around a curved space-time \( a^2(z) \hat{g}_{\mu\nu} \). By imposing a coordinate transformation \( dz = \frac{1}{a(z)} dy \), we write the bulk metric in the form
\[ ds^2 = a^2(z) \left( \hat{g}_{\mu\nu} + h_{\mu\nu} \right) dz^2 + d\bar{y}^2. \quad (42)\]

The interesting investigations of the tensor perturbation have already appeared in refs. [7,61] and in references therein. Then, under the TT gauge condition \( h_{\mu\nu}^\parallel = \hat{\nabla}_a h_{\mu\nu} = 0 \), the equation for the perturbation \( h_{\mu\nu} \) takes the following form:
\[ \left[ \partial_\bar{y}^2 + \frac{3}{4} \partial_\bar{y} a(z) \partial_\bar{y} a(z) - \frac{9}{4} \Lambda_4 \right] h_{\mu\nu}(x, z) = 0, \quad (43)\]

where \( \hat{\nabla} \) denotes the covariant derivative with respect to \( \hat{g}_{\mu\nu} \). By performing the KK decomposition \( h_{\mu\nu}(x, z) = a(z)^{-3/2} \varepsilon_{\mu\nu}(z) H(z) \) with \( \varepsilon_{\mu\nu}(z) \) satisfying the TT condition, we separate the perturbed equation (43) into the four-dimensional and extra-dimensional parts. Here, we can obtain two equations, i.e., \( \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \varepsilon_{\mu\nu}(z) = m^2 \varepsilon_{\mu\nu}(z) \) for the four-dimensional part, and the Schrödinger-like equation for the extra-dimensional sector
\[ m^2 \varepsilon_{\mu\nu}(z) H(z) = m^2 H(z). \quad (44)\]

Here, \( m \) is the mass of the KK mode, and the effective potential is derived as follows:
\[ V_{\text{eff}}(z) = -\frac{9}{4} \Lambda_4 + \frac{3}{4} \left[ \partial_y a(z) \right]^2 + \frac{3}{2} \partial_y \partial_y a(z), \quad (45)\]

for dS_4 geometry. The effective potential can also be transformed in terms of \( y \) coordinate,
\[ V_{\text{eff}}(z(y)) = -\frac{9}{4} \Lambda_4 + \frac{9}{4} \left[ \partial_y a(y) \right]^2 + \frac{3}{2} a(y) \partial_y a(y). \quad (46)\]

At the boundaries of the brane, the potential \( V_{\text{eff}}(z) \) tends to be negative for \( \Lambda_4 > 0 \). For the term \( -\frac{9}{4} \Lambda_4 < 0 \), this equation cannot be factorized. This result indicates that the tensor perturbation of dS_4 brane would not occur stably. For \( \Lambda_4 < 0 \), we obtain AdS_4 geometry, and the potential \( V_{\text{eff}}(z) \) tends to be positive at \( z \to \pm \infty \). Eq. (44) can be written as a factorizable equation, \( \mathcal{KK}'H(z) = m^2 H(z) \) with

![Figure 5](image-url)

Figure 5: The influence of \( \hat{b} \) on the dimensionless effective potential \( V_{\text{eff}}(\hat{y})k^2 \) (blue lines) and the zero-mode \( H_0(\hat{y}) \) (red lines) of the tensor perturbation for \( \Lambda_4 \) brane. (a) \( \hat{b} = 0.5 \), (b) \( \hat{b} = 3 \), (c) \( \hat{b} = 8 \).
which ensures the stability of the tensor perturbation.

4 Massive resonant modes

For the volcano-like effective potentials, the tensor perturbation has zero-mode and may also have resonant modes. Resonant frequencies further investigation of the metastable modes is necessary. Because the integral $z = \int_{m(y)}^{1} dy$ is difficult, we cannot obtain the analytical expressions of the warp factor $a(z)$ and the effective potential $V_{\text{eff}}(z)$. To solve the Schrödinger-like equation (38) for $H(z)$ numerically, we decompose $H(z)$ into an even parity mode and an odd parity mode, which are set to satisfy the following boundary conditions:

\begin{align}
H_{\text{even}}(0) &= 1, \quad \partial_z H_{\text{even}}(0) = 0; \\
H_{\text{odd}}(0) &= 0, \quad \partial_z H_{\text{odd}}(0) = 1.
\end{align}

To find the massive resonant states, we use the numerical method given in refs [62–65], where a relative probability was proposed as follows:

\begin{equation}
P = \frac{\int_{-z_c}^{z_{\text{max}}} |H_{\text{eff}}(z)|^2 \, dz}{\int_{-z_c}^{z_{\text{max}}} |H_{\text{eff}}(z)|^2 \, dz}.
\end{equation}

Here, $2z_c$ is about the width of the thick brane, and $z_{\text{max}}$ is set to $z_{\text{max}} = 10z_c$.

4.1 Flat brane

When the wave functions are either even-parity or odd-parity, the Schrödinger-like equation can be solved numerically. The dimensionless effective potential $V_{\text{eff}(z)}k^2$ is expressed in terms of $\bar{z} = kz$. Figure 6 shows the influence of $\tilde{b}$ on the effective potential and the resonant modes of gravity. The relative probability $P$ as a function of $m^2$ is obtained, and only the peak which satisfies $P > 0.1$ represents a resonance mode.

From Figure 6, we see that, with the increasing of the parameter $\tilde{b}$, the effective potential splits from a single-well into a three-well potential, which indicates that there are more resonant KK modes for larger $\tilde{b}$, and this can be confirmed by Figure 6(d)–(f). In Figure 6(d), there are no peaks of the relative probability, which means that

\[\text{Figure 6: The influence of parameter } \tilde{b} \text{ on the effective potential } V_{\text{eff}(z)}k^2 \text{ and the probabilities } P \text{ for the odd-parity (blue dashed lines) and even-parity (red lines) massive KK modes: (a) } \tilde{b} = 0.5, \text{ (b) } \tilde{b} = 3, \text{ (c) } \tilde{b} = 8, \text{ (d) } \tilde{b} = 0.5, \text{ (e) } \tilde{b} = 3, \text{ and (f) } \tilde{b} = 8.\]
there does not exist a resonant mode. In Figure 6(e), there is just one peak of the relative probability corresponding to the even-parity or the odd-parity wave function, and its wave functions with mass square \( m^2 = 0.31482 \) and \( m^2 = 1.96332 \) are plotted in Figure 7, which shows that the resonance is indeed quasi-localized on the sub-brane.

In Figure 6(f), we find that, there are five peaks corresponding to the even-parity or the odd-parity resonant modes that satisfy \( P > 0.1 \), and the corresponding wave functions of the first even-parity and odd-parity modes with mass square \( m^2 = 0.0177 \) and \( m^2 = 0.0176 \) are plotted in Figure 7. The numerical results of the mass spectrum, relative probability, full width at half maximum, and a lifetime of the resonance with \( \tilde{b} = 3, 8 \) are listed in Table 1. The resonances having a large lifetime can be quasi-localized on the brane for a long time. Note that the first even and odd resonance modes are not degenerate.

**Table 1:** The influence of the parameter \( \tilde{b} \) on the mass spectrum \( m_n \), the relative probability \( P \), the width of mass \( \Gamma \), and the lifetime \( \tau \) of the KK resonances for the flat brane (\( \tilde{\Lambda}_a = 0 \))

<table>
<thead>
<tr>
<th>( \tilde{b} )</th>
<th>( \tilde{\Lambda}_a )</th>
<th>( n )</th>
<th>Parity</th>
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<th>( P )</th>
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Figure 7: The wave functions for the first even- and odd-parity modes for the flat brane (\( \tilde{\Lambda}_a = 0 \)) with (a) \( \tilde{b} = 3 \) and (b) \( \tilde{b} = 8 \).
Figure 8: The influence of the parameter $\tilde{\Lambda}_4$ on the effective potential and the probabilities for both the odd-parity (blue dashed lines) and even-parity (red lines) massive KK modes with a fixed parameter $\tilde{b} = 3$. (a) $V_{\text{eff}}(\tilde{z})/k^2$, (b) $\tilde{\Lambda}_4 = -5$, (c) $\tilde{\Lambda}_4 = -8$, and (d) $\tilde{\Lambda}_4 = -10$.

Table 2: The influence of the parameter $\tilde{\Lambda}_4$ on the masses $m_n^2$ of resonances for the KK modes

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Figure 9: The influence of $\tilde{\Lambda}_4$ on the masses of the first even-parity and odd-parity modes with $\tilde{b} = 8$. The black dots are numerical results, the solid lines are the fit functions for the first even-parity (red line) and odd-parity (blue line) modes: (a) Odd-parity and (b) even-parity.
4.2 Bent branes

Because the tensor perturbation of dS\(_4\) brane would not be stable, we now consider AdS\(_4\) brane, which includes two parameters \(\tilde{b}\) and \(\tilde{\Lambda}_4\). Thus, the resonances are more involved and should be discussed specifically.

The effects of the parameters \(\tilde{\Lambda}_4\) on the effective potentials and relative probability are shown in Figure 8. The effective potentials have a multi-well with a minimum inside the brane and satisfy \(\tilde{V}_{\text{eff}}(\tilde{z}) \rightarrow -\frac{2}{3}\tilde{\Lambda}_4\) when \(\tilde{z} \rightarrow \pm\infty\).

The effective potential does not change its shape but shifts up and down with \(\tilde{\Lambda}_4\). Therefore, the number of the resonances, the relative probability \(P\), the width of mass \(\Gamma\), and the lifetime \(\tau\) of the KK resonances do not change with \(\tilde{\Lambda}_4\) for a fixed \(\tilde{b}\). Only the mass spectrum changes with \(\tilde{\Lambda}_4\). The specific values of masses of the resonances with different values of \(\tilde{\Lambda}_4\) (including \(\tilde{\Lambda}_4 = 0\)) are listed in Table 2, from which we obtain that all of the masses of the resonances depend linearly on the parameter \(\tilde{\Lambda}_4\). For instance, the

![Figure 10: The wave functions for the first even-parity and first odd-parity modes with \(\tilde{b} = 8\). (a) \(\tilde{\Lambda}_4 = -8\) and (b) \(\tilde{\Lambda}_4 = -12\).](image)

<table>
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<th>(\tilde{\Lambda}_4)</th>
<th>(n)</th>
<th>Parity</th>
<th>(m_n^2)</th>
<th>(P)</th>
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relations for the masses of the first even-parity and odd-parity modes with $\tilde{b} = 8$ can be expressed as follows:

$$m_{\text{odd}}^2 = 0.0176 - \frac{9}{4} \tilde{\Lambda}_4,$$

$$m_{\text{even}}^2 = 0.0177 - \frac{9}{4} \tilde{\Lambda}_4. \quad (52)$$

We plot the fit functions for masses of the first even-parity and odd-parity modes with different $\tilde{\Lambda}_4$ in Figure 9. The wave functions for the first even-parity and odd-parity modes with different $\tilde{\Lambda}_4$ are plotted in Figure 10, which shows that the wave function does not alter with $\tilde{\Lambda}_4$ when $\tilde{b}$ is fixed.

For AdS$_4$ brane, the potential well also becomes splitting, and the number of resonant modes increases with the parameter $\tilde{b}$. The influence of the parameter $\tilde{b}$ on the effective potential $V_{\text{eff}}(\tilde{\phi}, \tilde{k})^2$ and the probabilities $P$ are similar to the flat brane, so we do not discuss it repeatedly. Here, we list the numerical results for the mass spectrum, the relative probability, the width of mass, and the lifetime of the KK resonances with different $\tilde{b}$ in Table 3.

5 Conclusion

In this article, we investigate the linear tensor perturbation for $M_4$, $dS_4$, and AdS$_4$ branes in two-field mimetic gravity. We apply the reconstruction technique to find a set of thick brane solutions in asymptotical AdS$_5$ space-time, and derive the master equations for linear tensor perturbations under the TT gauge condition. The Schrödinger-like equations for the $M_4$ and AdS$_4$ branes are factorized into a supersymmetric form; therefore, the brane systems are stable against the tensor perturbations, while the dS$_4$ brane is unstable. For the flat and bent branes, the effective potentials of the corresponding Schrödinger-like equation behave as volcano-like or modified-volcano-like potentials, which may allow a localized zero-mode responsible for the four-dimensional Newtonian potential and a series of massive resonant modes. The thick branes have two parameters $\tilde{\Lambda}_4$ and $\tilde{b}$, one for the cosmological constant of the bent brane and the other for the inner structure of the domain wall. We investigate the effect of the two parameters on the thick branes, namely,

- When $\tilde{\Lambda}_4 \rightarrow -\tilde{\Lambda}_4$, the solution of dS$_4$ changes into that of AdS$_4$. In the limit $\tilde{\Lambda}_4 \rightarrow 0$, the bent brane solution is reduced to the flat brane solution. The number of the resonances does not change with $\tilde{\Lambda}_4$; however, the masses of resonant KK modes linearly decrease with the parameter $\tilde{\Lambda}_4$ for a fixed $\tilde{b}$.

Moreover, we would like to point out that the tensor perturbation of the two-field mimetic gravity model is the same as that of the original single-field mimetic theory [52] and GR. Nevertheless, the two mimetic scalar fields can generate different thick branes, leading to new types of effective potential and graviton resonant modes. There are some interesting extensions of this work. The resonant frequencies and spectroscopy of black holes are analyzed in refs [66,67]. The frequencies of resonant mode in braneworld are not discussed in this article, which will be left to our further study.

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References


