Abstract: This study systematically examines the behavior of multi-stacked long Josephson junctions (multi-sLJJs) under various magnetic inductances and a variety of drives. To investigate the localized modes in the multi-sLJJs, the tripled sine-Gordon equation (sGE) with a phase shift formation recognized as \$0-\pi-0\$ junction is analyzed. The amplitude equations are derived through asymptotic analysis. It is revealed that synchronized oscillations occur in the multi-sLJJs under both strong and weak couplings; however, the system experiences exponential decay in the absence of external drives. The stack naturally exhibits damping due to the energy produced by the oscillating mode through radiation, which arises from the higher harmonic excitation introduced into the system. It is observed that when an AC drive is present, the system exhibits synchronous oscillations for both strong and weak inductances but eventually becomes stable. However, for parametric drives under both coupling conditions, the system experiences exponential decay. Our findings suggest that weak coupling renders parametric drives ineffective. Nevertheless, numerical simulations demonstrate that a large driving amplitude induces oscillations in the system and causes it to approach a steady position under strong coupling.

Keywords: multi-stacked long Josephson junctions, asymptotic analysis, multiple scale expansions, tripled sine-Gordon equation, magnetic inductance, amplitude equations

1 Introduction

The Josephson effect is a fundamental concept in superconductivity, which Josephson confirmed both theoretically and experimentally in 1964 [1]. The Josephson junction, which involves two weakly interacting superconductors separated by a thin insulator, explains this phenomenon. This phenomenon reveals that a current can be induced even in the absence of any voltage change due to the phase variation of the wave functions of the two superconductors: the dc Josephson relation, which is proportional to the sinusoidal function of the phase difference between the electrodes, and the ac Josephson relation. It relates the time derivative of the phase difference to the voltage across the barrier, which have been demonstrated by Josephson [1]. The Josephson junctions (JJs) have promising applications, such as in the development of superconducting meta-materials and quantum information technologies [2–6].

The stacked long Josephson junctions (sLJJs) are currently generating significant interest where one junction is positioned directly above the other with a sufficiently thin isolating layer linked to the London penetration depth [7]. It is important to note that the coupling between junctions is uniform in the system. Each junction has one outermost terminal and an additional terminal that connects to another junction. However, it has been observed that such uniformity is disrupted in the case of substantial stacks [8]. Experimental evidence has shown that the most effective way to couple sLJJs is by stacking them vertically. The Josephson vortices moving in adjacent junctions interact with each other through their screening currents flowing in a thin superconducting layer between the junctions. These interactions can be observed in the current–voltage characteristics [9]. Furthermore, two dynamic regimes characterized by different soliton propagation velocities have been found in twofold stacks [10].

Multi-sLJJs represent a fascinating physical system where the nonlinearity and the interaction between junctions play a crucial role and reveal interesting features in...
the field of superconductivity. Compared to single JJs, multi-sLJJs have the potential to produce larger output results with a relatively compact width depending on the extent of combined junctions observed during experiments [10]. They also offer an opportunity to investigate physical outcomes that do not arise in single LJs [11]. Coupling the junctions has been shown to improve the performance of superconducting machines, particularly for the storage and communication of data [12]. Moreover, it has been observed that numerous properties of the LJs cannot be determined without considering the stack configuration, which includes the investigation of certain dynamical changes such as the quantum and classical dynamics, as well as the chaotic behavior of solitons [13–16].

Multi-sLJJs hold great promise for a wide range of applications in various fields, such as superconducting electronics, quantum computing, spin current control, and high-sensitivity terahertz electromagnetic wave detection. These applications have been demonstrated in various studies, e.g., [17–21]. Moreover, multi-sLJJs exhibit non-trivial properties in both low-\(T_c\) and high-\(T_c\) superconductors. For instance, the current locking by Josephson fluxons and the Cherenkov radiation in Nb/AlOx/Nb in low-\(T_c\), and BiSrCaCuO\(_2\) in high-\(T_c\), are some notable examples of stacked junctions reported in some studies [22–25].

2 The governing model

Several models have been employed to examine multi-stacked junctions experimentally using various models [26]. For example, Mineev et al. [27] first explored the fluxon dynamics in inductively sLJJs with multiple layers theoretically. In addition, double junctions have been utilized in single stacks to fabricate high-\(T_c\) superconductors as shown in the study by Benz et al. [12]. Likewise, coupled sine-Gordon equations (sGEs) have been considered to investigate the behavior of inductively sLJJs with different drives [28]. To study the effect of coupling in multi-sLJJs, we consider the nonlinear tripled sGE

\[
u_{xx}^l - u_{it}^l = \sin(u^l + \Theta) + S u_{xx}^{(2,1,2)} + hG(u^l, \omega(t), \Theta(x)). \tag{1}
\]

In Eq. (1), \(u(x, t)\) refers to the phase difference between the wave functions that identify the superconducting condensate in superconductors, \(S\) stands for the strength of the magnetic induction between the junctions, \(h\) is an amplitude of oscillations, and \(G\) represents an applied external drive.

The investigation of integrating the sGE complexity has a comprehensive variety of physical and mathematical utilizations in LJs [2,29], e.g., an ultra-short vibration transmission in a resonant medium, dynamics in weak ferromagnets [30], the dynamics of solitons and electro-acoustic interconnections in ferroelectric crystals [31], and quantum field theory of solitons [32]. Furthermore, the sGE also has very remarkable applications in many areas, in particular, the transport of information, the impact of solitary waves in the damped driven sG system [33], and continuous breather excitations, phase-pulling, and space-time complexity in an \(AC\)-driven sGE [34]. Similarly, coupled sGEs have been widely studied previously for different couplings to study a variety of physical phenomena [35], in particular, for investigations of non-equilibrium phenomena in superconductors, including voltage as well as phase-locking, and for demonstrations of superconducting devices in sLJJs [36].

The phase shift considered in Eq. (1)

\[
\Theta(x) = \begin{cases} 0, & |x| > b, \\ \pi, & |x| \leq b, \end{cases}
\]

was first proposed by Bulaevskii et al. [37] where a non-trivial ground state was realized by an unconfined fractional fluxon generation. It was also proposed that phase shift \(\pi\) may occur because of magnetic impurities in the system. There are numerous novelties of \(0 - \pi\) LJs (see [38,39] and the references therein). To solve the system of tripled sGE (Eq. (1)), we also consider the following continuity conditions [28]:

\[
\begin{align*}
    u(\pm b^+ (\Omega), 0) &= u(\pm b^- (\Omega), 0), \\
    u_t(\pm b^+ (\Omega), 0) &= u_t(\pm b^- (\Omega), 0),
\end{align*}
\]

where \(0 \leq b \leq \pi/4\).

The structure of the article is as follows: Section 3 presents the computation of the nonlinear amplitude equations for investigating defect modes in multi-layered LJs analytically. This section involves the analytical solution of the tripled sGE under small-scale integration with parametric \(AC\) drives. Furthermore, we examine coupling without drives by utilizing an asymptotic expansion combined with multiple scale analysis. In Section 4, we apply a similar analytical approach as in Section 3, but focusing on strong coupling. Section 5 compares the derived approximate amplitude equations with the corresponding numerical simulations of Eq. (1). Finally, Section 6 provides a summary and conclusion of our study with a future work.

3 Weakly coupled multi-sLJJs

This section employs analytical techniques to investigate multi-layered LJs using Eq. (1) under the influence of magnetic inductance with a phase shift. We restrict our
analysis to the state where the system’s natural frequency is nearly identical to the driving frequency, i.e., $\omega = \Omega$.

### 3.1 Undriven multi-sLJJs

When $G = 0$ with $S = O(\varepsilon^2)$, we obtain Eq. (1) in the form

$$u_{\varepsilon}^j - u_j^1 = \sin(u_j^1 + \Theta) + \varepsilon^2 u_{\varepsilon}^{1,2,j}$$

and $j = 1, 2, 3$. (4)

Introducing the asymptotic expansions $u_j^1 = \sum_i \varepsilon^n u_i^j$ together with $T_0 = e^{it}$ and $X_0 = e^{ix}$, we obtain a hierarchy of equations at each order of $\varepsilon$.

- Equations at $O(1)$: $\partial^2_\varepsilon u_0^j - D_0^2 u_0^j = \sin(u_0^j + \Theta)$. (5)

For simplicity, we consider $u_0^j = 0$, i.e., a uniform stable state solution at $O(\varepsilon^0)$.

- Equations at $O(\varepsilon)$: $\partial^2_\varepsilon u_1^j - D_0^2 u_1^j - \cos(u_0^j + \Theta)u_0^j$

$$= 2D_0 \partial_\varepsilon u_0^j + 2D_0 \partial_\varepsilon u_1^j.$$ (6)

Linearizing the above equation by $u_1^j = \mu_1^j e^{i\Omega t} + \mu_2^j e^{-i\Omega t}$ and continuity conditions stated in Eq. (3), we obtain

$$u_1^j = \mu_1^j (X_0) e^{i\Omega t} + \mu_2^j (X_0) e^{-i\Omega t},$$

where $A_j = A_j(X_1, ..., T_1, ...)$. (7)

$$\mu_j^1 (X_0) = \begin{cases} 
\cos(b\sqrt{1 + \Omega^2}) e^{i\Omega \sqrt{1 + \Omega^2}} & \text{if } X_0 < b, \\
\cos(X_0 \sqrt{1 + \Omega^2}) & \text{if } |X_0| \leq b, \\
\cos(b\sqrt{1 + \Omega^2}) e^{-i\Omega \sqrt{1 + \Omega^2}} & \text{if } X_0 > -b, 
\end{cases}$$ (8)

where $b = b(\Omega)$ is given as follows [28]:

$$b(\Omega) = \frac{1}{\sqrt{1 + \Omega^2}} \arctan \left( \frac{1 - \Omega^2}{1 + \Omega^2} \right), \quad \Omega^2 \ll 1.$$ (9)

It has been demonstrated that the system becomes unstable for $b_0 > \pi/4$.

In order to perform the stability analysis of the model under consideration subject to the additional phase shift given in Eq. (2), one needs to consider the static version of the sGE. It is straightforward to show that the given model admits two static solutions, namely $u_0 = 0$, $\pi \mod (2\pi)$.

To proceed further, we insert the ansatz $u = u_0 + e^{it}V(x)$, where $\varepsilon \ll 1$ and $\Omega \in \mathbb{C}$, into our given model. After a formal series of expansion and neglecting smaller terms in $\varepsilon$, one obtains the eigenvalue problem in the form of nonlinear Schrödinger equation as follows:

$$V_{xx} = [\Omega^2 + \cos(u_0 + \Theta(x))]V.$$ (10)

A uniform solution $u_0$ of the model under consideration is stable if and only if $\Omega^2 < 0$ and is unstable if $\Omega^2 > 0$. For the stability of the uniform solutions, one should consider the bounded solutions of Eq. (10) in the inner region and bounded and decaying to zero solutions in the outer regions. Following the procedure studied in detail in by Khan et al. [41] and Ahmad [40], we see that the uniform zero solution is stable when the facet length $b(\Omega)$ assumes values less than $\pi/4 = 0.4$. The uniform zero solution becomes unstable whenever the facet length takes values greater than $\pi/4$ [40,41].

For the uniform $u_0 = \pi$, the localized mode is given by the bounded solutions in all three regions. This implies that the localized mode of the uniform $\pi$ solution lies in the region where $\Omega^2 < 1$, or equivalently in the regions $\mu > 1$ and $\mu < 1$, where $\Omega = |\mu|, \mu \in \mathbb{R}$, and $i$ is the imaginary unit. This indicates that for some $\Omega, \Re(\Omega) \neq 0$. Hence, there exists a region in the localized spectrum of the constant $\pi$ solution where the eigenvalue $\Omega$ is not purely imaginary. Therefore, the stability ansatz implies that $u_0 = \pi$ is unstable [40,41].

- Equations at $O(\varepsilon^2)$: $\partial^2_\varepsilon u_2^j - D_0^2 u_2^j - \cos(\Theta)u_0^j$

$$= 2D_0 \partial_\varepsilon u_0^j + 2D_0 \partial_\varepsilon u_1^j + \frac{1}{2} \sin(u_0^j + \Theta)u_{1,2}^j.$$ (11)

Using the known functions $u_1^j$ and $u_0^j$, Eq. (11) can be obtained in the following form:

$$\partial^2_\varepsilon u_2^j - D_0^2 u_2^j - \cos(\Theta)u_0^j$$

$$= \begin{cases} 
2i\Omega \mu_1^j / \mu_1^j A_j \mp 2\sqrt{1 - \Omega^2} \mu_1^j / \partial_0 A_j & |X_0| > b, \\
2i\Omega \mu_1^j / \mu_1^j A_j - 2\sqrt{1 + \Omega^2} \partial_0 \mu_1^j / \partial_0 A_j & |X_0| \leq b. 
\end{cases}$$ (12)

In order to obtain bounded solutions for $u_2^j$, we utilize the Fredholm condition, which yields an equation of the following form [42]:

$$L u_2^j = G(X_0).$$ (13)

From Eq. (13), one can see that the left hand side is the second order differential operator which is self-orthogonal. The Fredholm theorem states that the necessary and sufficient conditions for Eq. (13) to have a bounded solution if the right-hand side be orthogonal to the complete system of linearly independent solutions of the corresponding homogeneous equation. Hence, we conclude that the above system has a bounded solution if and only if

$$\int_{-\infty}^{\infty} L G(X_0) dx = 0.$$ (14)

Applying the above condition, we obtain

$$D_0 A_j = 0.$$ (15)

It should be noted that our focus is to find the localized modes for the amplitude $A_j = A_j(T_1, T_2, ...)$, and therefore,
we consider $\partial_2 A_j = 0$ and $\partial_\nu A_j = 0$ in further computations. This conjecture results in Eq. (12), which is similar to the one obtained at $O(\epsilon)$. Therefore, we assume that $u_2^j = 0$.

- Equations at $O(\epsilon^3)$:
  \[
  \begin{align*}
  \partial_2^2 u_{(1)}^j - D_2^2 u_{(1)}^j - \cos(\theta + u_0^j)u_{(1)}^j & = 2(D_2D_2 - \partial_0^2)u_{(1)}^j + (D_2^2 - \partial_2^2)u_{(1)}^j - \frac{1}{6} u_{(1)}^j \cos(\theta + u_0^j) + \partial_2^2 u_{(2,1,2)}^j.
  \end{align*}
  \] (16)

Using the known values of $u_0^j$ and $u_2^j$, Eq. (16) implies that

\[
\begin{align*}
\partial_2^2 u_{(1)}^j - D_2^2 u_{(1)}^j - \cos(\theta)u_{(1)}^j &= \left\{ \begin{array}{ll}
F_{1,j} & |X_0| > b, \\
F_{2} & |X_0| \leq b,
\end{array} \right.
\end{align*}
\] (17)

where $F_{1,s}$ are given as follows:

\[
F_{1,j} = (2i\Omega D_2 A_j + A_{(1,2,1)}(1 - \Omega^2))u_{(1)}^j e^{i\Omega t} - \frac{1}{2} A_j^2 |A_j|^2 e^{i\Omega t} + \frac{1}{6} A_j^4 e^{3i\Omega t} u_{(1)}^j,
\]

\[
F_{2} = (2i\Omega D_2 A_j - A_{(1,2,1)}(1 + \Omega^2))u_{(1)}^j e^{i\Omega t} + \frac{1}{2} A_j^2 |A_j|^2 e^{\Omega t} + \frac{1}{6} A_j^4 e^{3\Omega t} u_{(1)}^j.
\]

The relationships mentioned above comprise two subharmonics. We obtain equations for the first harmonics by separating components for each harmonic

\[
\begin{align*}
\partial_2^2 u_{(1)}^j - D_2^2 u_{(1)}^j - \cos(\theta)u_{(1)}^j &= \left\{ \begin{array}{ll}
\left(2i\Omega D_2 A_j + A_{(1,2,1)}(1 - \Omega^2)\mu_{(1)}^j e^{i\Omega t} - \frac{1}{2} A_j^2 |A_j|^2 \mu_{(1)}^j + \frac{1}{6} A_j^4 e^{3i\Omega t} \mu_{(1)}^j & |X_0| > b, \\
\left(2i\Omega D_2 A_j - A_{(1,2,1)}(1 + \Omega^2)\mu_{(1)}^j e^{i\Omega t} + \frac{1}{2} A_j^2 |A_j|^2 \mu_{(1)}^j + \frac{1}{6} A_j^4 e^{3\Omega t} \mu_{(1)}^j & |X_0| \leq b.
\end{array} \right.
\end{align*}
\]

The solvability condition for the above equation is

\[
D_2 A_j = \zeta A_j |A_j|^4 i - \zeta A_{(1,2,1)} i.
\] (18)

where

\[
\zeta_1 = \frac{1}{4\Omega}, \quad \zeta_2 = \frac{1 - \Omega^2}{2\Omega}.
\]

For high harmonic oscillations, we consider the conditions $(3\Omega)^2 > 1$ to switch the system to a phonon band. This leads the system to oscillate. The bounded solutions of Eq. (17) for other harmonics are calculated as they will appear in further calculations when we go to higher orders. The bounded solutions of Eq. (17) for further harmonics are determined as they will appear in further calculations when we go to higher orders. For the sake of convenience, we do not add them in the article.

- Equations at $O(\epsilon^4)$:
  \[
  \begin{align*}
  \partial_2^2 u_{(1)}^j - D_2^2 u_{(1)}^j - \cos(u_0^j + \theta)u_{(1)}^j &= 2(D_2D_2 - \partial_0^2)u_{(1)}^j + (D_2^2 - \partial_2^2)u_{(1)}^j + \frac{1}{12} u_{(1)}^j \cos(\theta + u_0^j) + \partial_2^2 u_{(2,1,2)}^j.
  \end{align*}
  \] (19)

which gives the solvability condition as follows:

\[
D_2 A_j = 0.
\] (20)

Equations (19) and (20) give same result as obtained at $O(\epsilon^2)$. Therefore, we deduce that $u_2^j = 0$.

- Equations at $O(\epsilon^5)$:
  \[
  \begin{align*}
  \partial_2^2 u_{(1)}^j - D_2^2 u_{(1)}^j - \cos(\theta + u_0^j)u_{(1)}^j &= 2(D_2D_2 - \partial_0^2)u_{(1)}^j + (D_2^2 - \partial_2^2)u_{(1)}^j + \left[ (D_2^2 - \partial_2^2) + 2(D_0D_2 - \partial_0\partial_2) \right] u_{(1)}^j \\
  & + \left[ \frac{1}{120} u_{(1)}^j - \frac{1}{12} u_{(1)}^j \right] \cos(\theta + u_0^j) + \partial_2^2 u_{(2,1,2)}^j.
  \end{align*}
  \] (21)

We consider the relations for $A_j$ only in the above equations. Applying the Fredholm theorem, we obtain

\[
\begin{align*}
D_2 A_j &= a_2 A_j |A_j|^4 + a_2 A_{(1,2,1)} |A_{(1,2,1)}|^2 i \\
& + a_2 A_{(1,2,1)} |A_{(1,2,1)}|^2 i + a_2 A_{(1,2,1)} A_{(1,2,1)} i + a_2 A_{(1,2,1)} i,
\end{align*}
\] (22)

where the values of $a_2$’s are presented in Section 5. Since $a_1 = -0.00325 \pm 0.07216i \in \mathbb{C}$ in Eq. (22) is not purely imaginary, localized modes emerge at this point. Therefore, we opt not to assume the other terms at $O(\epsilon^5)$.

By combining and rescaling each of the solvability conditions, we obtain the amplitude equations as follows:

\[
\frac{d a_j}{d t} = a_j |a_j|^4 + \zeta a_j |a_j|^2 i + S(a_a a_{(1,2,1)} a_{(1,2,1)} i + a_a A_{(1,2,1)} A_{(1,2,1)} i + a_a A_{(1,2,1)} i).
\] (23)

The synchronized oscillations can be observed in the tripled amplitude equations of multi-sLJJs with weak coupling. The gradual decrease in the amplitudes of oscillations in Eq. (23) without drives is caused by the numerical value $a_1 = -0.00325 - 0.07216i$, where Re($a_1$) < 0 leads to a decay rate of $O(t^{-1/4})$. Furthermore, Eq. (23) displays synchronization in multi-sLJJs with similar amplitudes.

### 3.2 AC-driven multi-sLJJs

In this section, we focus on Eq. (1) with a direct AC drive where $G = \cos(\omega t)$. We employ analytical methods using the scaling $\Omega(1 + \kappa) = \omega$ such that $\kappa$ is a fitting variable. In
addition, we redefine $h = \varepsilon^2 H$ and $\kappa = \varepsilon^3 R$ and express $\omega t$ as $\Omega t$ to simplify Eq. (1) as follows:

$$u_{tt}^j - (1 + \kappa)^2 u_{xx}^j = \sin(u^j + \Theta(x)) + S^2 u_{xx}^{(2,1,2)} + h \cos(\Omega t), \quad j = 1, 2, 3. \quad (24)$$

The ground state for the aforementioned equations can be calculated using methods similar to those used in the Section 3.1 and is comparable to that found at Section 3.1. For simplicity and brevity, we limit our analysis to the solvability conditions derived from the hierarchy of equations at different order of $\varepsilon$ as follows:

- $O(\varepsilon^3) : D_2A_j = \zeta A_j |A_j|^2 i - \zeta A_{(2,1,2)} |A_j|^2 i + \phi H_i$, \quad (25)
- $O(\varepsilon^4) : D_3A_j = -\Omega A_j R_i$, \quad (26)
- $O(\varepsilon^5) : D_4A_j = \beta A_j |A_j|^4 + \beta A_j |A_j|^2 i + \beta A_{(2,1,2)} |A_j|^2 i + \beta A_{(2,1,2)}^2 i + \beta A_{(2,1,2)} A_j^2 i + \beta A_j^2 \bar{A}_{(2,1,2)} i + \beta A_j^2 \bar{A}_{(2,1,2)}^2 i + \beta A_j H_i + \beta A_j i$, \quad (27)

wherein the numerical values indicated in the aforementioned solvability conditions.

Combining Eqs. (25)-(27), we obtain

$$\frac{da_j}{dt} = \gamma a_j |a_j|^4 + \zeta a_j |a_j|^2 i - \Omega a_j \kappa i + (\beta a_j^2 + \beta |a_j|^2 i + \phi) H_i + S^2 a_j i + S(\gamma a_j |a_j|^2 i + \zeta a_{(2,1,2)}^2 i + \beta a_{(2,1,2)} |a_j|^2 i + \beta a_{(2,1,2)} a_j^2 i + \beta a_j^2 a_{(2,1,2)}^2 i + \beta a_j h - \zeta a_{(2,1,2)} i). \quad (28)$$

In the occurrence of a direct AC drive, the amplitude equations (Eq. (28)) indicate that the applied drive with an amplitude of $O(h)$ and the additional terms described in Eq. (28) obtained at $O(\varepsilon^2)$ supply the system with extra energy. As the transients disappear, the system continues to oscillate with constant amplitude for an extended period because the generated energy balances out the oscillation decay caused by radiative damping.

### 3.3 Parametrically driven multi-sLJJs

Here, we examine multi-sLJJs with parametric driven. Applying the previous scaling to the considered model, we obtain

$$u_{tt}^j - (1 + \kappa)^2 u_{xx}^j = \sin(u^j + \Theta(x)) + S^2 u_{xx}^{(2,1,2)} + h \sin(u^j + \Theta) \cos(\Omega t), \quad j = 1, 2, 3. \quad (29)$$

We restrict our analysis to the solvability conditions derived from the series of equations, without investigating into the details of the calculations. It is important to note that the conclusions drawn in Section 3.1 remain unchanged till $O(\varepsilon^2)$. The conditions for the continuing orders are outlined as follows:

- $O(\varepsilon^3) : D_2A_j = \zeta A_j |A_j|^2 i - \zeta A_{(2,1,2)} |A_j|^2 i$, \quad (30)
- $O(\varepsilon^4) : D_3A_j = -\Omega A_j R_i$. \quad (31)
- $O(\varepsilon^5) : D_4A_j = \gamma A_j |A_j|^4 + \gamma A_{(2,1,2)} |A_{(2,1,2)}|^2 i + \gamma A_{(2,1,2)}^2 A_j i + \gamma A_j \bar{A}_{(2,1,2)} i - (\gamma \bar{A}_{(2,1,2)} - \gamma A_j) H + \gamma A_j i$. \quad (32)

The numerical values can be found in the corresponding section dedicated to numerical analysis.

Combining Eqs. (30)-(32), we obtain

$$\frac{da_j}{dt} = \gamma a_j |a_j|^4 + \zeta a_j |a_j|^2 i - \Omega a_j \kappa i + S(\gamma a_j |a_j|^2 i + \zeta a_{(2,1,2)}^2 i + \beta a_{(2,1,2)} |a_j|^2 i + \beta a_{(2,1,2)} a_j^2 i + \beta a_j^2 a_{(2,1,2)}^2 i + \beta a_j h - \zeta a_{(2,1,2)} i) \quad (33)$$

In the case of applying a parametric drive, the effect of driving in Eq. (33) is relatively smaller, as the term $O(h^2)$ appears along with some additional terms. As a result, the system behaves similarly to when there is no drive. Therefore, we conclude that in multi-sLJJs with phase shift and weak coupling, the parametric drive has no significant impact on the system. It should be noted that synchronized oscillations are obtained when $\kappa = 0$ in the presence of drives.

### 4 Strongly coupled multi-sLJJs

In this part, we examine the dynamics of multi-sLJJs when $S \approx O(1)$ with phase shift. Here, we will analyze the system in the absence and presence of a variety of drives.

#### 4.1 Undriven multi-sLJJs

Here, we consider multi-sLJJs governed by Eq. (1) without drives as when $G = 0$. Using the previous scaling, we can obtain

$$u_{tt}^j - u_{xx}^j = \sin(u^j + \Theta) + S u_{xx}^{(2,1,2)}, \quad j = 1, 2, 3. \quad (34)$$
As stated in Section 3.1, we establish a similar conclusion at $O(1)$.

- Equations at $O(\varepsilon)$: $\partial^2_\omega u^I_0 - D^2_\omega u^I_0 = \cos(u^I_0 + \Theta)u^I_0 + S\partial^2_\omega u^{(2,1,2)}_0$. (35)

To normalize the above equation, we assume that $u^I_0$ (phase difference) are either odd or even, i.e.,

$$u^I_0 = u^I_1, \quad \text{or} \quad u^I_0 = -u^I_1,$$

and

$$u^I_0^2 = u^I_2, \quad \text{or} \quad u^I_0^2 = -u^I_2.$$ (36)

The above two assumptions (Eqs. (36) and (37)) have the same physical behavior. For simplicity and exactness, we consider Eq. (36) for odd and even parity and obtain

$$(1 \mp S)\partial^2_\omega u^I_1 - D^2_\omega u^I_1 = \cos(u^I_1 + \Theta)u^I_1.$$

Consider ansatz $u^I_1 = U(J_0)e^{\Theta J_0} + U((X)_0)e^{-\Theta J_0}$ with continuity conditions stated in Eq. (3), we have

$$u^I_1 = A_1(J_0, ..., J_1, ..., J_n)U(X_0)e^{\Theta J_0} + A_2(J_0, ..., J_1, ..., J_n)U(X_0)e^{-\Theta J_0},$$

with

$$U_1 = \begin{cases} \cos \left( \sqrt{1 + \Omega^2 J_0} \right) |X_0| > b, \\ \cos \left( X_0 \sqrt{1 + \Omega^2 J_0} \right) |X_0| \leq b. \end{cases}$$ (40)

Express the solutions for the ground state by assuming either an odd or even parity as follows:

$$u^I_1(X_0, J_0) = A_1U_0 e^{\Theta J_0} + A_2U_0 e^{-\Theta J_0},$$

$$u^I_0^2(X_0, J_0) = A_1U_0 e^{\Theta J_0} + A_2U_0 e^{-\Theta J_0},$$

where $\Theta$ can be derived from Eq. (40) by using continuity conditions stated in Eq. (3) and

$$b = \frac{1 + \Omega^2}{1 + \sqrt{1 + \Omega^2}} \arctan \left( \sqrt{1 + \Omega^2} \right), \quad \Omega^2 \ll 1,$$ (41)

with $\Omega = \Omega_3 \ll \Omega_2, 0 \leq \theta(\Omega) \leq \pi/4$, and $|S| < 1$. The value of $b$ corresponds to each of $\Omega_2 \rightarrow 0$, namely

$$b_{\text{eff}} = \frac{\pi}{4} \sqrt{1 + \Omega^2}. \quad (42)$$

- Equations at $O(\varepsilon^2)$:

$$\partial^2_\omega u^I_0 - D^2_\omega u^I_0 - \cos(u^I_0 + \Theta)u^I_0 = 2D_0D_1 - \partial_\omega \partial_\Theta^2 u^I_0 + S\partial^2_\omega u^{(2,1,2)}_0.$$ (43)

The equations can be written as

$$(1 \mp S)\partial^2_\omega u^I_0 - D^2_\omega u^I_0 - \cos(u^I_0 + \Theta)u^I_0 = 2(D_0D_1 - \partial_\omega \partial_\Theta^2 u^I_0).$$ (44)

By applying the condition stated in Eq. (14) to obtain solution of $u^I_0$, we have

$$D_1 A_1 = 0.$$ (45)

By Eq. (48), we find results similar to that as $O(\varepsilon)$. Hence, we deduce that $u^I_0 = 0$.

- Equations at $O(\varepsilon^3)$:

$$\partial^2_\omega u^I_0 - D^2_\omega u^I_0 - \cos(u^I_0 + \Theta)u^I_0 = 2(D_0D_2 - \partial_\omega \partial_\Theta^2 u^I_0) + \left( \frac{1}{6} \cos(u^I_0 + \Theta)u^I_0 + S\partial^2_\omega u^{(2,1,2)}_0 \right).$$ (46)

Here, we obtain the solvability condition for even parity as

$$D_1 A_1 = \eta_1 A_1 |A_1|^2 \text{ and } D_1 A_1 = \eta_2 A_1 |A_2|^2 \text{ for } \Omega = \Omega_3 \ll \Omega_1.$$ (47)

and for odd parity, we have

$$D_1 A_1 = \psi_1 A_1 |A_1|^2 \text{ and } D_1 A_1 = \psi_2 A_1 |A_2|^2 \text{ for } \Omega = \Omega_3 \ll \Omega_1.$$ (48)

The numerical values are presented in Section 5.

For continuous spectrum, we take $(3\Omega)^2 > 1$, which gives that $u^{(2,1)}_0 + c.c., u^{(2,2)}_0 + c.c.,$ and $u^I + c.c.$, signifying right and left radiations for $|X_0| > b$.

Moreover, the hypothesis $\Omega_2 = \Omega_3 > \Omega_1$ and the other harmonics $2\Omega_2 + \Omega_1, 2\Omega_2 + \Omega_2,$ and $2\Omega_2 + \Omega_3$ also oscillate for the continuous wave oscillation in multi-SLJJs. For the sake of simplicity, we omit these solutions.

- Equations at $O(\varepsilon^4)$:

$$\partial^2_\omega u^I_0 - D^2_\omega u^I_0 - \cos(u^I_0 + \Theta)u^I_0 = 2(D_0D_2 + D_0D_3 - \partial_\omega \partial_\Theta^2 u^I_0 + \left( \frac{1}{24} |u^I_0|^4 - u^I_0 \right) \sin(u^I_0) + \Theta) + S\partial^2_\omega u^{(2,1,2)}_0.$$ (49)

Using assumption considered in Eq. (36), it becomes

$$(1 \mp S)\partial^2_\omega u^I_0 - D^2_\omega u^I_0 - \cos(u^I_0 + \Theta)u^I_0 = 2(D_0D_2 - \partial_\omega \partial_\Theta^2 u^I_0).$$ (50)

Equation (53) gives solvability condition as follows:

$$D_2 A_2 = 0.$$ (51)

Substituting the above values in Eq. (53), we can infer that $u^I_0$ equals to zero, with the same conclusion as that obtained at $O(\varepsilon^3)$. 

(52)
In this section, we examine multi-sLJJs using Eq. (1) when \( G = \cos(\omega t) \). Considering \( S = \mathcal{O}(1) \) and applying the scaling considered in Section 3.2, with \( \omega_j = (1 + \kappa)\Omega_j \), then Eq. (1) becomes

\[
\dot{u}_j^{(k)} - (1 + \kappa)^2 u^{(k)}_j = \sin(u^{(k)}_j + \Theta(x)) + S u^{(2,1,2)}_j + h \cos(\Omega_j \tau), \quad j = 1, 2, 3.
\]

### 4.2 AC-driven multi-sLJJs

In this section, we examine multi-sLJJs using Eq. (1) when \( G = \cos(\omega t) \). Considering \( S = \mathcal{O}(1) \) and applying the scaling considered in Section 3.2, with \( \omega_j = (1 + \kappa)\Omega_j \), then Eq. (1) becomes

\[
\dot{u}_j^{(k)} - (1 + \kappa)^2 u^{(k)}_j = \sin(u^{(k)}_j + \Theta(x)) + S u^{(2,1,2)}_j + h \cos(\Omega_j \tau), \quad j = 1, 2, 3.
\]

For even parity,

\[
\mathcal{O}(e^5) : D_{2}A_j = \eta_2 A_j |A_j|^2 i + \eta_2 A_j |A_{2,1,2}|^2 i + \delta_j \Omega_j H_i, \quad (61)
\]

\[
\mathcal{O}(e^4) : D_{2}A_j = -\Omega_j A_j R_i, \quad (62)
\]

\[
\mathcal{O}(e^5) : D_{2}A_j = m_2 A_j |A_j|^4 + m_2 A_j |A_{2,1,2}|^4 + m_3 A_j |A_{2,1,2}|^2 |A_j|^2 + \delta_j \Omega_j H_i. \quad (63)
\]

Likewise, for odd parity,

\[
\mathcal{O}(e^3) : D_{2}A_j = \psi_2 A_j |A_j|^2 i + \psi_2 A_j |A_{2,1,2}|^2 i, \quad (64)
\]

\[
\mathcal{O}(e^4) : D_{2}A_j = \Omega_j A_j R_i, \quad (65)
\]

\[
\mathcal{O}(e^5) : D_{2}A_j = l_4 A_j |A_j|^4 + l_2 A_j |A_{2,1,2}|^4 + l_3 A_j |A_{2,1,2}|^2 |A_j|^2. \quad (66)
\]

Combining the Eqs. (61)–(63) for the even parity and using the original scaling, we obtain the amplitude equation as follows:

\[
\frac{d a_j}{d t} = -\frac{\Omega_j}{\mathcal{O}(e^5)} [\eta_2 A_j |a_j|^2 i + \eta_2 A_j |a_{2,1,2}|^2 i + \delta_j \Omega_j H_i - \Omega_j a_j i + m_2 A_j |a_j|^4 + m_2 A_j |a_{2,1,2}|^4 + m_3 A_j |a_{2,1,2}|^2 |a_j|^2 + h(4|a_j|^2 + m_3 |a_{2,1,2}|^2 + m_3 |a_{2,1,2}|^2)]. \quad (67)
\]

For odd parity, we combine Eqs. (64)–(66) and obtain

\[
\frac{d a_j}{d t} = \frac{\Omega_j}{\mathcal{O}(e^5)} [\psi_2 A_j |a_j|^2 i + \psi_2 A_j |a_{2,1,2}|^2 i - \Omega_j a_j i + l_2 A_j |a_j|^4 + l_2 A_j |a_{2,1,2}|^4 + l_3 A_j |a_{2,1,2}|^2 |a_j|^2]. \quad (68)
\]

When a direct AC drive is applied, the nonlinear amplitudes experience fast oscillations because of the driving effect. It is anticipated that in multi-sLJJs, an external drive will cause oscillations in the breathing mode, with the phase shift being determined by Eq. (67).

### 4.3 Multi-sLJJs subjected to parametric driving

In this section, we examine multi-sLJJs using Eq. (1) with parametric drive and strong coupling. Applying the same scaling as considered in Section 4.2. Eq. (1) becomes
\[ u_{xx}^{i} - (1 + \kappa)^2 u_{tt}^{i} = \sin(u^i + \Theta) + S u_{xx}^{(2,1,2)} + \h \sin(u^i + \Theta) \cos(\Omega \tau), \quad j = 1, 2, 3. \]  

(69)

Similar to previous section, the amplitude equations for even parity are

- \( O(\epsilon^3) \): \( D_2 A_j = \eta_1 |A_j|^2 i + \eta_2 |A_{(2,1,2)}|^2 i, \)  
- \( O(\epsilon^4) \): \( D_3 A_j = -\Omega A_j R_i, \)  
- \( O(\epsilon^5) \): \( D_4 A_j = n_2 |A_j|^4 + n_4 |A_{(2,1,2)}|^4 \)  
  \[ + n_4 |A_j|^2 |A_{(2,1,2)}|^2 + n_4 A_{(2,1,2)} A_{(2,1,2)} \]  
  \[ + n_4 H^2 A_j + n_4 H^2 A_j. \]  

(70)  

(71)  

(72)

Amplitude equations for odd parity are

- \( O(\epsilon^3) \): \( D_2 A_j = \psi_1 |A_j|^2 i + \psi_2 |A_{(2,1,2)}|^2 i, \)  
- \( O(\epsilon^4) \): \( D_3 A_j = -\Omega A_j R_i, \)  
- \( O(\epsilon^5) \): \( D_4 A_j = k_2 |A_j|^4 + k_4 |A_{(2,1,2)}|^4 \)  
  \[ + k_2 |A_j|^2 |A_{(2,1,2)}|^2 + k_4 A_{(2,1,2)} A_{(2,1,2)} \]  
  \[ + k_4 H^2 A_j. \]  

(73)  

(74)  

(75)

By combining Eqs. (70)–(72) for the even parity and employing the initial scaling, we derive

\[
\frac{\Omega_j}{\Omega} \frac{da_j}{dt} = \eta_1 |a_j|^2 i + \eta_2 |a_{(2,1,2)}|^2 i - \Omega_1 a_j \kappa i + n_2 |a_j|^4 + n_3 |a_{(2,1,2)}|^4 \]  
\[ + n_4 |a_j|^3 |a_{(2,1,2)}|^2 \]  
\[ + n_4 a_j^3 a_{(2,1,2)}^2 + n_4 H^2 a_j + n_4 H^2 a_j. \]  

(76)

Similarly, for odd parity, combining Eqs. (73)–(75), we define

\[
\frac{\Omega_j}{\Omega} \frac{da_j}{dt} = \psi_1 |a_j|^2 i + \psi_2 |a_{(2,1,2)}|^2 i - \Omega_1 a_j \kappa i + k_2 |a_j|^4 \]  
\[ + k_2 |a_j|^3 |a_{(2,1,2)}|^2 \]  
\[ + k_2 a_j^3 a_{(2,1,2)}^2 + k_2 H^2 a_j. \]  

(77)

For small driving amplitudes, the resulting oscillation amplitudes decay exponentially in response to parametric driving. As the driving amplitude in Eq. (76) is \( O(h^2) \), which is significantly small, the governing system does not reach equilibrium and displays behavior analogous to that seen in the absence of drives. This suggests that under limited driving amplitude, parametric driving does not affect the system. Nevertheless, we anticipate that the system may oscillate and approach a steady state (as revealed in the numerical section), similar to what is observed in the case of direct drives for relatively large driving amplitudes \( h \) and strong inductance parameters.

5 Numerical simulations and Discussion

Here, we solve the governing tripled sGE (1) with phase shift Eq. (2) for multi-sLJJs numerically and compare derived analytical results for weak and strong magnetic inductance. The Laplace differential operator was used to estimate the partial derivatives through finite difference method, and central difference formula was applied to integrate the sGE from Eq. (1). To discretize, we set \( \Delta t = \Delta x/t \) and \( \Delta x = 1 \times 10^{-2} \), and the magnetic inductance was held constant at \( S = 0.5 \) across all cases. To prevent the formation of standing waves caused by incoming radiations, the damping constants were utilized at the boundaries

\[
\vartheta(x) = \begin{cases} 
|x - X + x_0|/x_0, & |x| > (X - x_0), \\
0, & |x| \leq (X - x_0). 
\end{cases}
\]  

(78)

Consider different values of \( x_0 \), where \( \vartheta \) is the damping parameter. It should be emphasized that while comparing our numerical results, the choice of parameters has no impact because we took different damping parameters into account along with various boundary conditions. In the absence of variety of drives, the general analytical solutions are not present. Therefore, we use

\[ |a_i(0)| = A_i(0)/G, \quad i = 1, 2, 3. \]  

(79)

as the initial conditions where the approximation is formerly obtained by \( Ga_i(t/w) \) with \( G \) as a free-fitting real-valued constant. However, we have taken \( G = 1 \). By considering \( G \) as an unrestricted parameter, we experienced that the finest appropriate value may not be obtained by the above-mentioned value. We established that suitable values vary in different plots for initial condition stated in Eq. (79). The changes in the fitting parameter may be argued by the reason that asymptotic analysis used in analytical sections is only effective for infinitely long times, and therefore, there is a small preliminary fleeting that can be accentuated by letting \( G \neq 1 \). In general, it is also worth to mention that we incorporate \( \kappa = 0 \) as it was approximated analytically in the presence of drives. The discretization and calculation of the governing equations on a finite domain for numerical simulation, particularly when \( X = 40 \), is interesting to note since it demonstrates that the system's natural frequency should differ from the driving frequency. As a result, we draw the conclusion that \( \kappa \) may not be effectively zero in the numerical calculations. It should also be noted that, in the presence of drives, we cannot apply the direct value of \( h \), as this may blow because of the coupling, so we consider \( h = \eta_0(1 - e^{-it}) \) with \( t = e^{-15} \). This selection provides the ground state to gradually adjust itself to the existence of external drive. Other values of \( \tau \) have also been checked, and we did not observe any quantitative alteration.
The value of $\Omega$ is found to be 0.73825 for $b = 0.4$ and $S = 0.5$. Analytical values under the weak coupling condition are presented in the following table:

$$
\zeta_1 = 0.04333, \zeta_2 = 0.22234, \phi_1 = 0.60684,
\alpha_1 = -0.00325 - 0.07216i, \alpha_2 = 0.08625, \alpha_3 = 0.06905,
\beta_1 = -0.00325 + 0.18607i, \beta_2 = -0.55043,
\beta_3 = 0.00652, \beta_4 = 0.51667,
\beta_5 = -0.03845, \beta_6 = 0.10320, \beta_7 = -1.41367,
\gamma_1 = 0.54646, \gamma_2 = -0.41873,
\gamma_3 = -0.00325 + 0.05223i, \gamma_4 = 0.12731i,
\gamma_5 = 0.09640i,
\gamma_6 = 0.02596 - 0.22234i, \gamma_7 = -0.10662 - 0.17815i.
$$

By resolving Eq. (44), we derived the driving frequencies for facet lengths of $b(\Omega) = 0.4$ as $\omega_1 = 0.53342, \omega_2 = 0.81566$, and $\omega_3 = 0.53342$, respectively. The numerical values for the other parameters that were used in the analytical computations with strong coupling with even parity are presented in the following table:

$$
\eta_1 = 0.16985, \eta_2 = 0.26011, \delta_1 = 0.79108,
\delta_2 = 0.56503, \delta_3 = 0.62142,
l_1 = -0.00643 - 0.18433i,
l_2 = -0.05359 + 0.40965i, l_3 = -0.01879 - 2.38812i,
m_1 = -0.00643 + 0.91037i,
m_2 = -0.05359 + 0.40965i, m_3 = -0.01879 - 1.91557i,
m_4 = -2.50794, m_5 = 4.95644, m_6 = -0.00948,
n_1 = -0.00643 - 0.14270i,
n_2 = -0.05359 + 0.9216i, n_3 = -0.01878 + 1.84317i,
n_4 = 0.00221 + 0.04614i,
n_5 = 0.00301 - 0.43837i, n_6 = 0.00776 - 0.51417i.
$$

The numerical results for computations performed analytically under the condition of strong coupling with odd parity are likewise presented in the following table:

$$
\psi_1 = -0.00251, \psi_2 = 0.08354,
l_1 = -0.00221 - 0.03199i,
l_2 = -0.00302 - 0.12495i, l_3 = -0.03442 - 1.49371i,
k_1 = 0.08617i, k_2 = 2.80458i, k_3 = 0.09457i,
k_4 = 1.9541i, k_5 = 0.07617i.
$$

The numerical results and the systematically computed approximations in the form of amplitude are shown in Figures 1 and 2. The left panel of Figure 1 displays circles and stars that were formed under weak coupling conditions, while the black curve was generated using Eq. (23). It is evident that the obtained amplitudes are synchronized and decay exponentially over a period of time. In both panels of Figure 2 and the right panel of Figure 1, we exhibit $a_2(t)$ (i.e., the envelope). The black curve, circles, and stars demonstrate the approximation obtained from Eqs. (23) and (58) while the red curves represent numerical results derived from Eqs. (4) and (34), respectively. The two figures demonstrate that the synchronized oscillations observed in multiple-sLJJs experience an exponential decay when there are no external drives.

Figures 3 and 4 demonstrate the numerical results derived from Eqs. (24) and (62), respectively, with red curves. Meanwhile, the black curves signify the amplitude solutions provided in Eqs. (28) and (67) for the weakly and strongly tripled multi-sLJJs with phase shift under direct drives. The driving amplitudes used in the two figures with phase shift were considered as $h = 0.06$ and 0.2. It is clear that the naturally present nonlinear amplitudes synchronously decrease over time as the oscillatory excitation mode produces radiations due to the nonlinearity in the system.

Figures 5 and 6 illustrate the numerical simulations derived from Eqs. (29) and (69), respectively, with red curves. The approximate solutions are shown as black curves obtained in Eqs. (33) and (76) in the existence of parametric drives for both strong and weakly multi-sLJJs with phase shift. In comparison to direct AC drives, it is observed that the parametric drive has no influence on the governing system. The amplitude decreases exponentially more quickly over time to attain synchronized oscillation. In the left panel of Figure 5, we can deduce that the system oscillates for a while when the driving amplitude is large but ultimately tends to a constant state after some time. Figure 7 illustrates the numerical simulation of the system in the existence of parametric drives (when $S = 1$) since analytical approximations are not available. It is observed that in the presence of parametric drives, the system oscillates and eventually tends to a stable state, similar to that obtained with direct drives.

### 6 Conclusion

We investigated the behavior of a tripled sGE with various drives forming multi-sLJJs with a $0-\pi-0$ phase shift. Using asymptotic analysis along with multi-scale expansions, we analytically solved the system and obtained ordinary
Figure 1: Profile for the two SLJs that are weakly connected in the absence of drives. The synchronized oscillation of Eq. (23) is displayed in the left panel. The red curve in the right panel displays the numerical outcomes of Eq. (4), while the black curve in the panel represents the approximation derived in Eq. (23).

Figure 2: Profile for the two SLJs that are strongly connected in the absence of drives. The synchronized oscillation of Eq. (58) is displayed in the left panel. The red curve in the right panel displays the results of numerical simulations of the governing system Eq. (34), while the black curve shows the approximation derived in Eq. (58).

Figure 3: Profile for the two SLJs that are weakly connected in the existence of ac-drives. The red curves in the two panels indicate the numerical findings of the system from Eq. (24), while the black curves in the two panels show the amplitude of oscillations derived in Eq. (28).
**Figure 4:** Profile for the two sLJJs that are strongly connected with direct ac-drives. The red curves in the two panels indicate the numerical findings of the system from Eq. (62), while the black curves in the two panels show the amplitude of oscillations derived in Eq. (67).

**Figure 5:** Profile for the two sLJJs that are weakly connected in the presence of parametric drives. The red curves in the two panels indicate the numerical findings of the system from Eq. (29), while the black curves in the two panels show the amplitude of oscillations derived in Eq. (33).

**Figure 6:** Profile for the two sLJJs that are strongly connected in the presence of parametric drives. The red curves in the two panels indicate the numerical findings of the system from Eq. (69), while the black curves in the two panels show the amplitude of oscillations derived in Eq. (76).
differential equations for the nonlinear amplitudes for both weak and strong couplings. The derived approximate solutions represent synchronized oscillations in the multi-sLJJs with both weak and strong magnetic inductance, considered as $S = O(e^2)$ and $S = O(1)$, respectively. In the absence of drives, it is observed that the synchronized oscillations with $a_i = O(1)$ decay rapidly. It is important to note that the coupling between the junctions in multi-sLJJs depends on both geometrical and physical factors included in the model.

It has been shown that the coupled-mode amplitudes derived from the occurrence of AC-drive multi-sLJJs decrease over time, indicating that the stack goes through damping, which principally occurs because the breathing modes produce radiation as the assumed high excitations are analytical approximated at different orders in the system together with a coupling. It has also been discussed that the radiative annihilation in tripled sGE arises from exponential decay in the dynamics of breather, which may be suitable to attain super-radiant emission of radiations in the form of energy from coupled oscillators [43].

Furthermore, we explored the impact of parametric drives on the coupling effect in multi-sLJJs. Our findings indicate that when the driving amplitude is small enough, the oscillation is not affected by parametric drives. However, in the case of high driving amplitude with strong magnetic inductance, the system oscillates but eventually reaches a stable state due to the driving effect, as illustrated in the numerical section. Similar findings have been reported for single LJs under external drives [44]. Therefore, it is determined that the parametric drive has no impact on the synchronization of the system when the applied drive is sufficiently small. On the other hand, we speculate that due to the existence of applied drives, the system can cause localized mode oscillations in multi-sLJJs with high dynamic amplitude as observed in the case of direct AC drive.

Coherent super-radiant emission has been previously discussed in coupled systems, with experimental studies actively exploring terahertz emission from stacked intrinsic JJs in cuprate superconductors under zero applied magnetic fields [45–47]. In this work, we conducted an analytical and numerical investigation of a triple system, leading to the observation of synchronized oscillation. It would be of interest to extend the present study to $n$-stacks to enable the production of super-radiant terahertz emission in the form of energy.

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