Joint optimization of two-dimensional warranty period and maintenance strategy considering availability and cost constraints

Abstract: Warranty can improve customer satisfaction and increase product sales, but it will bring additional economic burden to manufacturers, so making reasonable warranty decisions is particularly important. The current research focuses more on the interests of manufacturers, while there is less attention paid to indicators that consumers are concerned about, such as system availability. To solve this problem, a joint optimization model of the two-dimensional warranty period and preventive maintenance (PM) strategy considering availability and cost constraints was established. Based on the failure dependence analysis of multi-component systems, the actual failure rate function is constructed by using the failure dependence coefficient matrix. Based on the comprehensive consideration of imperfect PM and minimum maintenance, the two-dimensional warranty cost model and system availability model are established, respectively. Aiming at maximum availability and considering the warranty cost budget constraint, a joint optimization model of the two-dimensional warranty period and PM strategy was built, and a model-solving algorithm combining grid search and binary search was introduced. The effectiveness of the model is verified by the case analysis, and the necessity of the PM strategy and considering failure dependence are reflected in the model comparative analysis and parameter sensitivity analysis. Finally, reasonable suggestions are put forward for manufacturers’ warranty decisions through the result analysis.

Keywords: two-dimensional warranty, preventive maintenance strategy, joint optimization, availability, binary search

1 Introduction

1.1 Background and motivation

Warranty refers to the maintenance service provided by the manufacturer for the failure of the sold products due to quality problems within a certain period [1]. As the structure and function of products become more and more complex, the corresponding warranty cost and difficulty also continue to increase [2]. Product warranty service has become one of the key factors affecting consumers’ decision to buy products [3]. The warranty period specifies the time limit for the manufacturer (seller) to maintain, replace, and return the product. Under the condition of the market economy, the length of the warranty period has become a key factor affecting the customer’s satisfaction and loyalty to the product [4]. So manufacturers need to determine a reasonable warranty period and make a scientific warranty decision scheme to control cost, improve reputation, and retain more users. Warranties can be divided into one- and two-dimensional warranties based on the number of variables used to define warranties. The one-dimensional warranty is usually defined by age or usage, while a two-dimensional warranty can be characterized as a two-dimensional region, usually represented by age and usage. Now, two-dimensional warranties have been adopted in many complex products [5].

For a long time, scholars have often regarded the research object as a whole and neglected the dependence between various components within the system when researching two-dimensional warranty decision-making models [6]. In fact, with the progress of manufacturing technology and the improvement of system functionality, the dependence between various components within the system is becoming increasingly close. According to the mainstream view in the current academic community, the dependence between components can be divided into economic dependence, structural dependence, and failure dependence, among which the research on failure dependence is the most
extensive [7]. Failure dependence widely exists in multi-component systems. Taking a gearbox as an example, if the gearbox spindle fails, it will enhance the vibration of the gearbox and further exacerbate the gear wear, which means that there is a failure dependence between the spindle and the gear. From this perspective, the existence of failure dependence changes the inherent failure patterns of components, thus affecting the formulation of the optimal warranty decision plan. However, the current two-dimensional warranty decision-making model neglects the exploration of failure dependence in multi-component systems, and this study attempts to fill this gap.

The high availability of systems has always been an important goal pursued by engineers [8], which is also closely monitored by users. High availability means that equipment has a higher proportion of the normal working time and is less prone to malfunctions. Availability is crucial for large and complex equipment. For example, the breakdown of the production workshop will cause immeasurable economic losses, so the high availability of the manufacturing systems will bring greater profits to manufacturing enterprises. The breakdown of the weaponry will threaten the safety of soldiers and lead to war failure, so the high availability of the weaponry will increase the probability of successful warfare. Two-dimensional warranty equipment is usually capital intensive [9], and users often have high requirements for the availability of two-dimensional warranty equipment. However, most of the current research on two-dimensional warranty decision-making has not considered the availability of equipment but focuses more on product prices, profits, and costs.

To address the above challenges, this study proposes a two-dimensional warranty period and maintenance strategy joint optimization model considering availability and cost constraints. The innovation of this article mainly lies in the following:

1) This article takes maximizing system availability as the decision objective and establishes a decision model with budget as the constraint condition.
2) Considering the failure dependence between multiple components, a multi-component system failure rate model is established based on a failure chain model.
3) A novel binary method has been developed for model solving.

The research of the article is mainly under the umbrella of social and economic physics, statistics, and nonlinear physics. In this article, mathematical modeling and algorithm development are mainly used to solve problems in engineering physics, and important suggestions are provided for manufacturers to carry out warranty activities. The fitting of the failure rate function of power plants involves probability theory, mathematical statistics, and random process knowledge. The solution of the entire model involves nonlinear programming problems. The developed binary method is a commonly used sorting algorithm in computer algorithms, which has enlightening significance for solving this engineering problem. So, the article is under the umbrella of social and economic physics, statistics, and nonlinear physics.

1.2 Relevant literature

The determination of warranty period has always been a concern of scholars. Wu et al. established a model of free renewal warranty cost for components before the operation time reaches the specified value, which was used to determine the optimal warranty period and cost length when maximizing profit [10]. Chien divided the failure of repairable products into minor failure and serious failure, determined the optimal warranty period based on the minimization of the cost function, and obtained the age replacement interval outside the optimal warranty period from the perspective of consumers and manufacturers [11]. Huang et al. established a model of sales price, warranty period, and product reliability under the strategy of free replacement or maintenance [12]. Liu et al. optimized the product's warranty period and reliability based on the free replacement or maintenance strategy for repairable products [13]. Xie et al. developed a model to evaluate the overall profitability of components within and outside the warranty period [14]. He et al. defined an attraction index based on available warranty areas to describe the degree of attraction of the two-dimensional warranty period to customers with different usage rates. From the manufacturer's point of view, the optimal two-dimensional basic warranty period was designed to maximize the expected profit, and the effectiveness of the method is demonstrated by numerical examples [15]. Chen and Popova proposed a two-dimensional warranty strategy, which is free replacement within the limited area and minimal maintenance beyond the limited area but within the warranty area. With the lowest cost as the goal, the warranty period and the limited area were obtained [16]. Baik et al. put forward two models: the faulty system is always carried out with minimum maintenance, and the faulty system is always replaced in the context of the two-dimensional warranty, and the model is used to carry out cost analysis and give the optimal warranty period [17]. Taleizadeh and Mokhtarzadeh designed a value-risk method for dual-channel (online and offline) sales manufacturers to
optimize the two-dimensional warranty scheme and pricing simultaneously, and higher profits were obtained by considering the covariance between product warranty claims [18]. Park et al. studied the method for determining the optimal warranty period of products from the manufacturer’s perspective under specific lemon law regulations [19].

It can be seen that most studies only stand on the manufacturer’s point and take the lowest warranty cost or the highest profit as the goal to make the warranty period decision, ignoring the needs of consumers to some extent. At present, the two-dimensional warranty decision model based on the win–win perspective of both manufacturers and consumers has been preliminarily studied. Huang et al. classified customers according to their usage conditions in the basic warranty period and provided two-dimensional extended warranty schemes for different types of customers to reduce warranty costs and improve marketing competitiveness [20]. He et al. established a two-dimensional extended warranty cost model based on the product failure process considering consumers’ different usage rates, purchase time, and maintenance options of the extended warranty and obtained the isoline of win–win warranty area and win–win extended warranty interval [21]. Dong et al. considered the consumer utility function of different warranty periods and obtained a two-dimensional warranty period that both manufacturers and consumers won [22]. Salmasnia et al. studied the optimal two-dimensional warranty policy for second-hand products by considering the two-dimensional warranty areas in both L-shaped and rectangular cases [23]. Peng et al. studied the optimal cost allocation ratio for preventive maintenance (PM) of two-dimensional warranty equipment from the user’s perspective [24]. Wang studied the design and pricing of customized two-dimensional extended warranty solutions from the perspective of suppliers, allowing each consumer to match suitable warranty solutions based on their usage rate [25]. Gupta et al. designed a new optimal extended warranty policy based on the price curve and demonstrated the advantages of the two-dimensional warranty policy compared to the one-dimensional warranty policy [26]. Ruan et al. considered the learning effectiveness of users and manufacturers during the warranty period and studied the problem of “the PM and the two-dimensional warranty design,” proving that considering both the PM planning and the two-dimensional warranty design is a win–win strategy for both manufacturers and users [27]. Zhang et al. considered the differences in user usage rates and studied the optimal two-dimensional extended warranty policy in stable and dynamic markets [28]. Although the above research paid attention to the interests of consumers to some extent, the essence of their decision is to hope that manufacturers can sell more products or warranty services. For users, the improvement of the availability of warranty objects means the reduction of unexpected failure, which is the ideal state expected by users [29]. As an important performance index measuring the probability of systems being in the working state, system availability can effectively guide maintenance decisions and has always been a hot topic in the field of reliability engineering [30].

The application of different maintenance strategies during the warranty period is another research focus besides the decision on the warranty period. Varnosfaderani and Chukova divided the warranty area into several sub-areas, used the failure rate rollback method to describe the maintenance degree of imperfect maintenance, and stipulated that minimum maintenance should be carried out in the first sub-area and the last sub-area, imperfect maintenance should be carried out in the first failure of the remaining sub-areas, and minimum maintenance should be carried out in the subsequent failure. The sub-areas were identified by minimizing manufacturer maintenance costs [31]. Based on the imperfect PM strategy, Shahanaghi et al. established a two-dimensional extended warranty decision model intending to minimize the cost of the two-dimensional extended warranty to obtain the optimal number and degree of PM [32]. Yu and Chen divided the warranty area into three sub-areas, adopted the minimum maintenance strategy, and proposed the optimal two-dimensional warranty scheme for products with the goal of the lowest warranty cost [33]. Wang and Song designed a two-dimensional PM and replacement strategy. Under the strategy, PM was carried out according to calendar time or usage, and if the product withstood (n − 1)th shock, it was replaced with a new product at the nth shock. From the manufacturer’s point, the average warranty cost for the whole warranty period was obtained by using the renewal theory [34]. Gupta and Chattopadhyay proposed the labor code priority index and studied the identification of early reliability problems in two-dimensional warranty equipment [35]. Wei et al. studied a new burn-in policy for repairable products with a two-dimensional combination warranty based on the assumption of product heterogeneity [36]. PM has been paid more and more attention to by researchers because it can prevent failure or serious consequences of failure and reduce the loss caused by failure downtime. However, there are relatively few studies on the joint optimization of the warranty period and PM interval.

At present, most of the research objects are single-component systems, and the dependence between the components of multi-component systems is rarely considered. For multi-component systems, there are three forms of dependence between components, namely, economic dependence, failure dependence, and structural dependence [37,38]. Failure
dependence, also known as stochastic dependence, means that the failure of one component affects the status of other components in the system. Generally, it can be divided into three categories [39,40]: Class I failure dependence refers to that when one component fails in the system, other components will fail with a certain probability \( a (0 \leq a \leq 1) \); Class II failure dependence is failure rate dependence in essence, that is, when a component fails, it will increase the failure rate of other components to a certain extent, but it will not directly cause the component failure; and Class III failure dependence is related to impact damage, that is, when a component fails, it will cause random damage to other components of the system to a certain extent. When the impact damage accumulates to a certain extent, it will lead to the component failure. The current study does not provide sufficient support for the warranty decision of failure-dependence multi-component systems.

1.3 Overview of the study

Based on the above analysis, this study introduced a PM strategy into complex product warranty. For the multi-component system with failure dependence, a joint optimization model of two-dimensional warranty and PM strategy based on cost and availability is established, which effectively integrated PM and corrective maintenance and provided a decision scheme for complex products warranty considering PM strategy, to achieve the product warranty decision scientific and effective.

The main contributions of this work are listed as follows:

1) Taking the multi-component system as the warranty object, the warranty decision-making model of the multi-component system is established based on the analysis of the failure dependence of each component in the multi-component system.

2) The warranty decision model integrates corrective maintenance and PM effectively, considering the warranty cost and the system availability simultaneously.

3) The joint optimization of the two-dimensional warranty period and PM interval for a failure-dependence multi-component system is carried out.

The remainder of this article is organized as follows. Section 2 introduces the model formulation, including six aspects: problem description, working assumptions and notations, the system failure rate function, the imperfect PM strategy, the system availability, and the expected warranty costs. Section 3 presents the model optimization analysis including joint optimization model and algorithms. Section 4 provides the corresponding case study. Finally, the conclusions and research perspectives are offered in Section 5.

2 Model formulation

2.1 Problem description

The multi-component system with failure dependence adopts the two-dimensional warranty policy. \( K_1^B \) and \( K_2^B \) represent time and usage limits, respectively, as shown in Figure 1. During the warranty period, the manufacturer shall carry out post-failure corrective maintenance and regular PM for the multi-component system. The study assumes that the usage rate of the multi-component system by different consumers is a random variable, but for a given consumer, it is a certain value during the warranty period [41]. The distribution function of the usage rate is \( G(r) = P(R \leq r), r_{\min} \leq r \leq r_{\max} \). The distribution of usage rate can be determined by records of similar products or customer surveys [42]. During the product design phase, the desired reliability of the product is ensured in the nominal usage rate \( r_0 \).

During the warranty period, the manufacturer expects to control the warranty cost, and the consumer expects to obtain higher product availability. Therefore, how to take into account the interests and needs of both manufacturers and consumers and scientifically determining the warranty period and PM interval of the failure dependence multi-component system is the problem that the manufacturer needs to solve.

![Figure 1: Two-dimensional warranty.](image-url)
2.2 Assumptions and notations

To construct the model, the assumptions of this article are as follows:

1) The failure-dependence multi-component system consists of Y single components in series.
2) The failure-dependence multi-component system is a repairable system, and its failure rate increases with time and usage [43].
3) Corrective maintenance is minimum maintenance, and PM is imperfect maintenance.
4) Each component in a multi-component system has unidirectional failure dependence, which belongs to Class II failure dependence.
5) Failure under the nominal usage rate and the consumer usage rates are subject to Weibull distribution [44].

The notations that appeared in the study are presented in Table 1.

### 2.3 System failure rate function

In the two-dimensional warranty, the calendar time and usage of the product are the key factors affecting the failure rate. There are three main fitting methods for the two-dimensional warranty product failure rate function, namely, the bivariate method, composite method, and marginal approach. The marginal approach transforms the usage rate into a function of calendar time, effectively changes the two-dimensional problem into a one-dimensional problem, and is widely used at present. The accelerated failure time (AFT) model is one of the marginal approaches. In this study, the AFT model is used to fit the failure rate function of a single component.

$I_1$ and $I_2$ indicate the first failure time under the nominal usage rate and actual usage rate, respectively. For component $I$, the probability density function, cumulative distribution function, and failure rate function of $I_1$ are $f_{I_1}(t)$, $F_{I_1}(t)$ and $h_{I_1}(t)$, respectively, then

$$f_{I_1}(t) = \frac{a}{\beta} \left( \frac{t}{\beta} \right)^{a-1} e^{-\left( \frac{t}{\beta} \right)^{a}},$$  

$$F_{I_1}(t) = 1 - e\left( \frac{t}{\beta} \right)^{a},$$  

$$h_{I_1}(t) = \frac{a}{\beta} \left( \frac{t}{\beta} \right)^{a-1},$$

where $a$ is the shape parameter and $\beta$ is the scale parameter. According to the AFT model [45], the quantitative relation between $I_1$ and $I_2$ is

$$\frac{I_1}{I_2} = \left( \frac{r_0}{r} \right)^{\gamma'},$$

where $\gamma'$ is the AFT parameter and $\gamma' \geq 1$. Under the actual usage rate $r$, the scale parameter turns into

$$\beta_r = \beta \left( \frac{r_0}{r} \right)^{\gamma'}. $$

Then, the cumulative distribution function and failure rate function of $I_2$ for component $i$ are

$$F_{I_2}(t) = 1 - \exp \left[ - \left( \frac{r}{r_0} \right)^{\gamma'} \frac{t^a}{\beta_r^a} \right],$$  

$$h_{I_2}(t) = \frac{F_{I_2}(t)}{F_{I_1}(t)} = \frac{d}{r_0} \left( \frac{r}{r_0} \right)^{\gamma'} \frac{t^{a-1}}{\beta_r^a}. $$

### Table 1: Model notations

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>The time limit of the two-dimensional warranty</td>
</tr>
<tr>
<td>$r$</td>
<td>The usage limit of the two-dimensional warranty</td>
</tr>
<tr>
<td>$F_r$</td>
<td>The nominal usage rate of the system</td>
</tr>
<tr>
<td>$g(r)$</td>
<td>The distribution function of the usage rate</td>
</tr>
<tr>
<td>$F_{r_{min}}$</td>
<td>The minimum value of usage rates</td>
</tr>
<tr>
<td>$F_{r_{max}}$</td>
<td>The maximum value of usage rates</td>
</tr>
<tr>
<td>$r$</td>
<td>The actual usage rate of the system</td>
</tr>
<tr>
<td>$I_1$</td>
<td>The first failure time under the nominal usage rate</td>
</tr>
<tr>
<td>$I_2$</td>
<td>The first failure time under the actual usage rate</td>
</tr>
<tr>
<td>$f_{I_1}(t)$</td>
<td>The probability density function of $I_1$ for component $i$</td>
</tr>
<tr>
<td>$F_{I_1}(t)$</td>
<td>The cumulative distribution function of $I_1$ for component $i$</td>
</tr>
<tr>
<td>$h_{I_1}(t)$</td>
<td>The failure rate function of $I_1$ for component $i$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>The AFT parameter</td>
</tr>
<tr>
<td>$\beta_r$</td>
<td>The scale parameter under the actual usage rate</td>
</tr>
<tr>
<td>$F(t(r))$</td>
<td>The cumulative distribution function of $I_2$ for component $i$</td>
</tr>
<tr>
<td>$h(t(r))$</td>
<td>The failure rate function of $I_2$ for component $i$</td>
</tr>
<tr>
<td>$h_{I_2}(t(r))$</td>
<td>The actual failure rate of component $i$</td>
</tr>
<tr>
<td>$\tau_0$</td>
<td>PM interval</td>
</tr>
<tr>
<td>$d$</td>
<td>The level of imperfect PM</td>
</tr>
<tr>
<td>$\gamma(d)$</td>
<td>The PM improvement factor under level $d$</td>
</tr>
<tr>
<td>$y$</td>
<td>The number of components in the system</td>
</tr>
<tr>
<td>$n_s$</td>
<td>The number of PM during the two-dimensional warranty for the system</td>
</tr>
<tr>
<td>$A_e$</td>
<td>The expected availability of the system</td>
</tr>
<tr>
<td>EC</td>
<td>The expected warranty cost for the system</td>
</tr>
</tbody>
</table>
The failure chain model is used to describe the unidirectional failure dependence of components \([46]\). The failure chain model is shown in Figure 2.

Each component is numbered, as shown in Table 2.

The failure dependence coefficient is introduced to describe the degree of failure dependence between components. The failure dependence coefficient \(\phi_{ij}\) means the impact of the failure rate of component \(j\) on the failure rate of component \(i\). The failure dependence matrix in Figure 2 can be constructed as

\[
\phi = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\phi_{21} & 0 & 0 & 0 & 0 \\
0 & \phi_{32} & 0 & 0 & 0 \\
\phi_{41} & 0 & 0 & 0 & 0 \\
0 & 0 & \phi_{53} & \phi_{54} & 0
\end{bmatrix}
\] (8)

Then, considering the failure dependence, the actual failure rate of component \(i\) can be expressed as follows:

\[
h_{i,\text{real}}(t|r) = (I_t h(t|r)) + \{\phi\{h(t|r)\},
\]

where \(h_{i,\text{real}}(t|r)\) is the matrix of \(Y \times 1, 1 \leq i \leq Y\), which represents the actual failure rate of component \(i\), \(I_t\) is the \(Y\)-order unit matrix, and \(h(t|r)\) is the matrix of \(Y \times 1, 1 \leq i \leq Y\), which represents the inherent failure rate of each component. Then, the failure rate function of the system is

\[
h_{s}(t|r) = \sum_{i=1}^{Y} h_{i,\text{real}}(t|r).
\]

2.4 Imperfect PM strategy

The manufacturer shall carry out regular imperfect PM of the system within the warranty period and the PM interval is \(T_0\). Imperfect PM leaves the repaired system in a state between as good as new and as bad as old. Imperfect PM is widely used because the maintenance cost is reasonable and the system state can be improved \([32]\). The effect of imperfect PM is modeled by the virtual age method and the failure-rate regression method. \(d\) represents the level of imperfect PM, and \(0 \leq d \leq D\). \(d = 0\) corresponds to the state of the system without PM. \(M\) is the upper limit of PM. Let PM improvement factor be \(\chi(d)\). According to Gupta et al. \([35]\):

\[
\chi(d) = (1 + d) \cdot e^{-d}.
\]

The virtual age of the system after the qth PM is \([47]\)

\[
\theta_q = \theta_{q-1} + \chi(d) \cdot T_0,
\]

where \(\theta_{q-1}\) represents the virtual age of the system after the \((q - 1)\)-th PM. The change in system failure rate under imperfect PM is shown in Figure 3.

2.5 System availability

Based on the calculation method of the average availability proposed by Li et al. \([48]\), the long-term availability of the system in this study was modeled. The combination of the PM interval and the level of imperfect maintenance is \([T_0, d]\). The minimum maintenance time is \(T_1\), and the PM time under different PM levels is \(T_0(d)\). As shown in Figure 1, the actual length of the system warranty varies with different usage rates. The actual length of the system warranty is

\[
K_r = \min(K_r^1, K_r^2|r).
\]

Then, during the two-dimensional warranty period, the number of PM under usage rate \(r\) of the system is

![Figure 2: The failure chain model.](image)

![Table 2: Component numbers](table)

<table>
<thead>
<tr>
<th>Component</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
</tr>
</tbody>
</table>

![Figure 3: The system failure rate under imperfect PM.](image)
During the two-dimensional warranty period, the number of corrective maintenance under usage rate $r$ of the system is

$$L = \sum_{u=0}^{a_u} \int_{\gamma(d) \cdot T_0}^{n_x \cdot \gamma(d) \cdot T_0 + K_r - n_x \cdot T_0} h_u(t(r))dt + \int_{n_x \cdot \gamma(d) \cdot T_0}^{n_x \cdot \gamma(d) \cdot T_0 + K_r - n_x \cdot T_0} h_u(t(r))dt.$$  

Under usage rate $r$, the expected downtime of the system is

$$T_d = T_1 \cdot L + n_a \cdot T_p(d).$$

Then, the expected availability of the system during the warranty is

$$A_\alpha = \frac{K_r - T_d}{K_r}.$$  

Based on the above analysis, the expected availability of all consumers within the two-dimensional warranty period can be expressed as

$$A_\alpha = \frac{K_r - T_d}{K_r}.$$  

2.6 Expected warranty costs

The minimum maintenance cost is $C_f$, and the PM time under different PM levels is $C_p(d)$. Under usage rate $r$, the expected warranty cost of the system is

$$C_d = C_f \cdot L + n_a \cdot C_p(d).$$

The expected warranty cost of all consumers within the two-dimensional warranty period can be expressed as

$$EC = \int_{r_{\min}}^{r_{\max}} C_d \cdot g(r)dr.$$  

3 Model optimization analysis

Given the warranty cost budget, this study is devoted to finding the two-dimensional warranty period that maximizes availability. Then, by changing the PM interval and level, the final plan of the two-dimensional warranty period, PM interval, and level that maximizes system availability are determined, thus achieving joint optimization of the two-dimensional warranty period, PM interval, and level. This section first establishes a joint optimization model and then provides a detailed introduction to the algorithm for solving the model.

3.1 Joint optimization model

Let the manufacturer’s warranty cost budget be $C_a$. Under equal cost $C_a$, the two-dimensional warranty period optimization model aiming at maximizing system availability is as follows:

$$\arg \max_{K_L, K_R} A_\alpha$$

subject to:

$$EC = C_a$$

$$C_d = \int_{r_{\min}}^{r_{\max}} C_f \cdot L + n_a \cdot C_p(d) \cdot g(r)dr,$$

Then, the expected availability of the system during the warranty is

$$A_\alpha = \frac{K_r - T_d}{K_r}.$$  

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The expected warranty cost of all consumers within the two-dimensional warranty period can be expressed as

$$EC = \int_{r_{\min}}^{r_{\max}} C_d \cdot g(r)dr.$$
Step 1: Given the PM strategy and warranty cost budget, the two-dimensional warranty period scheme that maximizes system availability under the PM strategy and the constraint of warranty cost is solved.

Step 2: Since the PM level is limited and the PM interval can be discretized, the optimal two-dimensional warranty period of the system can be obtained by transforming the PM strategy and applying the same method and principle in Step 1.

Step 3: The PM strategy and two-dimensional warranty period, which make the system reach the highest availability, are selected, thus realizing the joint optimization of the two-dimensional warranty period and PM strategy.

The main process of solving the model is shown in Figure 4.

The key to Step 1 is the calculation of expected warranty cost and system availability. As can be seen from Figure 1, when the usage rate is greater than $K^2_0/K^1_0$, the warranty period of the system varies with the usage rate. It is difficult to directly obtain the expected warranty cost and availability of the system under all usage rates by the calculus method. Therefore, when the usage rate is greater than $K^2_0/K^1_0$, the method of transforming continuous to discrete is adopted, and the usage rate is divided into 20 equal minizones on the interval $[K^2_0/K^1_0, r_{\text{max}}]$. The probability of each usage rate and the warranty cost and availability under the utilization rate are calculated, respectively. Then, the warranty cost and expected availability of the system with a usage rate greater than $K^2_0/K^1_0$ are obtained according to the calculation method of mathematical expectation. Finally, the total expected warranty cost and availability of the system are obtained by considering the usage rate greater than $K^2_0/K^1_0$ and less than $K^2_0/K^1_0$ comprehensively. The pseudo-code for warranty costs is shown in Appendix A. The pseudo-code for system availability is shown in Appendix B [49].

Next, we need to find all two-dimensional warranty periods under this cost budget. This article mainly adopts Binary Search to solve this problem. Binary search is a
search algorithm that looks for a specific element in an ordered array. The basic idea is to divide the array into two parts, determine which part the target element is in, and then continue searching in that part until you find the target element or determine that it does not exist [50]. Combined with this model, the basic logic of binary search is shown in Figure 5.

The solution process shown in Figure 5 can be used to obtain the optimal two-dimensional warranty period under a specific PM strategy and warranty cost budget. Since the PM interval and maintenance level are discrete, the optimal PM strategy and two-dimensional warranty period under a specific cost budget can be obtained by changing the PM strategy a limited number of times [51,52]. By repeating Steps 1 to Step 3, the joint optimization of the two-dimensional warranty period and maintenance strategy can be achieved.

### 4 Case analysis

In this section, the study illustrates the application of the proposed model through a case study of a power plant in China. The power plant has adopted the two-dimensional warranty, whose one dimension is calendar time (years), and one dimension is usage (km). It is necessary to decide

![Figure 5: The application process of binary search.](image-url)
the optimal warranty period and PM strategy to realize the improvement of system availability based on controlling the warranty cost. The power plant can be regarded as a multi-component system consisting of three components. There is failure dependence between components, and the failure chain model is shown in Figure 6.

The failure dependence coefficient matrix is

\[
\Phi = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0.02 & 0.01 & 0
\end{bmatrix}
\]

Consumer usage rates \( r \) in \([0, 6]\) follows Weibull distribution, and

\[
r : W(\eta = 2.2, \theta = 0.5)
\]

The first failure time of each component under the nominal usage rate is subject to Weibull distribution, and

\[
I_1 : W(\alpha = 1.5, \beta = 0.6).
\]

The unit usage rate is \(10^4\) km/year.

The imperfect PM level \(d\) is \([0, 1, 2, 3, 4, 5]\), the corresponding imperfect PM cost \(C_p\) is \([0, 10, 30, 60, 100, 160]\), and the corresponding imperfect PM time \(T_p\) is \([0, 5, 10, 15, 20, 30]\). The corrective maintenance cost \(C_f = 250\), and the corrective maintenance time \(T_f = 40\). The unit of \(T_p\) and \(T_f\) is “Day” and the unit of \(C_p\) and \(C_f\) is “CNY.”

4.1 Model validity analysis

The PM strategy \(d = 3 T_0 = 9\) months is taken as an example to test the validity of the model. The value range \(K_{B1}\) is \([1, 10]\), and the value step is 0.2. The value range \(K_{B2}\) is \([1, 5]\), and the value step is 0.2. Costs and availability under different warranties can be calculated separately. The variation trend three-dimensional graph of cost and availability under different warranty periods can be drawn by using the data obtained in the calculation, as shown in Figures 7 and 8.

The cost budget for \(C_a\) is 700 CNY. First, the binary search method is used to find all the combinations \([K_{B1}, K_{B2}]\) satisfying the conditions on the equal cost curve \((C_a = 700)\), as shown in Figure 9.

In Figure 9, the blue dots represent the combination of \([K_{B1}, K_{B2}]\) with budget 700 CNY, and the red dot indicates that no \(K_{B2}\) matches \(K_{B1}\) so that meets the budget. The variation trend of availability \(K_{B1}\) is shown in Figure 10.

By calculating the availability of each valid point, the optimal two-dimensional warranty period \([K_{B1}, K_{B2}]\) is found. That is \([9.9, 1.33]\), and the availability is 0.9625. The PM strategy is changed, that is, the value of PM level \(d\) and PM interval \(T_0\) are changed, and the optimal two-dimensional warranty period and corresponding system availability are found under different PM strategies. The value

Figure 6: The failure chain model of the power plant.

Figure 7: The availability trends under different warranty periods.

Figure 8: The cost trends under different warranty periods.
of $T_0$ is in the range of [1 month, 15 months], and its step is 1 month. Figure 11 shows the variation of maximum system availability under different PM strategies.

As can be seen from Figure 11, when the PM interval is small, the system availability is low, mainly because frequent PM leads to increased system downtime. This suggests that manufacturers should determine the PM interval scientifically and reasonably to prevent the maintenance surplus. The combination of the optimal two-dimensional warranty period and system availability under the partial PM strategy is shown in Table 3.

According to the actual situation of manufacturers and consumers, the alternative PM strategy $(d, T_0)$ is determined as (5, 6), (4, 15), (3, 4), (2, 9), and (1, 2). As shown in Figure 9, the optimal strategy is to choose a PM interval of 4 months and a warranty period of 6 years. The combination of the optimal two-dimensional warranty period and system availability under the partial PM strategy is shown in Table 3.

**Table 3: Optimization of the two-dimensional warranty period and availability**

<table>
<thead>
<tr>
<th>Preventive strategy</th>
<th>$C_a = 700$ CNY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>$T_0$ (months)</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
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<td>4</td>
<td>6</td>
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</tr>
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<td>2</td>
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<td>3</td>
<td>4</td>
</tr>
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<td>3</td>
<td>11</td>
</tr>
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<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
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<td>2</td>
<td>9</td>
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<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
</tr>
</tbody>
</table>

Bold values represent the optimal solution among all candidate solutions.
Table 3, the optimal PM strategy can be determined as (4, 15), and the optimal two-dimensional warranty is (10 years, $1.35 \times 10^4$ km). In this case, the availability is 0.9668.

4.2 Model comparative analysis

When the failure dependence among components of a multi-component system is not considered, the failure dependence coefficient matrix $\phi$ becomes a zero matrix. The same parameter settings as 4.1 when considering failure dependence are adopted. The variation trend three-dimensional graph of cost and availability under different warranty periods are shown in Figures 12 and 13, respectively.

The binary search is used to solve the optimal two-dimensional warranty period and availability without considering failure dependence. The optimal two-dimensional warranty period $[K_{B1}^2, K_{B2}^2]$ is [9.8, 1.34], and the availability is 0.9607. Compared with considering failure dependence, the optimal two-dimensional warranty period changes, indicating that the existence of failure dependence will affect the two-dimensional warranty decision of the system, so managers should consider the impact of failure dependence when making decisions. Tables 4 and 5, respectively, show the system availability and warranty cost under different two-dimensional warranties.

It can be seen from Tables 4 and 5 that the existence of failure dependence will increase the warranty cost and reduce the system availability. Therefore, in the design and manufacturing stage of the system, efforts should be made to reduce the failure dependence coefficient between components, reduce the warranty cost, and increase the system availability as much as possible.

When PM strategies are not carried out during the warranty period, i.e., when $T_0 = 10$, the warranty cost and availability of the system during the warranty period can be calculated separately. Then, the average warranty cost at all sampling points is obtained and compared with the PM strategy adopted during the warranty period, as shown in Figure 14. Through comparison, it can be found that PM reduces warranty costs; therefore, it is necessary to implement PM during the warranty period.

![Figure 13: The cost trends when failure dependence is not considered.](image)

Table 4: System availability under different two-dimensional warranties

<table>
<thead>
<tr>
<th>$K_{B1}^2$ (10$^4$ km)</th>
<th>$K_{B2}^1$ (years)</th>
<th>Failure dependence is not considered</th>
<th>Failure dependence is considered</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1$</td>
<td>$1.2$</td>
<td>$1.4$</td>
</tr>
<tr>
<td>$1$</td>
<td>0.9034</td>
<td>0.9089</td>
<td>0.9126</td>
</tr>
<tr>
<td>$1.2$</td>
<td>0.9032</td>
<td>0.9079</td>
<td>0.9102</td>
</tr>
<tr>
<td>$1.4$</td>
<td>0.9032</td>
<td>0.9077</td>
<td>0.9093</td>
</tr>
<tr>
<td>$1.6$</td>
<td>0.9032</td>
<td>0.9077</td>
<td>0.9091</td>
</tr>
<tr>
<td>$1.8$</td>
<td>0.9032</td>
<td>0.9077</td>
<td>0.9091</td>
</tr>
<tr>
<td>$2$</td>
<td>0.9032</td>
<td>0.9077</td>
<td>0.9090</td>
</tr>
</tbody>
</table>
4.3 Parameter sensitivity analysis

In the optimization of the two-dimensional warranty period, the distribution of consumers’ usage rates will have a great impact on the final two-dimensional warranty period. In Section 4.1, consumer usage rates follow the Weibull distribution with shape parameter $\eta = 2.2$ and scale parameter $\theta = 0.5$. The values of shape parameters and scale parameters were changed, and the other parameters were set in the same way as in Section 4.1. The change of optimal two-dimensional warranty period with usage rate distribution was studied.

Figures 15–18, respectively, show all qualified two-dimensional warranties and the optimal two-dimensional warranties on the $C_a = 700$ curve under different usage rate distributions. It can be seen that the optimal two-dimensional warranty period under different usage rate distributions is not the same. So, the usage rate distribution of consumers has an important effect on the optimization of the two-dimensional warranty period. This tells us in two-

<table>
<thead>
<tr>
<th>$K_a^2$ (10^1 km)</th>
<th>$K_a^1$ (years)</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>$K_a^2$ (10^1 km)</th>
<th>$K_a^1$ (years)</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
</tr>
</thead>
</table>

Table 5: System warranty cost under different two-dimensional warranties (unit: CNY)

![Figure 14: Comparison of warranty costs.](image1)

![Figure 15: Schematic diagram of two-dimensional warranty combination ($\eta = 2.3, \theta = 0.4$).](image2)
dimensional warranty period design, effective prediction and distribution fitting of consumers’ usage rate is an important basis for optimization of the two-dimensional warranty period.

The warranty cost budget is another important factor affecting the optimization of the two-dimensional warranty period. In Section 4.1, the warranty cost budget is 700 CNY. The cost is [700, 2,100], and the step size is 200. The optimal two-dimensional warranty period and system availability under different warranty cost budgets are shown in Table 6.

Figure 19 shows the trend of $K_B$ and $A_z$ changing with warranty cost budgets. From Figure 18, it can be seen that a higher warranty cost budget can lead to a longer warranty period for the mileage dimension. Simultaneously, it can be seen that system availability decreases with the increase of warranty cost budget, indicating that more warranty cost

<table>
<thead>
<tr>
<th>$C_a$</th>
<th>$K_B^1$ (years)</th>
<th>$K_B^2$ (10^4 km)</th>
<th>$A_z$</th>
<th>$A_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>700 CNY</td>
<td>9.9</td>
<td>1.33</td>
<td>0.625</td>
<td></td>
</tr>
<tr>
<td>900 CNY</td>
<td>10</td>
<td>1.68</td>
<td>0.529</td>
<td></td>
</tr>
<tr>
<td>1,100 CNY</td>
<td>10</td>
<td>2.28</td>
<td>0.932</td>
<td></td>
</tr>
<tr>
<td>1,300 CNY</td>
<td>10</td>
<td>3.32</td>
<td>0.922</td>
<td></td>
</tr>
<tr>
<td>1,500 CNY</td>
<td>10</td>
<td>4.95</td>
<td>0.893</td>
<td></td>
</tr>
</tbody>
</table>
budget is not always better. Therefore, determining a reasonable warranty cost budget is also particularly important for manufacturers.

5 Conclusion

In this article, a joint optimization model of the two-dimensional warranty period and PM interval for failure-dependence multi-component systems was established. The model integrated PM and corrective maintenance effectively and considered the warranty cost and availability simultaneously, giving concern to the interests of both consumers and manufacturers. First, the failure dependence analysis of multi-component systems was conducted, and the actual failure rate model of the multi-component system was constructed. On this basis, the multi-component system warranty cost model and availability model were established. Aiming at maximum availability and considering the warranty cost budget constraint, a joint optimization model of the two-dimensional warranty period and PM strategy was built, and a model-solving algorithm combining grid search and binary search was introduced. The case analysis results indicated that the model established in this article could provide technical support for the decision-making of the two-dimensional warranty period.

The research done in the article enriches the study of social and economic physics, statistics, and nonlinear physics. The mathematical model and the algorithm developed have potential applications in the fields of engineering physics and computational physics. There are still some problems related to our analysis that could be further studied. The study only considers failure dependence. The situation is more interesting when economic dependence and structural dependence are considered. And, joint optimization of the two-dimensional extended warranty period and PM interval is also a very interesting direction to be studied in the future.

Acknowledgments: The authors are very grateful to the associate editor and anonymous reviewers for their valuable and constructive comments and suggestions.

Funding information: The authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

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Appendix A  Pseudo-code for warranty costs

Function: The warranty cost is obtained through this program
Parameters: \( K_B^1, K_B^2, C_t, C_p(d), d, T_0, \) etc.

if \( r_{\text{min}} \leq r \leq K_B^2/K_B^1 \)
\( n_a = \text{floor}(K_B^1/T_0) \)
\[ \begin{align*}
\text{EC}_1 &= \int_{r_{\text{min}}}^{K_B^1/K_B^2} C_d \cdot g(r)dr \\
\text{EC}_2 &= \int C_d[r_m(q)] \cdot g[r_m(q)]d
\end{align*} \]
end if

else
\( r_{\text{low}} = K_B^2/K_B^1; \)
\( M = 20; \)
\( r_c = (r_{\text{max}} - r_{\text{low}})/M; \)
\( r_1 = r_{\text{low}} : r_c : (r_{\text{low}} + (M - 1) * r_c); \)
\( r_2 = r_1 + r_c; \)
\( r_m = r_{\text{low}} : r_c : (r_{\text{low}} + M * r_c); \)
for \( q = 1: M \)
\( n_a = \text{floor}(K_B^2/r_m(q)T_0) \)
\[ \begin{align*}
\text{EC}_2 &= \int C_d[r_m(q)] \cdot g[r_m(q)]d \\
\text{EC}_3 &= \sum M \text{EC}_2 \\
\end{align*} \]
end for
end if

EC = EC_1 + EC_3
print EC
end procedure

Appendix B  Pseudo-code for system availability

Function: The system availability is obtained through this program
Parameters: \( K_B^1, K_B^2, T_t, T_p(d), d, T_0, \) etc.

if \( r_{\text{min}} \leq r \leq K_B^1/K_B^2 \)
\( n_a = \text{floor}(K_B^1/T_0) \)
\[ \begin{align*}
A_{z1} &= \int A_k \cdot g(r)dr \\
A_{z2} &= \sum A_m - \int A_k \cdot g(r)dr \\
A_{z3} &= \sum M A_{z2} \\
\end{align*} \]
end if

else
\( r_{\text{low}} = K_B^2/K_B^1; \)
\( M = 20; \)
\( r_c = (r_{\text{max}} - r_{\text{low}})/M; \)
\( r_1 = r_{\text{low}} : r_c : (r_{\text{low}} + (M - 1) * r_c); \)
\( r_2 = r_1 + r_c; \)
\( r_m = r_{\text{low}} : r_c : (r_{\text{low}} + M * r_c); \)
for \( q = 1: M \)
\( n_a = \text{floor}(K_B^2/r_m(q)T_0) \)
\[ \begin{align*}
A_{z1} &= \sum [K_B^2/r_m(q)] - T_d[r_m(q)] \cdot g[r_m(q)]/K_B^2/r_m(q)T_0 \\
A_{z3} &= \sum M A_{z2} \\
\end{align*} \]
end for
end if

\( A_z = A_{z1} + A_{z3} \)
print \( A_z \)
end procedure