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Fisher Concord: Efficiency of Quantum Measurement

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Abstract: By comparing measurement-induced classical Fisher information of parameterized quantum states with quantum Fisher information, we study the notion of Fisher concord (as abbreviation of the concord between the classical and the quantum Fisher information), which is an information-theoretic measure of quantum states and quantum measurements based on both classical and quantum Fisher information. Fisher concord is defined by multiplying the inverse square root of quantum Fisher information matrix to measurement-induced classical Fisher information matrix on both sides, and quantifies the relative accessibility of parameter information from quantum measurements (alternatively, the efficiency of quantum measurements in extracting parameter information). It reduces to the ratio of the classical Fisher information to quantum Fisher information in any single parameter scenario. In general, Fisher concord is a symmetric matrix which depends on both quantum states and quantum measurements. Some basic properties of Fisher concord are elucidated. The significance of Fisher concord in quantifying the interplay between classicality and quantumness in parameter estimation and in characterizing the efficiency of quantum measurements are illustrated through several examples, and some information conservation relations in terms of Fisher concord are exhibited.

Keywords: quantum measurement, quantum Fisher information, measurement-induced classical Fisher information, Fisher concord

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1 Introduction

The notion of Fisher information plays a pivotal role in parameter estimation and signal detection, in both the classical and the quantum scenarios [1–5]. In the classical scenario, the celebrated Crámer-Rao bound sets a fundamental limit to the precision of parameter estimation, and the limit is achievable via the maximum likelihood estimator in the asymptotic sense. However, in the quantum scenario, the situation changes radically due to interference between different optimal measurements for different parameters [3–5]. In quantum information theory, signals as parameters are mathematically described by numbers and physically encoded in quantum states (operators). To extract the signal information from parameterized quantum states, we have to perform quantum measurements on quantum states and process the data obtained from the outcome probabilities [3–21]. The fundamental difference between the classical and the quantum scenarios lies in that the operators representing various quantum states and quantum measurements are not commutative in general.

For parameterized quantum states, quantum Fisher information sets a fundamental bound to the estimation precision via the quantum Cramér-Rao inequality, which may not be achievable in multiparameter cases [3–5]. In this work, we will employ quantum Fisher information as a basic quantity of the total information about the parameter, including both accessible and inaccessible ones. The accessible one can be extracted by quantum measurements, while the inaccessible one cannot be extracted by any quantum measurement (i.e., has to be destroyed by quantum measurement). The quantum Fisher information will serve as a prior information and will be compared with measurement-induced classical Fisher information in assessing efficiency of information extraction via quantum measurement.

For parameterized quantum states with single parameter, when we perform quantum measurements on the states, the measurement-induced classical Fisher information quantifies the information amount that is accessible through quantum measurements. Taking optimization over all quantum measurements, we get the maximal accessible information. Braunstein and Caves proved that quantum Fisher information sets a fundamental upper bound to the measurement-induced classical Fisher information, and this bound is achievable [8]. That is, for the single parameter case, quantum Fisher information is just the maximal accessible information in the sense that we can find measurements such that its measurement-induced classical Fisher information equals to its quantum Fisher information. However, the optimal measurements may depend on the parameter, and thus are not in general uniform (i.e., not independent of the parameter).

Inspired by the notion of quantum discord [22–24], which is a measure of the quantumness of correlations defined by the difference between the total correlations (quantum mutual information) and the classical correlations (the correlations extractable via quantum measurements), we exploit the relation between quantum Fisher information (which quantifies the total information in parameters) and the measurement-induced classical Fisher information (which quantifies the accessible information in parameters) in order to quantify the efficiency of quantum measurements in extracting the parameter information. Instead of subtracting measurement-induced classical Fisher information from quantum Fisher information which leads to Fisher discord (i.e., Fisher information analogue of quantum discord), here we use a ratio of the former to the later to define a measure for relative classicality of Fisher information in parameterized quantum states, and call it Fisher concord, whose precise definition will be given in the next section. For the single parameter case, Fisher concord is a dimensionless quantity lying in the unit interval, achieving the maximal value 1 when the measurement-induced classical Fisher information equals the quantum Fisher information, i.e., when these two quantities of Fisher information are in maximal concord, and achieves the minimal value 0 when these two quantifies of Fisher information are in minimal concord (maximal discord). Its complement (i.e., 1 minus Fisher concord) quantifies the intrinsic loss of Fisher information in quantum measurement and may also be regarded essentially as a kind of Fisher discord.

For the single parameter scenario, by the elegant Braunstein-Caves theorem [8], Fisher concord can indeed achieve the maximal value 1. That is, the parameter information encoded in the states can be fully extracted via optimal quantum measurements (which may depend on the parameter) as precise as possible.

For the multi-parameter scenario, both quantum Fisher information and measurement-induced classical Fisher information are matrices. For each parameter, the quantum Cramér-Rao bound can be saturated asymptotically. However, due to the noncommutativity of different optimal measurements for different parameters, in general, there does not exist a uniform measurement that can fully extract the information about all parameters simultaneously [3, 12, 13]. We have to consider the interference between different measurements for different parameters. Inspired by the idea of conditional density operators [26–28], we may interpret Fisher concord as a kind of conditional density operator, with the prior being the quantum Fisher information. The off-diagonal elements in the matrix of Fisher concord encode the interference between different parameters.

The remainder of the article is organized as follows. In Section 2, we investigate Fisher concord and discuss its basic properties. In Section 3, we illustrate Fisher concord through several examples including pure qubit states, mixed qubit states with three different kinds of parameterizations, and two-qubit states. Then we derive several information conservation relations in terms of Fisher concord. Finally, we conclude with discussion in Section 4.

2 Fisher Concord

To motivate our approach to quantifying accessibility of parameter information via quantum measurements in terms of Fisher information, let us first recall quantum discord [22–25]. Consider a bipartite state ρ^{ab} shared by two parties a and b with reduced states $\rho^a = \mathrm{tr}_b\rho^{ab}$, $\rho^b = \mathrm{tr}_a\rho^{ab}$, the total amount of correlations in it are well quantified by the quantum mutual information [29–32]

$$I(\rho^{ab}) = S(\rho^a) + S(\rho^b) - S(\rho^{ab})$$

where $S(\rho^a) = -\mathrm{tr}\rho^a\log\rho^a$ is the von Neumann entropy. However, the quantum mutual information may not be fully extractable via quantum measurement. The amount of extractable correlations via the von Neumann measurement $\Pi^b = \{\Pi_i^b\}$ on party b is the classical correlations defined as $I(\Pi^b(\rho^{ab}))$, where

$$\Pi^b(\rho^{ab}) = \sum_i (\mathbf{1}^a \otimes \Pi_i^b)\rho^{ab}(\mathbf{1}^a \otimes \Pi_i^b)$$

is the post-measurement state. Now the measurement-dependent discord

$$D(\rho^{ab}|\Pi^b) = I(\rho^{ab}) - I(\Pi^b(\rho^{ab}))$$

is the difference between the total correlations and the classical correlations, and quantifies the inevitable loss of correlations caused by the quantum measurements. The quantum discord is further defined as the minimal discord $D(\rho^{ab}) = \min_{\Pi^b} D(\rho^{ab}|\Pi^b)$ where the minimization is over all von Neumann measurements Π^b over party b. In this paper, we will only be concerned with the measurement-dependent discord $D(\rho^{ab}|\Pi^b)$.

In the parameter estimation scenario, the quantum Fisher information resembles the quantum mutual information, while the measurement-induced classical Fisher information resembles the classical correlations. To make this analogy more explicit, consider the parameterized quantum states $\rho(\theta)$, which depend on parameters $\theta = (\theta_1, \ldots, \theta_p)$. Recall that quantum Fisher information (matrix) of parameterized states $\rho = \rho(\theta)$ (we suppress θ for later convenience) is the $n \times n$ real symmetric matrix $\mathbf{Q}(\rho) = (Q_{ij})$ with matrix elements Q_{ij} defined as [4, 5]

$$Q_{ij} = \frac{1}{2} \operatorname{tr} \rho (L_i L_j + L_j L_i).$$

Here the symmetric logarithmic derivative operators L_i for the parameter θ_i are determined implicitly by

$$\frac{\partial}{\partial \theta_i} \rho = \frac{1}{2}(L_i \rho + \rho L_i).$$

Quantum Fisher information sets a fundamental upper bound to the estimation precision via the celebrated Cramér-Rao inequality [3–5], and can be regarded as a measure to quantify the total information concerning the parameters $\theta = (\theta_1, \ldots, \theta_p)$ encoded in the states $\rho = \rho(\theta)$.

To extract the parameter information, we have to perform quantum measurements on the states. A quantum measurement M is described by a positive-operator-valued measure (POVM) such that $M = \{M_l | M_l \geq 0, \sum_l M_l = \mathbf{1}\}$. If we perform quantum measurement M on the states $\rho(\theta)$, then a parameterized classical probability $p_l = p_l(\theta)$ arises with

$$p_l(\theta) = \operatorname{tr} \rho M_l.$$

For this family of measurement-induced classical probability distributions, we have the classical Fisher information (matrix) $\mathbf{C}(\rho|M) = (C_{ij})$ which is also a $p \times p$ real symmetric matrix with matrix elements [1, 3, 8]

$$C_{ij} = \sum_l p_l \frac{\partial \ln p_l}{\partial \theta_i} \frac{\partial \ln p_l}{\partial \theta_j} = \sum_l \frac{1}{p_l} \frac{\partial p_l}{\partial \theta_i} \frac{\partial p_l}{\partial \theta_j}.$$

The Braunstein-Caves theorem states that [8]

$$\mathbf{C}(\rho|M) \leq \mathbf{Q}(\rho), \tag{1}$$

Furthermore, when $p = 1$, this bound is saturated after optimizing over all POVMs. In contrast to the quantum Fisher information as a measure of total information for the parameters, the measurement-induced classical Fisher information can be naturally interpreted as the accessible information of the parameters information via the POVM M.

A natural question arises: How to quantify the efficiency of quantum measurements in extracting the parameter? Or alternatively, how to quantify the information loss caused by the quantum measurements? In the single parameter case for which both the classical Fisher information and quantum Fisher information are numbers, motivated by quantum discord, it is tempting to take $\mathbf{Q} - \mathbf{C}$ or $\mathbf{C}\mathbf{Q}^{-1}$ as candidates. Here we choose the latter, a dimensionless quantity, to quantify the efficiency of quantum measurements.

For the single parameter scenario, when $\mathbf{Q}(\rho) \neq 0$, by dividing $\mathbf{Q}(\rho)$ from both sides of equation (1) directly, we have

$$\frac{\mathbf{C}(\rho|M)}{\mathbf{Q}(\rho)} \leq 1. \tag{2}$$

By use of the ratio of $\mathbf{C}(\rho|M)$ to $\mathbf{Q}(\rho)$, we can assign a rate to any POVM M, which signifies the relative efficiency to extract the parameter information from the quantum states. In this way, the power of different measurements is comparable. For example, the measurement N that satisfies $\mathbf{C}(\rho|N)/\mathbf{Q}(\rho) = 1/2$ has only half the power of the measurement M with $\mathbf{C}(\rho|M)/\mathbf{Q}(\rho) = 1$. If we regard quantum Fisher information as a representative of prior information of this parameter encoded in the states, then the ratio $\mathbf{C}(\rho|M)/\mathbf{Q}(\rho)$ may be interpreted as a kind of conditional "probability".

When passing to the multi-parameter scenario, quantum Fisher information plays the role of prior "density operator" (we ignore the unit trace condition of density operator). The problem is how to incorporate this prior information into a quantity of measurement efficiency. In general, the naive expression of $\mathbf{C}(\rho|M)\mathbf{Q}^{-1}(\rho)$ is not Hermitian. As inspired by different quantum extensions of conditional probability [26–28], to obtain a Hermitian matrix, we modify the single parameter case by splitting $\mathbf{Q}^{-1}(\rho)$ into two equal parts and putting them on both sides of $\mathbf{C}(\rho|M)$ in a symmetric fashion, that is, $\mathbf{Q}^{-\frac{1}{2}}(\rho)\mathbf{C}(\rho|M)\mathbf{Q}^{-\frac{1}{2}}(\rho)$. Of course, another version is $\mathbf{C}^{\frac{1}{2}}(\rho|M)\mathbf{Q}^{-1}(\rho)\mathbf{C}^{\frac{1}{2}}(\rho|M)$. In this work, we choose $\mathbf{Q}^{-\frac{1}{2}}(\rho)\mathbf{C}(\rho|M)\mathbf{Q}^{-\frac{1}{2}}(\rho)$ as our candidate. Consequently, we come to an information measure depending on a parameterized states $\rho = \rho(\theta)$ and a POVM M performed on the states:

$$\mathbf{F}(\rho|M) := \mathbf{Q}^{-\frac{1}{2}}(\rho)\mathbf{C}(\rho|M)\mathbf{Q}^{-\frac{1}{2}}(\rho). \tag{3}$$

We call it Fisher concord, which encodes the difference between quantum Fisher information and measurement-induced classical Fisher information. If the quantum Fisher information matrix is degenerate, then there must exist some parameters whose quantum Fisher information is zero, which means that these parameter information can not be encoded into this states. Therefore, we just omit these parameters, and only consider the case that the quantum Fisher information is invertible. Note that Zhu has studied such a kind of quantity as the metric-adjusted complementarity chamber from a different perspective in Ref. [33]. In this work, we focusing on the interpretation of this quantity as a concord of Fisher information and a measure of efficiency for quantum measurements, and investigate its fundamental properties.

Fisher concord enjoys the following properties:

(a) When $p = 1$ (i.e., single parameter case), $\mathbf{F}(\rho|M)$ is a scale, and ranges from 0 to 1. For different POVMs and the same states $\rho(\theta)$, the larger $\mathbf{F}(\rho|M)$ is, the larger efficiency the POVM M has in extracting the parameter information from $\rho(\theta)$. If $\mathbf{F}(\rho|M) = 1$, the POVM M has the maximal capability.

(b) (Monotonicity) If the measurement N refines the measurement M, then

$$\mathbf{F}(\rho|M) \leq \mathbf{F}(\rho|N). \tag{4}$$

That is, the refined measurement N has larger efficiency in extracting parameter information from the same states ρ than the measurement M.

(c) (Nontrivial bound of its trace) There is a general nontrivial bound for the trace of Fisher concord:

$$\operatorname{tr}\mathbf{F}(\rho|M) \leq \min\{p, d-1\}.$$

Here d is the dimension of the system space of the parameterized states $\rho(\theta)$.

We now outline the reasoning leading to the above results.

(a) This follows directly from equation (2) and the non-negativity of classical Fisher information and quantum Fisher information [34].

(b) Let $N = \{N_{l,\eta}\}$ be a refinement of the measurement $M = \{M_l\}$ in the sense that $M_l = \sum_\eta N_{l,\eta}$. The outcomes of M are $\{l\}$, while the outcomes of N are $\{(l, \eta)\}$. Each outcome l for M is further split into several outcomes $\{(l, \eta)\}$ indexed by (l, η) in the measurement N. The completeness relation for the measurement M is $\sum_l M_l = \mathbf{1}$, while that for the refined measurement N is $\sum_{l,\eta} N_{l,\eta} = \mathbf{1}$.

Because of the monotonicity of quantum Fisher information [34], we know that the measurement-induced classical Fisher information increases when measurement is refined, that is, if the measurement N refines the measurement M, then

$$\mathbf{C}(\rho|M) \leq \mathbf{C}(\rho|N). \tag{5}$$

Since $\mathbf{Q}^{-\frac{1}{2}}(\rho)$ is positive definite, by multiplying $\mathbf{Q}^{-\frac{1}{2}}(\rho)$ to both sides of the inequality (5) from both the left-hand side and the left-hand side, we get equation (4), the monotonicity of Fisher concord.

(c) By multiplying $\mathbf{Q}^{-\frac{1}{2}}(\rho)$ on both sides of inequality (1), we have

$$\mathbf{F}(\rho|M) = \mathbf{Q}^{-\frac{1}{2}}(\rho)\mathbf{C}(\rho|M)\mathbf{Q}^{-\frac{1}{2}}(\rho) \leq \mathbf{1}_p.$$

Here p is the parameter number, and $\mathbf{1}_p$ is the identity matrix on the p-dimensional Hilbert space. Hence the trace of Fisher concord has a trivial bound:

$$\begin{aligned}\operatorname{tr}\mathbf{F}(\rho|M) &= \operatorname{tr}\mathbf{Q}^{-\frac{1}{2}}(\rho)\mathbf{C}(\rho|M)\mathbf{Q}^{-\frac{1}{2}}(\rho)\\ &= \operatorname{tr}\mathbf{Q}^{-1}(\rho)\mathbf{C}(\rho|M) \leq p.\end{aligned} \tag{6}$$

On the other hand, form an result of Gill and Massar [35], we have the following nontrivial bound for the trace of Fisher concord:

$$\operatorname{tr}\mathbf{F}(\rho|M) = \operatorname{tr}\mathbf{Q}^{-1}(\rho)\mathbf{C}(\rho|M) \leq d-1, \tag{7}$$

with d the dimension of the quantum state space.

Combining inequalities (6) and (7), we obtain a general bound for the trace of Fisher concord:

$$\operatorname{tr}\mathbf{F}(\rho|M) \leq \min\{p, d-1\}. \tag{8}$$

For any mixed density operators in d-dimensional Hilbert space, the number of necessary parameters to characterize the states is $p = d^2 - 1$, and for pure states, $p = 2d-2$. For this kind of parameterized states, the bound (7) is obviously nontrivial, and Gill and Massar proved that for any POVM M, inequality (7) could be saturated if and only if all the POVM elements M_l are rank one, and satisfy $\operatorname{tr}\rho M_l \neq 0$ [35, 36].

3 Qubit Systems and Information Conservation in Terms of Fisher Concord

In this section, we evaluate Fisher concord explicitly for single as well as two-qubit systems, which are important building blocks for quantum information processing. From these results we gain a more intuitive understanding of various features of Fisher concord.

For qubit systems, we can parameterize any rank-one POVM, which includes the optimal measurement for Gill-Masaar bound [35]. Therefore, we will calculate Fisher concord with respect to any rank-one POVM for pure qubit states, mixed qubit states with three different kinds of parameterizations. However, for two-qubit system, it is difficult to write down any POVM. So we only study two special POVMs investigated in Ref. [21] to calculate the Fisher concord for parallel spins and antiparallel spins, respectively, and the Bell measurement for Bell-diagonal states around maximally mixed states.

3.1 Single qubit systems

We start from pure qubit states. For a pure qubit state $\rho = |\mathbf{n}\rangle\langle\mathbf{n}|$ with

$$|\mathbf{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

in the standard basis with amplitude parameter $\theta \in [0, \pi]$ and phase parameter $\phi \in [0, 2\pi)$, the quantum Fisher information can be evaluated as

$$\mathbf{Q}(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$

which is a diagonal matrix. Consequently, $Q_\theta(\rho) = 1$ and $Q_\phi(\rho) = \sin^2\theta$.

To evaluate the measurement-induced classical Fisher information, we have to write down the POVMs on qubit system. Since rank-one POVMs are the most refined POVM for qubit systems and can saturate the Gill-Massar inequality (7), we only consider these POVMs. Any rank-one POVM $M = \{M_l\}$ on a qubit can be parameterized as [37]

$$M_l = a_l \begin{pmatrix} \cos^2\frac{\theta_l}{2} & e^{-i\phi_l}\cos\frac{\theta_l}{2}\sin\frac{\theta_l}{2} \\ e^{i\phi_l}\cos\frac{\theta_l}{2}\sin\frac{\theta_l}{2} & \sin^2\frac{\theta_l}{2} \end{pmatrix}, \quad (9)$$

with $a_l \in (0, 1]$, $\theta_l \in [0, \pi]$, and $\phi_l \in [0, 2\pi)$. In terms of these parameters, the completeness condition reduces to

$$\sum_l a_l = 2, \quad (10a)$$

$$\sum_l a_l \cos\theta_l = 0, \quad (10b)$$

$$\sum_l a_l \sin\theta_l \sin\phi_l = 0, \quad (10c)$$

$$\sum_l a_l \sin\theta_l \cos\phi_l = 0. \quad (10d)$$

The probability p_l of the outcome labeled by l after performing the POVM M on the states ρ is

$$\begin{aligned} p_l &= \mathrm{tr}\rho M_l \\ &= \frac{a_l}{2}\left(1 + \cos\theta\cos\theta_l + \sin\theta\sin\theta_l\cos(\phi_l - \phi)\right). \end{aligned}$$

We denote the parameter (θ, ϕ) by (θ_1, θ_2) for later convenience. By definition, the first diagonal element of the measurement-induced classical Fisher information $\mathbf{C}(\rho|M) = (C_{ij})$ is just the classical Fisher information of the probability distribution $\{p_l\}$ with respect to the parameter θ:

$$\begin{aligned} C_{11} &= C_\theta(\rho|M) \\ &= \sum_l \frac{a_l^2}{4p_l}(\sin\theta\cos\theta_l - \cos\theta\sin\theta_l\cos(\phi_l - \phi))^2. \end{aligned}$$

The other diagonal element is the classical Fisher information with respect to the parameter ϕ:

$$C_{22} = C_\phi(\rho|M) = \sum_l \frac{a_l^2}{4p_l}\sin^2\theta\sin^2\theta_l\sin^2(\phi_l - \phi).$$

Hence, we have the Fisher concord $\mathbf{F}(\rho|M) = (F_{ij})$ of the pure qubit state $\rho = |\mathbf{n}\rangle\langle\mathbf{n}|$ after performing the POVM M, with matrix elements

$$F_{11} = \sum_l \frac{a_l^2}{4p_l}(\sin\theta\cos\theta_l - \cos\theta\sin\theta_l\cos(\phi_l - \phi))^2,$$

$$F_{22} = \sum_l \frac{a_l^2}{4p_l}\sin^2\theta_l\sin^2(\phi_l - \phi),$$

$$F_{12} = F_{21} = 0.$$

By use of the completeness conditions (10), we obtain for any rank-one POVM M,

$$\mathrm{tr}\mathbf{F}(\rho|M) = \frac{C_\theta(\rho|M)}{Q_\theta(\rho)} + \frac{C_\phi(\rho|M)}{Q_\phi(\rho)} = 1, \quad (11)$$

which is consistent with the Gill-Massar result (7). This exhibits a complementary relation between the two parameters θ and ϕ and the conservation of Fisher concord: The Fisher concord for θ and that for ϕ sum to one. For the optimal POVM M with respect to the parameter θ, we have $C_\theta(\rho|M)/Q_\theta(\rho) = 1$, which means that the information about θ can be fully extracted. However, in this situation, we can gain nothing about the other parameter ϕ in the sense that $C_\phi(\rho|M)/Q_\phi(\rho) = 0$. This can be regarded as a kind of uncertainty relation with respective to different parameters. Furthermore, this tradeoff relation (11) holds for any rank-one POVM. Thus typically different measurements could be chosen in order to choose parameter which should be estimated with larger precision at the expense of the other one.

Now consider a general mixed qubit state

$$\sigma = \frac{1}{2}\begin{pmatrix} 1 + r\cos\theta & re^{-i\phi}\sin\theta \\ re^{i\phi}\sin\theta & 1 - r\cos\theta \end{pmatrix}$$

with $r \in [0, 1]$ being the Bloch vector norm (purity). We denote the parameter (r, θ, ϕ) by $(\theta_1, \theta_2, \theta_3)$ for later convenience. The quantum Fisher information can be directly evaluated as

$$\mathbf{Q}(\sigma) = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & \frac{1}{1-r^2} \end{pmatrix}.$$

Using the parametrization of any rank-one POVM M specified in equation (9), we have the Fisher concord $\mathbf{F}(\sigma|M) = (F_{ij})$ with matrix elements

$$F_{11} = \sum_l \frac{a_l}{2} \frac{(\sin\theta\cos\theta_l - \cos\theta\sin\theta_l\cos(\phi_l - \phi))^2}{1 + r\cos\theta\cos\theta_l + r\sin\theta\sin\theta_l\cos(\phi_l - \phi)},$$
$$F_{22} = \sum_l \frac{a_l}{2} \frac{\sin^2\theta_l \sin^2(\phi_l - \phi)}{1 + r\cos\theta\cos\theta_l + r\sin\theta\sin\theta_l\cos(\phi_l - \phi)},$$
$$F_{33} = \sum_l \frac{a_l}{2} \frac{(1-r^2)(\cos\theta\cos\theta_l + \sin\theta\sin\theta_l\cos(\phi_l - \phi))^2}{1 + r\cos\theta\cos\theta_l + r\sin\theta\sin\theta_l\cos(\phi_l - \phi)},$$
$$F_{ij} = 0, \quad i \neq j.$$

By use of the completeness conditions (10), we have the Fisher concord conservation relation

$$\mathrm{tr}\mathbf{F}(\sigma|M) = \frac{C_\theta(\sigma|M)}{Q_\theta(\sigma)} + \frac{C_\phi(\sigma|M)}{Q_\phi(\sigma)} + \frac{C_r(\sigma|M)}{Q_r(\sigma)} = 1,$$

which demonstrates the tradeoff relation between estimating different parameters.

Consider the same state as the above example, but parameterized differently by amplitude θ, phase ϕ, and phase diffusion δ, as investigated in Ref. [37],

$$\tau = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi-\delta^2}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi-\delta^2}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}.$$

We denote the parameter (θ, ϕ, δ) by $(\theta_1, \theta_2, \theta_3)$ and get the quantum Fisher information

$$\mathbf{Q}(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2\delta^2}\sin^2\theta & 0 \\ 0 & 0 & \frac{4\delta^2\sin^2\theta}{e^{2\delta^2}-1} \end{pmatrix}.$$

For any rank-one POVM M expressed as equation (9), we have the Fisher concord $\mathbf{F}(\tau|M) = (F_{ij})$ with matrix elements

$$F_{11} = \sum_l \frac{a_l}{2} \frac{(-\sin\theta\cos\theta_l + e^{-\delta^2}\cos\theta\sin\theta_l\cos(\phi_l - \phi))^2}{1 + \cos\theta\cos\theta_l + e^{-\delta^2}\sin\theta\sin\theta_l\cos(\phi_l - \phi)},$$
$$F_{22} = \sum_l \frac{a_l}{2} \frac{\sin^2\theta_l\sin^2(\phi_l - \phi)}{1 + \cos\theta\cos\theta_l + e^{-\delta^2}\sin\theta\sin\theta_l\cos(\phi_l - \phi)},$$
$$F_{33} = \sum_l \frac{a_l}{2} \frac{(1 - e^{-2\delta^2})\sin^2\theta_l\cos^2(\phi_l - \phi)}{1 + \cos\theta\cos\theta_l + e^{-\delta^2}\sin\theta\sin\theta_l\cos(\phi_l - \phi)},$$
$$F_{ij} = 0, \quad i \neq j.$$

By use of the completeness conditions (10), we have the following conservation of Fisher concord for different parameters

$$\mathrm{tr}\mathbf{F}(\tau|M) = \frac{C_\theta(\tau|M)}{Q_\theta(\tau)} + \frac{C_\phi(\tau|M)}{Q_\phi(\tau)} + \frac{C_r(\tau|M)}{Q_r(\tau)} = 1.$$

In addition to the above parameterizations of a general qubit state, we further consider the parametrization involving the Bloch vector (x, y, z) satisfying $r^2 = x^2 + y^2 + z^2 \leq 1$:

$$\gamma = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}.$$

Noting that here (x, y, z) plays the role of $(\theta_1, \theta_2, \theta_3)$, we get the quantum Fisher information

$$\mathbf{Q}(\gamma) = \mathbf{1} + \frac{1}{1-r^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}(x, y, z),$$

which is not diagonal. By straightforward calculation, we have

$$\mathbf{Q}^{-\frac{1}{2}}(\gamma) = \mathbf{1} - \frac{1-\sqrt{1-r^2}}{r^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}(x, y, z).$$

For any rank-one POVM M as expressed in equation (9), the outcome probabilities are

$$q_l = \frac{a_l}{2}(1 + z\cos\theta_l + x\sin\theta_l\cos\phi_l + y\sin\theta_l\sin\phi_l).$$

The matrix elements of the measurement-induced classical Fisher information $\mathbf{C}(\rho|M) = (C_{ij})$ can be evaluated as

$$C_{11} = \sum_l \frac{a_l^2}{4q_l}\sin^2\theta_l\cos^2\phi_l, \quad C_{22} = \sum_l \frac{a_l^2}{4q_l}\sin^2\theta_l\sin^2\phi_l,$$
$$C_{12} = \sum_l \frac{a_l^2}{4q_l}\sin^2\theta_l\sin\phi_l\cos\phi_l,$$
$$C_{13} = \sum_l \frac{a_l^2}{4q_l}\sin\theta_l\cos\theta_l\cos\phi_l,$$
$$C_{23} = \sum_l \frac{a_l^2}{4q_l}\sin\theta_l\cos\theta_l\sin\phi_l, \quad C_{33} = \sum_l \frac{a_l^2}{4q_l}\cos^2\theta_l.$$

Then we can calculate the Fisher concord $\mathbf{F}(\gamma|M)$ directly. We omit the tedious expressions. It is remarkable that even though the quantum Fisher information is not diagonal, we still get the following information conservation relation

$$\mathrm{tr}\mathbf{F}(\gamma|M) = 1.$$

3.2 Two-qubit systems

Now we turn to two-qubit systems, We first consider parallel spins $|\mathbf{n},\mathbf{n}\rangle$ with $|\mathbf{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$, which can be represented in vector form

$$|\mathbf{n},\mathbf{n}\rangle = \begin{pmatrix} \cos^2\frac{\theta}{2} \\ \frac{1}{2}e^{i\phi}\sin\theta \\ \frac{1}{2}e^{i\phi}\sin\theta \\ e^{2i\phi}\sin^2\frac{\theta}{2} \end{pmatrix}.$$

under the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Put $\rho_{\|} = |\mathbf{n},\mathbf{n}\rangle\langle\mathbf{n},\mathbf{n}|$, the quantum Fisher information can be directly calculated as

$$\mathbf{Q}(\rho_{\|}) = \begin{pmatrix} 2 & 0 \\ 0 & 2\sin^2\theta \end{pmatrix}.$$

Since it is difficult to parameterize all the POVMs (even rank-one POVMs) on two-qubit systems, we consider some special but important POVMs as investigated in Refs. [11, 21]. One of them is the von Neumann measurement $\Phi = \{|\Phi_l\rangle : l = 1, 2, 3, 4\}$ with mutually orthogonal projectors [21]

$$|\Phi_l\rangle = \frac{\sqrt{3}}{2}|\mathbf{n}_l,\mathbf{n}_l\rangle + \frac{1}{2}|\Psi^-\rangle, \qquad (12)$$

where $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ denotes the singlet state, and

$$|\mathbf{n}_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\mathbf{n}_1\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix},$$

$$|\mathbf{n}_2\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 \\ \frac{-1+i\sqrt{3}}{\sqrt{2}} \end{pmatrix}, \quad |\mathbf{n}_3\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix} -1 \\ \frac{1+i\sqrt{3}}{\sqrt{2}} \end{pmatrix},$$

with the corresponding Bloch vectors

$$\begin{aligned} \mathbf{n}_0 &= (0, 0, 1), \\ \mathbf{n}_1 &= \frac{1}{3}(\sqrt{8}, 0, -1), \\ \mathbf{n}_2 &= -\frac{1}{3}(\sqrt{2}, -\sqrt{6}, 1), \\ \mathbf{n}_3 &= -\frac{1}{3}(\sqrt{2}, \sqrt{6}, 1) \end{aligned}$$

actually pointing to the four vertices of a tetrahedron. Note that the phases of $|\mathbf{n}_l\rangle$ are so chosen such that $|\Phi_l\rangle$ defined by equation (12) are mutually orthogonal.

The measurement-induced classical Fisher information $\rho_{\|}$ with respect to the parameters θ and ϕ can be directly evaluated as

$$C_\theta(\rho_{\|}|\Phi) = \sum_l \frac{1}{u_l}\left(\frac{\partial u_l}{\partial\theta}\right)^2 = 1,$$

$$C_\phi(\rho_{\|}|\Phi) = \sum_l \frac{1}{u_l}\left(\frac{\partial u_l}{\partial\phi}\right)^2 = \sin^2\theta,$$

respectively. Consequently, Fisher concord of the parallel spins $\rho_{\|}$ is

$$\mathbf{F}(\rho_{\|}|\Phi) = \mathbf{Q}^{-\frac{1}{2}}(\rho_{\|})\,\mathbf{C}(\rho_{\|}|\Phi)\mathbf{Q}^{-\frac{1}{2}}(\rho_{\|}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies that

$$\operatorname{tr}\mathbf{F}(\rho_{\|}|\Phi) = 1.$$

As we can see, the POVM Φ has the same efficiency in extracting information from the two parameters θ and ϕ from the states $\rho_{\|}$.

Next we consider antiparallel spins

$$|\mathbf{n},-\mathbf{n}\rangle = \begin{pmatrix} -\frac{1}{2}\sin\theta \\ e^{i\phi}\cos^2\frac{\theta}{2} \\ -e^{i\phi}\sin^2\frac{\theta}{2} \\ \frac{1}{2}e^{2i\phi}\sin\theta \end{pmatrix}$$

expressed in the standard base $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Let $\rho_\perp = |\mathbf{n},-\mathbf{n}\rangle\langle\mathbf{n},-\mathbf{n}|$. The quantum Fisher information of the antiparallel spins can be evaluated as

$$\mathbf{Q}(\rho_\perp) = \begin{pmatrix} 2 & 0 \\ 0 & 2\sin^2\theta \end{pmatrix}.$$

For an alternative von Neumann measurement $\Psi = \{|\Psi_l\rangle : l = 1, 2, 3, 4\}$ with [21]

$$|\Psi_l\rangle = \frac{\sqrt{3}}{2}|\Omega_l\rangle + \frac{1}{2}|\Psi^-\rangle, \qquad (13)$$

where $|\Omega_l\rangle = (|\mathbf{n}_l,-\mathbf{n}_l\rangle + |-\mathbf{n}_l,\mathbf{n}_l\rangle)/\sqrt{2}$, and $|\mathbf{n}_l\rangle$ are the same as before, while

$$|-\mathbf{n}_0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |-\mathbf{n}_1\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix},$$

$$|-\mathbf{n}_2\rangle = \frac{-i}{\sqrt{3}}\begin{pmatrix} \frac{1+i\sqrt{3}}{\sqrt{2}} \\ 1 \end{pmatrix}, \quad |-\mathbf{n}_3\rangle = \frac{i}{\sqrt{3}}\begin{pmatrix} \frac{1-i\sqrt{3}}{\sqrt{2}} \\ 1 \end{pmatrix},$$

which guarantee that $|\Psi_l\rangle$ defined by equation (13) are mutually orthogonal. The measurement-induced classical Fisher information of $\rho_\perp$ with respect to the parameter θ and ϕ can be evaluated as

$$C_\theta(\rho_\perp|\Psi) = 2, \quad C_\phi(\rho_\perp|\Psi) = 2\sin^2\theta.$$

The Fisher concord can be directly calculated as

$$\mathbf{F}(\rho_\perp|\Psi) = \mathbf{Q}^{-\frac{1}{2}}(\rho_\perp)\mathbf{C}(\rho_\perp|\Psi)\mathbf{Q}^{-\frac{1}{2}}(\rho_\perp) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The POVM Ψ has the maximal efficiency for extracting the information of the parameters θ and ϕ simultaneously. In this case,

$$\operatorname{tr}\mathbf{F}(\rho_\perp|\Psi) = 2.$$

Thus all the $p = 2$ parameters can be estimated at their quantum limit, which stands in sharp contrast to the parallel spins case.

Finally, we consider the Bell-diagonal states of two qubits which have the density operators

$$\rho^{ab} = \frac{1}{4}\Big(\mathbf{1} + \sum_{i=1}^{3} \theta_i \sigma_i \otimes \sigma_i\Big), \tag{14}$$

where σ_i are the three Pauli matrices, and $\theta = (\theta_1, \theta_2, \theta_3)$ is the parameter vector.

Now we regard Bell diagonal states (14) as a kind of parametrization around the point $\theta = (0, 0, 0)$, and we want to know the optimal Fisher concord at this center point.

By direct calculations, we know the quantum Fisher information at this point is the identity matrix $\mathbf{Q}(\rho^{ab}) = \mathbf{1}$. For the Bell measurement $\beta = \{|\beta_{kl}\rangle : k, l = 0, 1\}$ with $|\beta_{kl}\rangle := \Big(|0, l\rangle + (-1)^k |1, 1 \oplus l\rangle\Big)/\sqrt{2}$ being the eigenvectors of the Bell-diagonal states, the measurement-induced classical Fisher information with respect to the three parameters θ_i are $C_{ii} = 1$ for $i = 1, 2, 3$. Hence, the Fisher concord of the Bell-diagonal states at the origin $\theta = (0, 0, 0)$ can be directly calculated as

$$\mathbf{F}(\rho^{ab}|\beta) = \mathbf{Q}^{-\frac{1}{2}}(\rho^{ab})\mathbf{C}(\rho^{ab}|\beta)\mathbf{Q}^{-\frac{1}{2}}(\rho^{ab}) = \mathbf{1}.$$

The Bell measurement has the maximal efficiency for all the parameters $\theta_1, \theta_2, \theta_3$ simultaneously at the point $\theta = (0, 0, 0)$. Correspondingly,

$$\mathrm{tr}\mathbf{F}(\rho^{ab}|\beta) = 3.$$

The trace of Fisher concord at the point $\theta = (0, 0, 0)$ achieves its maximal value, and all the $p = 3$ parameters can be estimated at their quantum limit.

4 Conclusions

Motivated by quantum discord and conditional density operator, we have introduced the notion of Fisher concord, which is a measure of efficiency of quantum measurement in extracting parameter information. It is defined by comparing measurement-induced classical Fisher information with quantum Fisher information, and depends on both the parameterized states and the quantum measurement performed on the states. In contrast to quantum discord, which is intended to quantify the quantumness of correlations in bipartite states, Fisher concord quantifies the relative accessibility of Fisher information, and as such, it is inversely related to the discord measure. Large Fisher concord implies small interference between measurements for different parameters. Several examples are explicitly analyzed in terms of Fisher concord, which exhibit interesting conservation relations of Fisher concord.

In summary, Fisher concord plays a dual role: Firstly, with focus on quantum states, it quantifies the accessible Fisher information in a relative fashion with the quantum Fisher information as the prior. In this sense, it is a kind of conditional density operator. Secondly, with focus on quantum measurements, it quantifies the efficiency of quantum measurement in extracting parameter information, and summarizes uncertainty relation from an informational perspective. It may shed light on the measurement-disturbance tradeoff relations [38–46] and may be a useful notion in quantum metrology [47–61].

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