Research Article

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Influence of coupling effects on analytical solutions of functionally graded (FG) spherical shells of revolution

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Abstract: Due to the lack of commercially available finite elements packages allowing us to analyse the behaviour of porous functionally graded (FG) structures in this paper, axisymmetric deformations of coupled FG spherical shells are studied. The analytical solution is presented by using complex hypergeometric polynomial series. The results presented agree closely with the reference results for isotropic spherical shells of revolution. The influence of the effects of material properties is characterized by a multiplier characterizing an unsymmetric shell wall construction (stiffness coupling). The results can be easily adopted in design procedures. The present results can be treated as the benchmark for finite element investigations.

Keywords: spherical shells, analytical solutions, porous functionally graded materials, construction of material properties

1 Introduction

The analytical description of axisymmetric isotropic shells of revolution is demonstrated in monographs [1–3]. Galletly and Muc [4,5] and Muc [6] studied buckling and deformations of laminated shells of revolution considering only laminated unidirectional symmetric configurations. Due to coupling effects between the membrane and bending stress resultants, the analysis of arbitrarily laminated shells was usually carried out using numerical methods, e.g. Czebyshev expansions [7].

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Considering the structural deformations in only the elastic range, the analysis of composite constructions can be divided into three categories:

- Unidirectional laminates – the potential abrupt variation of mechanical properties between laminae.
- Porous functionally graded materials (FGM) – a smooth variation of the properties from the bottom to the top surface.
- Nanostructures reinforcing isotropic matrix – they are made of a polymer matrix reinforced with nanoplatelets or carbon nanotubes; the material properties are derived with the use of homogenization theories, e.g. the Mori–Tanaka method.

Due to the possible variations of material characteristics along the spatial coordinates (x, y, z – the thickness direction), it is rather difficult or impossible to find analytical solutions characterizing structural deformations even for the simplest shells as, e.g. cylindrical, conical, spherical, etc. The variations along the thickness direction z can be eliminated by the 2D approximations (power series, Legendre, Czebyshev polynomials – Muc [8], Fantuzzi et al. [9]). Considering two geometrical variables ($x = R\varphi, y = R\theta$, where R is the radius of the sphere, φ the meridional coordinate, and θ the circumferential coordinate) defining the variations along the shell reference midsurface for axisymmetric shell constructions, it is possible to formulate and solve three independent problems: (1) shells closed at the vertex, (2) shells opened at the vertex and (3) shell panels (Figure 1).

For the 2D Love–Kirchhoff kinematical hypothesis, the stress–strain relations can be expressed in the following way: $N_i = A_{ij}\varepsilon_j + B_{ij}\kappa_j$, $M_i = B_{ij}\varepsilon_j + D_{ij}\kappa_j$, where N denotes the membrane forces, M the bending moments, ε the membrane strains and κ the changes in curvatures. Each of the above values has three components. A, B and D denotes the matrices $[3 \times 3]$, and their explicit form will be presented in the next section.

For anisotropic shell structures (unidirectional laminates, nanoplates and nanotubes embedded in a matrix),

Figure 1: Geometry of the spherical shells: (a) hemispherical shells $0 \leq \varphi \leq \pi/2$, $0 \leq \theta \leq 2\pi$, (b) shells opened at the vertex $0 < \varphi_1 \leq \varphi \leq \varphi_2 < \pi/2$, $0 \leq \theta \leq 2\pi$, and (c) shell panels $0 < \varphi_1 \leq \varphi \leq \varphi_2 < \pi/2$, $0 < \theta_1 \leq \theta \leq \theta_2 < 2\pi$.

the terms of the stiffness matrices A , B and D are functions of two global coordinates x and y and define the structural configuration along the thickness coordinate z . Considering porous FGM, the variability of the terms of the matrices A , B and D depends on the definition of the thickness distributions $t(x, y, z)$ (assuming that the Young modulus E is a function of the z coordinate). If the shell thickness t is the function of the z coordinate only, the B terms are not equal to zero for unsymmetric configurations of pores (Figure 2).

Note that in the literature a variety of papers exists solving different problems for 2D axisymmetric shell constructions (anisotropic or porous FGM); see Table 1. In general, these problems can be divided into two classes:

1. Solutions are functions of two geometrical variables x and y for shells with unsymmetric loading and boundary conditions and shell panels (Figure 1) and shells having unsymmetric wall configurations; see Muc and Muc-Wierzgoń [10]. This class deals also with buckling and free vibrations problems where the expansion along the circumferential direction is required.

2. Solutions are functions of one meridional variable only for symmetric shell wall configurations, axisymmetric boundary and loading conditions for constructions plotted in Figures 1a and b; the above condition is also satisfied for unsymmetric shell wall constructions made of FG materials (Figure 1c).

Although a great number of studies have been done for static, buckling and free vibration analysis of axisymmetric shells using isotropic and laminated materials, as pointed out in Table 1, a general lack of information for axisymmetric structures made of FGM is observed since the problems were solved with the use of numerical methods only (finite element [11,12] or finite difference methods [13–15] including also the generalized differential quadrature method GDQ [16–19]). Note that the applied method of solutions is directly connected with the used numerical method – for instance, the problems with singular points are solved with the use of finite difference methods [13–15], whereas the GDQ method was adopted to the discussion of problems without singular points only (see e.g. Sayyad and Ghugal [20]).

Figure 2: Shell wall construction for the uniform thickness $t(x, y, z) = t$: (a) uniform distribution of pores, (b) symmetric distribution, and (c) unsymmetric distribution.

Table 1: Classification of approaches used in the analysis of spherical shells

Material characteristics of the shell wall	Forms of a shell geometry		
	Closed shell at the vertex (two regular singular points: $\varphi = 0, \varphi = \pi/2$)	Open shell at the vertex (without singular points)	Shell panel (without singular points)
Unidirectional laminates	[13–15, 20, 28, 29, 33, 39, 46–49, 54, 70]	[7, 24, 30, 51]	[26, 31, 32, 38, 52]
Nanostructures	[50, 55–69]		[53, 74–77]
Porous functionally graded materials (FGM)	[12, 23, 35, 37, 40, 45]	[25, 27, 41–44]	[34, 36, 71–73]

The aim of the present article is to extend the analytical solutions for axisymmetric spherical shells made of FGM and subject to the normal uniform pressure. The solutions are obtained with the use of hypergeometric functions and compared with the available published results for isotropic shells. Since such a class of problems has not been considered and discussed in the literature, to the best of our knowledge, the demonstrated herein solutions are the novelty in the existing literature and in addition they can constitute and can be easily adopted to the design codes of composite spherical shells. It has been proved analytically that for FGM shells the influence of the unsymmetric shell wall construction (the non-zero terms of the stiffness matrix B , Figure 1c) can be described by only one parameter (function – multiplier) and is independent of the form of boundary value problems.

2 Formulation of the problem

The shells of revolution can be defined in the curvilinear orthogonal coordinates φ and θ of a point on the shell midsurface. It is convenient to take the spherical coordinates, where the angle φ defines the location of a point along the meridian, and θ describes the location of a point along the parallel circle – the circumferential coordinate (see Figure 3). $R_1 = R_1(\varphi)$ is the principal radius of the meridian and $R_2 = R_2(\varphi)$ is the principal radius of the parallel circle. Note that $r = R_2 \sin(\varphi)$ and the Lamé parameters are defined as follows: $A = R_1$, $B = r$.

When an arbitrary shell of revolution is subjected to rotationally symmetrical loads, its deformations and stress resultants and couples do not depend upon the circumferential variable θ . The axisymmetric deformations of shells of revolution are described by two displacement components of the shell midsurface u and w in the meridional and normal directions to the shell cross-section, respectively (Figure 3).

For shells of revolution under axisymmetrical loads, the strain displacement relations take the following form:

$$\varepsilon_1 = \frac{1}{R_1} \left(\frac{du}{d\varphi} + w \right), \quad \varepsilon_2 = \frac{1}{R_2} (u \cot \varphi + w), \quad (1)$$

$$\kappa_1 = \frac{1}{R_1} \frac{d\beta}{d\varphi}, \quad \kappa_2 = \frac{1}{R_2} \beta \cot \varphi, \quad \beta = \frac{1}{R_1} \left(u - \frac{dw}{d\varphi} \right), \quad (2)$$

where ε denotes the membrane strain, κ the change in curvature, and 1, 2 correspond to the meridional φ and circumferential directions θ , respectively. The constitutive equations are written in the following way:

$$N_r = A_{rs}\varepsilon_s + B_{rs}\kappa_s, \quad M_r = B_{rs}\varepsilon_s + D_{rs}\kappa_s, \quad r, s = 1, 2, \quad (3)$$

where

$$A_{rs} = \int_{-t/2}^{t/2} Q_{rs} dz, \quad B_{rs} = \int_{-t/2}^{t/2} Q_{rs} z dz, \quad (4)$$

$$D_{rs} = \int_{-t/2}^{t/2} Q_{rs} z^2 dz, \quad r, s = 1, 2,$$

$$Q_{11} = Q_{22} = \frac{E(z)}{1 - \nu^2}, \quad Q_{12} = Q_{21} = \nu Q_{11},$$

where N_r are the stress resultants and M_r the moment resultants. The elastic modulus E variation characterizes the distribution of porosity along the thickness direction

Figure 3: Cross-section of an axisymmetric shell of revolution.

z and is defined as

$$E(z)/E_b = [(E_t/E_b - 1)f(z) + 1], \quad f(z) = \left(\frac{z}{t} + \frac{1}{2}\right)^n, \quad (5)$$

where t and b refer to the material properties on top and bottom surfaces, n is the power index and ν is Poisson's ratio.

To complete the set of fundamental relations, we add the equilibrium equations derived using the principle of virtual work [1–3]:

$$\begin{aligned} \frac{d}{d\varphi}(rN_1) - N_2R_1 \cos \varphi + rT &= 0, \\ \frac{d}{d\varphi}(rT) - \left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right)rR_1 - rR_1q &= 0, \\ \frac{d}{d\varphi}(rM_1) - M_2R_1 \cos \varphi - rR_1T &= 0. \end{aligned} \quad (6)$$

where T denotes the transverse shear force and q is the normal pressure to the shell midsurface.

Note that the present formulation (the Love–Kirchhoff hypothesis) is valid for thin FG plates or shells, i.e. for $t/R < 0.05$. For thicker structures, it is necessary to adopt more complicated forms of kinematical approximations taking into account possible deformations of the normal to a shell/plate midsurface, e.g. in the form proposed by Vlasov [21]. A broader discussion of these problems is presented by Muc and Flis [22]. In the buckling, free vibration analysis the use of geometrically nonlinear relations is required, e.g. in the form proposed by Muc [8,15].

Both for complex kinematical hypotheses and nonlinear relations, the stress–strain relations (3) become more complex as defined in refs [8,9,15].

3 Solution of governing relation for axisymmetric spherical shells

Let us examine deformations of spherical shells ($R_1 = R_2 = R$) having constant thickness, i.e. $t(\varphi) = \text{const}$. Using the classical approach [1–3] for isotropic shells, the system of equations is reduced to the following form:

$$L(T) + \nu T = (B + RA)\beta/R + \Phi(\varphi)/R, \quad (7)$$

$$L(\beta) - \nu\beta = \frac{R(B + RA)}{B^2 - AD}T, \quad (8)$$

$$L = \frac{d^2}{d\varphi^2} + \cot \varphi \frac{d}{d\varphi} - \cot^2 \varphi, \quad A = A_{11}, \quad (9)$$

$$B = B_{11}, \quad D = D_{11},$$

$$\begin{aligned} \Phi(\varphi) &= (1 + \nu) \cot \varphi \frac{F(\varphi)}{\sin^2 \varphi} - q(1 + \nu)R^2 \cot \varphi \\ &\quad - \frac{d}{d\varphi} \left[(1 + \nu) \frac{F(\varphi)}{\sin^2 \varphi} - qR^2 \right] \end{aligned} \quad (10)$$

$$F(\varphi) = R^2 \int_{\varphi_e}^{\varphi} q \sin \bar{\varphi} \cos \bar{\varphi} d\bar{\varphi} + C,$$

φ_e is a coordinate of the shell edge and the constant C characterizes an axial coordinate of the external load applied at the top edge.

By inserting $B = 0$, relations (7) and (8) are reduced to the classical Meissner equations for isotropic shells. Elimination of each variable (T or β) in equations (7) and (8) leads to fourth-order differential equations, i.e.

$$LL(\beta) - \nu^2\beta - \frac{(B + RA)^2}{B^2 - AD}\beta - \frac{(B + RA)}{B^2 - AD}\Phi(\varphi) = 0, \quad (11)$$

$$LL(T) - \nu^2T - \frac{(B + RA)^2}{B^2 - AD}T + \frac{1}{R}[\nu\Phi(\varphi)] - L[\Phi(\varphi)] = 0. \quad (12)$$

Each of equations (11) and (12) can be solved for one of the dependent variables. The solutions can be represented as

$$\beta = \beta^c + \beta^p, \quad T = T^c + T^p, \quad (13)$$

where superscripts c and p denote the complementary ($\Phi = 0$) and particular ($\Phi \neq 0$) solutions, respectively. The complementary solutions satisfy the relation

$$\begin{aligned} [L(\beta^c) + i\mu^2\beta^c][L(\beta^c) - i\mu^2\beta^c] &= 0, \\ \mu^2 &= \sqrt{\frac{(B + RA)^2}{AD - B^2} - \nu^2}. \end{aligned} \quad (14)$$

Each of the equations has complex solutions ($AD > B^2$). The hypergeometric equation (14) has three regular singular points: 0 ($\varphi = 0$), 1 ($\varphi = \pi/2$) and ∞ . Usually, for spherical shells, the analytical solutions of complementary equations are formulated for two regions corresponding to singularity. Using the Mathematica package, the solutions of the fourth-order differential equations for complementary variables can be derived analytically with the aid of one command *DSolve*; see the general formulation presented by Polyanin and Zaitsev [78]. The general solution is represented by the complex hypergeometric functions and have the following form:

$$T^c \text{ (or } \beta^c) = C_1Z_1 + C_2Z_2 + C_3Z_3 + C_4Z_4, \quad (15a)$$

the regular singular point at $\varphi = 0$

$$\begin{aligned}
Z_1 &= {}_2F_1[0.25(3 - \alpha_1), 0.25(3 + \alpha_1), 0.5, \cos^2 \varphi] \sin \varphi, \\
Z_2 &= {}_2F_1[0.25(3 - \alpha_2), 0.25(3 + \alpha_2), 0.5, \cos^2 \varphi] \sin \varphi, \\
Z_3 &= {}_2F_1[0.25(5 - \alpha_1), 0.25(5 + \alpha_1), 1.5, \cos \varphi] \sin \varphi \cos \varphi, \\
Z_4 &= {}_2F_1[0.25(5 - \alpha_2), 0.25(5 + \alpha_2), 1.5, \cos \varphi] \sin \varphi \cos \varphi,
\end{aligned} \quad (15b)$$

the regular singular point at $\varphi = \pi/2$

$$\begin{aligned}
Z_1 &= {}_2F_1[0.25(3 - \alpha_1), 0.25(3 + \alpha_1), 2, \cos^2 \varphi] \sin \varphi, \\
Z_2 &= {}_2F_1[0.25(3 - \alpha_2), 0.25(3 + \alpha_2), 2, \cos^2 \varphi] \sin \varphi, \\
Z_3 &= {}_2F_1[0.5 + 0.25(5 - \alpha_1), 0.5 \\
&\quad + 0.25(5 + \alpha_1), 2, \cos^2 \varphi] \sin \varphi \cos \varphi, \\
Z_4 &= {}_2F_1[0.5 + 0.25(5 - \alpha_2), 0.5 \\
&\quad + 0.25(5 + \alpha_2), 2, \cos^2 \varphi] \sin \varphi \cos \varphi, \\
\alpha_1 &= \sqrt{5 - 4\mu^2\sqrt{-1}}, \alpha_2 = \sqrt{5 + 4\mu^2\sqrt{-1}},
\end{aligned} \quad (15c)$$

where $C_1 \dots C_4$ are the complex constants of integration. Since $\text{Re}(Z_1) = \text{Re}(Z_2)$, $\text{Im}(Z_1) = -\text{Im}(Z_2)$, $\text{Re}(Z_3) = \text{Re}(Z_4)$, and $\text{Im}(Z_3) = -\text{Im}(Z_4)$, the solutions of equation (15a) can be represented as real variables with real constants of integration $\tilde{C}_1, \dots, \tilde{C}_4$. For FGM spherical shells, the deformation functions are controlled by the value of the parameter μ^2 (equation (14)). For isotropic shells, this value is the function of the ratio R/t :

$$\mu_{\text{isotr}}^2 = \sqrt{\frac{(B + RA)^2}{AD - B^2} - \nu^2} = \sqrt{12 \frac{R^2}{t^2} - \nu^2} \approx 2\sqrt{3} \frac{R}{t}, \quad (16)$$

but for FGM shells, it is also the function of the coupling effects expressed by the term B (equation (14)). Figure 4 illustrates the influence of the coupling effects on the variations of the parameter μ^2 . It is represented by two values: the ratio E_t/E_b and the porosity index n (equation (5)).

Figure 4: Variations of the dimensionless controlling parameter $\mu^2/\mu_{\text{isotr}}^2$ (1 – isotropy; 2 – $E_t/E_b = 1$; 3 – $E_t/E_b = 10$).

The evaluation of T^c and β^c variables is identical and therefore the value of the parameter μ affects the solutions of the complementary equation (14) in a similar way. The results are presented in Figure 5 for a spherical segment where $0 < \varphi_1 \leq \varphi \leq \varphi_2 < \pi/2$. In such a case, the solutions (equation (15b)) have no singular points and demonstrate only the coupling effects. For isotropic and porous FG structures, the difference in the complementary solution $\text{Re}[Z_1]$, $\text{Im}[Z_1]$ (equation (15b)) can reach even almost 10%; however, for the solution, $\text{Im}[Z_1]$ is almost negligible. The identical plots can be drawn for equation (15c), which are the solutions for $\varphi_1 \leq \varphi \leq \pi/2 < \varphi_2$ (the singular point at $\pi/2$; Figure 6).

Comparing the complementary solutions for isotropic and FGM shells (Figures 5 and 6), one can observe the effects of the $\mu^2/\mu_{\text{isotr}}^2$ ratio. Both the changes in the index parameter n and of the E_b/E_t ratio affect the solutions expressed by the solution functions. In general, the results illustrate the localized effects at the lower boundary (i.e. $\varphi = \varphi_2$, $\varphi = \pi/2$).

From the equilibrium equation (17), one can obtain the analytical form of the stress resultants N_φ^c , N_θ^c :

$$RN_\theta^c = \frac{1}{\sin(\varphi)} \left[\frac{d}{d\varphi}(rT^c) - rN_\varphi^c \right], \quad N_\varphi^c = T^c \cot(\varphi). \quad (17)$$

Finally, the kinematic complementary displacements u^c , w^c can be derived in the closed analytical way with the use of the relations (17), the fundamental definitions (1) and (2) and the constitutive relations (3):

$$\begin{aligned}
u^c &= A_u \sin \varphi + \sin \varphi \int_{\varphi_e}^{\varphi} d\psi \frac{R(\varepsilon_\varphi^c - \varepsilon_\theta^c)}{\sin \psi}, \\
w^c &= R\varepsilon_\theta^c - u^c \cot(\varphi),
\end{aligned}$$

Figure 5: Solution functions for the FGM spherical shell segment $0 < \varphi_1 \leq \varphi \leq \varphi_2 < \pi/2$ (1 – isotropy; 2 – $E_b/E_t = 10$, $n = 2$; 3 – $E_b/E_t = 10$, $n = 10$).

Figure 6: Solution functions for FGM hemispherical shells (1 – isotropy; 2 – $E_b/E_t = 10$, $n = 2$; 3 – $E_b/E_t = 10$, $n = 10$).

$$\begin{aligned}\varepsilon_\varphi^c &= -\frac{B}{AR} \frac{d\beta^c}{d\varphi} + \frac{N_\varphi^c - \nu N_\theta^c}{A(1-\nu^2)}, \\ \varepsilon_\theta^c &= -\frac{B}{AR} \beta^c \cot \varphi + \frac{N_\theta^c - \nu N_\varphi^c}{A(1-\nu^2)}.\end{aligned}\quad (18)$$

A_u denotes a real constant of integration. The computations can be made easily applying the symbolic packages, e.g. Mathematica.

It is convenient and necessary to distinguish two classes of problems that should be introduced independently, referring directly to the convenience and simplicity of analytical solutions given for two types of singular points (equation (15b) and (15c)).

4 Spherical shells under a uniform internal pressure

Let us consider the case of the spherical shell loaded by the uniform external pressure q . For particular solutions, the appropriate differential equations (11) and (12) are homogeneous and

$$T^p = \beta^p = w^p = u^p = N_t^p = M_t^p = 0. \quad (19)$$

Now, we consider two solutions.

4.1 Clamped spherical segment $0 < \varphi_1 \leq \varphi \leq \varphi_2 < \pi/2$

The clamped edge conditions are the following:

$$u = w = \frac{dw}{d\varphi} = 0 \quad \text{at } \varphi = \varphi_1 \text{ and } \varphi = \varphi_2. \quad (20)$$

Conditions (20) correspond to $\beta^c = 0$ at the edges (equation (2)).

Using the equilibrium equation (6) and taking into account the form of particular solutions (20), one can find that

$$\begin{aligned}N_1 &= -\frac{dT^c}{d\varphi} + \frac{Rq \cos 2\varphi - \cos 2\varphi_1}{4 \sin^2 \varphi} - Rq, \\ N_2 &= -T^c \cot \varphi - \frac{Rq \cos 2\varphi - \cos 2\varphi_1}{4 \sin^2 \varphi}.\end{aligned}\quad (21)$$

The bending moments can be derived from the constitutive relations (3):

$$\begin{aligned}M_1 &= B(\varepsilon_1 + \nu \varepsilon_2) + D(\kappa_1 + \nu \kappa_2), \quad M_2 = B(\varepsilon_2 + \nu \varepsilon_1) \\ &\quad + D(\kappa_2 + \nu \kappa_1).\end{aligned}\quad (22)$$

For clamped spherical shells, it is convenient to express the boundary conditions (20) in the equivalent form

$$\beta = \beta^c = 0, \quad \varepsilon_2 = 0 \quad \text{at } \varphi = \varphi_1, \varphi = \varphi_2. \quad (23)$$

The solution of equation (11) has four real constants of integration $\tilde{C}_1, \dots, \tilde{C}_4$. They are evaluated from equation (23). T^c is represented by β^c in equation (8), and ε_2 is computed from equations (3), (21), and (22), where all unknown functions are expressed by functions β^c .

Figure 7 illustrates the distributions of circumferential membrane forces N_2 for spherical segments. These results are similar to those presented by Muc [70] for shallow shell segments. The maximal difference between isotropic and porous FG spherical shell segments occurs at the lower φ_1 boundary and corresponds to the results of complementary solutions plotted in Figure 5.

4.2 Clamped hemispherical shell $0 \leq \varphi \leq \pi/2$

For the structure, the boundary conditions can be written in the following way:

$$\begin{aligned}u &= \frac{dw}{d\varphi} = \kappa_1 = 0 \quad \text{at } \varphi = 0 \text{ and } \beta = \beta^c = 0, \\ \varepsilon_2 &= 0 \quad \text{at } \varphi = \pi/2.\end{aligned}\quad (24)$$

Two solutions (equation (15c)) Z_3, Z_4 are singular at the shell apex and they would have to be suppressed [2] and one can find that

$$T^c = A_1 S_1 + A_2 S_2 \quad \text{and} \quad S_1 = \text{Re}[Z_1], \quad S_2 = \text{Im}[Z_1], \quad (25)$$

Figure 7: Distributions of the dimensionless circumferential forces N_2 ($t/R = 0.02$).

and clamped at the edge $\varphi = \pi/2$:

$$w = w^c + w^p = 0, \quad \beta = \beta^c + \beta^p = 0. \quad (26)$$

The value of the particular solutions:

$$T^p = \beta^p = 0. \quad (27)$$

Since equations (11) and (12) are homogeneous. However, the effects of the distributed load q are brought by the equilibrium equation (6), i.e.

$$N_\varphi^p = N_\theta^2 = \frac{qR}{2}. \quad (28)$$

The stress resultants can be derived from the equilibrium conditions (6) and the relations (24) and (26)–(28):

$$N_\varphi = \frac{qR}{2} [(P_1 S_1 + P_2 S_2) \cot \varphi + 1], \quad (29)$$

$$N_\theta = \frac{qR}{2} \left[P_1 \frac{dS_1}{d\varphi} + P_2 \frac{dS_2}{d\varphi} + 1 \right]. \quad (30)$$

Using the above relations and the definitions (2) and (3) and the relation (7), the stress couples can be written in the following way:

$$M_\varphi = \frac{qR}{2\mu^4} \left[(\nu P_1 + 2\mu^2 P_2) \left(\frac{dS_1}{d\varphi} + \nu S_1 \cot \varphi \right) + (\nu P_2 - 2\mu^2 P_1) \left(\frac{dS_2}{d\varphi} + \nu S_2 \cot \varphi \right) \right] + \frac{B}{RA} N_\varphi, \quad (31)$$

$$M_\theta = \frac{qR}{2\mu^4} \left[(\nu P_1 + 2\mu^2 P_2) \left(\nu \frac{dS_1}{d\varphi} + S_1 \cot \varphi \right) + (\nu P_2 - 2\mu^2 P_1) \left(\nu \frac{dS_2}{d\varphi} + S_2 \cot \varphi \right) \right] + \frac{B}{RA} N_\varphi. \quad (32)$$

Inserting the coupling stiffness $B = 0$, one can find the relations (29)–(32) for isotropic shells [2,3]. The symbols P_1, P_2 are integration constants and, for the boundary conditions (26), they are equal to

$$P_1 = \frac{2C_i}{qR^2}, \quad P_1 = P_2 \frac{2\mu^2 J_1 + \nu J_2}{2\mu^2 J_2 - \nu J_1},$$

$$P_2 = \frac{-(1-\nu)(2\mu^2 J_2 - \nu J_1)}{J_1'(2\mu^2 J_1 + \nu J_2) + J_2'(2\mu^2 J_2 - \nu J_1)}, \quad (33)$$

$$J_1 = S_1(\varphi = \pi/2), \quad J_2 = S_2(\varphi = \pi/2),$$

$$J_1' = \frac{dS_1(\varphi = \pi/2)}{d\varphi}, \quad J_2' = \frac{dS_2(\varphi = \pi/2)}{d\varphi}.$$

Figure 8 demonstrates the distributions of the dimensionless circumferential forces $N_2/(qt)$ along the shell meridian. The results demonstrate that the localized effects at the clamped edge increase with the decrease of the thickness ratio. The coupling effects expressed by the controlling parameter μ^2 (14) have a significant influence on the values of the circumferential forces at the clamped edge – the growth of this value results in the increase of bending effects.

Figure 8: Distributions of the dimensionless circumferential forces N_2 ($t/R = 0.02$).

The distributions of the stress resultants N_2 show that these values are almost equal to $0.5qR$ (the membrane state).

5 Concluding remarks

The analytical solutions presented herein demonstrate evidently the significant influence of the coupling terms B illustrating the unsymmetry in the construction of shell walls of porous FGM. These effects are characterized by the controlling parameter μ . It is worth mentioning that a similar controlling parameter is proposed and introduced in the description of flutter problems for structures made of porous FGM or plates reinforced by nanostructures (nanoplates or nanotubes) (see Muc et al. [10,22]).

It is very important to formulate precisely the assumptions of the analysis, particularly in view of the used approaches. In our opinion, the classification presented in Table 1 can be useful in understanding the limitations (or generality) of the results and not only for spherical shells.

Let us note that the distributions of the stress resultants and bending moments are sensitive to the variations of the thickness ratio t/R . In the present work, the classical Love–Kirchhoff hypothesis is employed but, for thicker structures ($t/R > 0.1$), the higher-order 2D shell theories should be used, e.g. in the form proposed in the Appendix of ref. [22].

The present results are derived for spherical shells; however, the identical analysis can be easily extended to the analysis of shells or pressure vessels having various forms: paraboloidal, hyperbolical, elliptical, and torispherical discussed from the numerical and optimization point of view, e.g. in ref. [79,80].

The results presented demonstrate that the analytical results can be treated as a benchmark for finite element analysis.

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