Online Appendix

This appendix contains proofs of the theorems and assertions made in the paper. Appendix A contains details of the neutral model focusing on the first two stages of the game (investment and CP pricing), which are the most involved. Appendix B does the same for the non-neutral model. Appendix C.1 shows the welfare comparison results.

More specifically, Appendix A.1 describes and proves the lemma that shows that when one platform invests more than the other, the perceived quality from the consumers’ point of view is higher. Appendices A.2 and A.3 show the steps involved in solving for the CP price SPE. In Appendix A.2, we derive candidate equilibrium prices for the different market configurations, together with conditions for each configuration. We use best response dynamics as the primary approach to determine the equilibrium pairs.

Appendix A.3 shows the existence of a pure-strategy Nash Equilibrium in the price subgame. Specifically, for the candidate equilibrium prices established in Appendix A.2, we determine if they are best replies across all the market configurations. This section contains a proof of Theorem 3.1, together with the associated lemmas.

Appendix A.4 shows the existence and uniqueness of the SPE in the investment stage of the game. We find best responses for both platforms and determine points of intersection to arrive at the SPE. We also present Theorem 3.2, which characterizes an SPE in the investment stage. Section A.6 presents the proof of Theorem 3.3 which shows the existence of the SPE.

Appendix B contains the critical steps needed to solve for the non-neutral model. Many of the techniques employed in the neutral model are also applicable in the non-neutral model. This section, like the previous one, applies to the first three stages of the game. Appendix B.1 offers a characterization of the possible CP connection profiles for each price pair $w_1$ and $w_2$. These profiles determine which relation/s defined in Section 4.1.2 hold.

In Appendix B.2, we show which relation in Section 4.1.2 does not occur in the equilibrium path. This partly motivates why we make Assumption 1. Appendix B.3, shows the existence, uniqueness and characterization of the CP price subgame. It also presents the proof of Theorem 4.1 together with its required lemmas. The rest of the sections are used to show the existence of an SPE in the investment stage of the game. Here, we make use of best response dynamics, similarly to those used in the neutral model. The section also presents Theorem 4.2 and its accompanying lemmas.

Appendix C.1 uses the SPE prices and investment levels discussed earlier to determine welfare metrics. This appendix highlights differences across the different social welfare metrics by comparing the neutral and non-neutral regimes.

A.1. Technical Details for the Neutral Model

A.1. A Lemma. The following lemma shows that if a platform has a higher platform quality in the neutral regime then consumers joining it perceive it to be of a higher value.

**Lemma A.1.** If $y_\alpha > y_\beta$, then $F_i(y_\alpha, \cdot) > F_i(y_\alpha, \cdot)$.

**Proof.** After some algebraic manipulations the explicit expression for $F_i(y_{\phi(i)}, y_{\phi(j)}, \tau, r_\alpha, r_\beta)$ in terms of average CP content quality, platform quality and the mass of content providers on both platforms is as given below for both $\phi(i) = \alpha$ and $\phi(i) = \beta$;

$$F_i(y_{\phi(i)}, y_{\phi(j)}, \tau, r_\alpha, r_\beta) = \int_0^1 E \left[ \max \{u_{ij}(y_{\phi(i)}, y_{\phi(j)}, \gamma_j, c_{\phi(i)}, r_{\phi(j)}, 0) \} \right] dj,$$

$$= y_\alpha (\gamma (r_\alpha + 1) + a(1 - r_\alpha)) + y_\beta (\gamma (r_\beta + 1) + a(1 - r_\alpha - r_\beta)),$$

$$F_i(y_{\phi(i)}, y_{\phi(j)}, \tau, r_\alpha, r_\beta) = \int_0^1 E \left[ \max \{u_{ij}(y_{\phi(i)}, y_{\phi(j)}, \gamma_j, c_{\phi(i)}, r_{\phi(j)}, 0) \} \right] dj,$$

$$= y_\beta (\gamma (r_\alpha + 1) + a(1 - r_\alpha)) + y_\beta (\gamma (r_\beta + 1) + a(1 - r_\alpha - r_\beta)).$$
It immediately follows from above that \( F_i(y_{\alpha}^*, \cdot) > F_i(y_{\alpha}, \cdot) \).

A.2. Candidate Equilibrium Prices for Different Markets. This section derives the candidate equilibrium pairs for the different market configurations as well as the necessary conditions required for the configuration to occur.

**Uncovered Market - \( C_I \).** In this case we suppose *ex ante* that the market is uncovered with only the high quality platform serving the market. We identify the equilibrium prices for this market configuration and the conditions on \((\gamma, a, f)\) for which this market configuration is feasible. We first derive the best price responses of each platform to the price set by its rival. The condition for an uncovered market where only the high-quality platform participates in the market is given by,

\[
\frac{w_\beta}{(q_\beta + q_{\alpha})y_\beta} \geq \frac{w_\alpha}{y_\beta q_\beta + y_{\alpha} q_{\alpha}} > \gamma - a.
\]

In this configuration, both platform’s profit do not depend on \(w_\beta\). Thus given \(w_\alpha\) that satisfies \((w_\alpha)/(y_\beta q_\beta + y_{\alpha} q_{\alpha}) > \gamma - a\), any \(w_\beta\) that satisfies the following condition

\[
w_\beta \geq y_\beta \frac{w_\alpha(q_\alpha + q_\beta)}{y_\beta q_\beta + y_{\alpha} q_{\alpha}}
\]

is a best response by platform \(\beta\). This follows from condition (6). On the other hand given \(w_\beta > (\gamma - a)(q_\alpha + q_\beta)y_\beta\) the best response is given by the optimal solution of the following problem

\[
\max \pi_{\alpha}^{ui}(w_\alpha, w_\beta)
\]

s.t. \(w_\alpha \in \left((\gamma - a)(y_\beta q_\beta + y_{\alpha} q_{\alpha}), \frac{w_\beta(y_\beta q_\beta + y_{\alpha} q_{\alpha})}{y_\beta(q_\alpha + q_\beta)}\right)\).

From the first order conditions of (7) we infer that the best response is characterized as follows,

\[
w_\alpha = \begin{cases} w_\alpha^* & \text{if } w_\beta \geq \frac{w_\alpha^* y_\beta(q_\alpha + q_\beta)}{(y_\beta q_\alpha + y_{\alpha} q_{\alpha})} \\ \frac{w_\beta}{y_\beta(q_\alpha + q_\beta)} & \text{if } w_\beta < \frac{w_\alpha^* y_\beta(q_\alpha + q_\beta)}{(y_\beta q_\alpha + y_{\alpha} q_{\alpha})}, \end{cases}
\]

where \(w_\alpha^* = \frac{f}{9}(5a + \gamma)y_{\alpha} - \frac{f}{18}(a - 7\gamma)y_\beta\) and is the unrestricted solution to problem 7. Denoting the equilibrium price pair in this configuration by \((w_\alpha^{ui}, w_\beta^{ui})\) we note that any price combination,

\[
\frac{w_\alpha^{ui}}{w_\beta^{ui}} = \frac{y_\beta q_\beta + y_{\alpha} q_{\alpha}}{y_\beta(q_\alpha + q_\beta)},
\]

where \((\gamma - a)(q_\alpha y_{\alpha} + y_\beta q_\beta) < w_\alpha^{ui} \leq w_\alpha^*\) and \((\gamma - a)(q_\alpha + q_\beta)y_\beta < w_\beta^{ui} \leq \frac{w_\alpha^* y_\beta(q_\alpha + q_\beta)}{(y_\beta q_\alpha + y_{\alpha} q_{\alpha})}\), is an equilibrium price pair in this configuration. In addition, when \(w_\alpha^{ui} = w_\alpha^*\) any price combination such that,

\[
w_\alpha^{ui} = w_\alpha^*,
\]

\[
w_\beta^{ui} \geq \frac{w_\alpha^* y_\beta(q_\alpha + q_\beta)}{(y_\beta q_\beta + y_{\alpha} q_{\alpha})},
\]

is an equilibrium price too. It remains to specify the necessary condition for this configuration to occur. From condition (6), configuration \(C_I\) occurs only if \((\gamma - a)(y_\alpha q_{\alpha} + y_\beta q_{\beta}) - w_\alpha^{ui} < 0\). This results in the following necessary condition,

\[
\frac{\gamma}{a} < \frac{4f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta}
\]

**Uncovered Market - \( C_{II} \).** In this case we suppose *ex ante* that the market is uncovered with both platforms serving the market. We first identify the equilibrium prices and then the values of \((\gamma, a)\) for which this
market configuration is feasible. The condition for an uncovered market in which both the high-quality and low-quality platforms serve the market is given by,

$$\gamma - a < \frac{w_\beta}{(q_\alpha + q_\beta) y_\beta} < \frac{(w_\alpha - w_\beta)}{q_\alpha (y_\alpha - y_\beta)} < \gamma + a. \quad (12)$$

The best reply functions of the respective platforms are obtained from the first order conditions of the platforms profit functions and are given below.

$$w_\alpha(w_\beta) = \frac{5f}{9}(y_\alpha - y_\beta) + \frac{f}{9}(y_\alpha - y_\beta) + \frac{1}{2}w_\beta, \quad (13)$$

$$w_\beta(w_\alpha) = \frac{1}{6}y_\beta(f(\gamma - a)(y_\alpha - y_\beta) + 9w_\alpha)}{(2y_\alpha + y_\beta)}. \quad (14)$$

Note that the above functions are linear. Solving the above two simultaneous equations yields the following unique equilibrium prices,

$$w_\alpha^e = \frac{f((8\gamma + 40a)y_\alpha^2 - (23\gamma + a)y_\beta y_\alpha - (17a + 7\gamma)y_\beta^2)}{9(y_\beta + 8y_\alpha)}, \quad (15)$$

$$w_\beta^e = \frac{4f y_\beta(\gamma + 2a)(y_\alpha - y_\beta)}{3(y_\beta + 8y_\alpha)}. \quad (16)$$

The reaction functions and their intersection point are shown in Figure 4. Since the market is not covered the lowest quality content provider does not join the lower quality platform. Therefore a necessary condition for the above configuration to hold is \((\gamma - a)(q_\alpha + q_\beta)y_\beta - w_\beta^e < 0\). Substituting for \(w_\beta^e\) the above condition can be rewritten as,

$$\frac{\gamma}{a} < \frac{2f(y_\alpha - y_\beta) + 30y_\alpha - 3y_\beta}{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}. \quad (17)$$

**Covered market- C_{III}.** We now suppose *ex ante* that the market is covered with both platforms serving the market. We again identify the equilibrium prices and then the values of \((\gamma, a)\) for which this market
configuration is feasible. Proceeding as we did in the previous market configurations, to derive the equilibrium prices, we first derive best response prices of each platform to the price set by the other platform. The condition for a covered market in which both the high-quality and low-quality platforms serve the market is

\[
\frac{w_\beta}{(q_\alpha + q_\beta)y_\beta} \leq \overline{\gamma} - a < \frac{(w_\alpha - w_\beta)}{q_\alpha(y_\alpha - y_\beta)} < \overline{\gamma} + a.
\]

The first order conditions associated with the profit functions for both platforms yield the following best reply functions,

\[
w_\alpha(w_\beta) = (y_\alpha - y_\beta)(\frac{1}{3}f(\overline{\gamma} + a) - \frac{2}{9}f^2(\overline{\gamma} - a)) + \frac{1}{2}w_\beta,
\]

\[
w_\beta(w_\alpha) = \begin{cases} 
(y_\alpha - y_\beta)(\frac{1}{7}f(a - \overline{\gamma}) - \frac{2}{15}f^2(\overline{\gamma} - a)) + \frac{1}{2}w_\alpha & \text{if } w_\alpha < w_\alpha^* \\
(\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta & \text{if } w_\alpha \geq w_\alpha^*
\end{cases}
\]

where \( w_\alpha^* = y_\alpha(\frac{2}{7}f^2(a - \overline{\gamma}) + \frac{1}{3}f(\overline{\gamma} + a)) - y_\beta(\frac{2}{7}f^2(\overline{\gamma} + a) + \frac{1}{3}f(5a - \overline{\gamma})) \).

**Interior Solution.** From the above best response functions we get the following unique equilibrium prices in the case of an interior solution.

\[
w_\alpha^{ci} = (y_\alpha - y_\beta)(\frac{1}{27}f(7f(\overline{\gamma} - a) - 6(3a - \overline{\gamma}))),
\]

\[
w_\beta^{ci} = (y_\alpha - y_\beta)(\frac{2}{27}f(f(\overline{\gamma} - a) + 3(3a - \overline{\gamma}))).
\]

A market is covered with an interior solution in the price subgame if the price charged by the lower quality platform is lower than the value derived by the lowest quality content provider, i.e., \((\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta - w_\beta^{ci} > 0\). In this market configuration the lowest quality content provider prefers the lowest quality platform, otherwise we have a preempted market. Moreover, the lowest-quality content provider’s net utility must also be positive. Thus the following condition has to hold in equilibrium,

\[
(\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta - w_\beta^{ci} > \max\{w_\alpha^{ci} - (\overline{\gamma} - a)(y_\beta q_\beta + y_\alpha q_\alpha), 0\}.
\]
By plugging the equilibrium prices in (19) and (20) into the above inequality, we obtain the following necessary conditions on the tuple \((y_\alpha, y_\beta, \tau, a)\) for this configuration to exist.

\[
\frac{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{2f(y_\alpha - y_\beta) + 6y_\alpha + 21y_\beta} < \tau < \frac{5f + 18}{5f + 6}.
\]

**Corner solution.** We denote the content provider market to be covered with a corner solution in the price subgame if the lower quality platform quotes a price that is just sufficient so that the lowest quality content provider joins the platform. In this case, a corner solution occurs and we have the following price charged by platform \(\beta\),

\[
w_\beta^{cc} = (\tau - a)(q_\alpha + q_\beta)y_\beta.
\]

From the first order conditions of the high quality profit function we deduce that the equilibrium price is given by,

\[
w_\alpha^{cc} = \frac{1}{18}(f4f(a - \tau) + 6(a - \tau))y_\alpha + \frac{1}{18}(4f(\tau - a + 3(\tau - 5a))y_\beta.
\]

For configuration \(C_{III}\) to occur with a corner solution the following three conditions need to hold,

\[
(\tau - a)(q_\alpha + q_\beta)y_\beta - w_\beta^{cc} > (\tau - a)(y_\beta q_\beta + y_\alpha q_\alpha) - w_\alpha^{cc},
\]

\[
w_\beta^{ci} \geq (\tau - a)(q_\alpha + q_\beta)y_\beta,
\]

\[
w_\beta^{ci} \leq (\tau - a)(q_\alpha + q_\beta)y_\beta.
\]

The above inequalities yield the following necessary and sufficient conditions on \(\tau, y_\alpha, y_\beta\) for the above equilibrium prices to yield Configuration \(C_{III}\),

\[
\frac{2f(y_\alpha - y_\beta) + 30y_\alpha - 3y_\beta}{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta} \leq \tau \leq \min \left\{ \frac{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{2f(y_\alpha - y_\beta) + 6y_\alpha + 21y_\beta}, \frac{4f(y_\alpha - y_\beta) + 18y_\alpha - 9y_\beta}{4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta} \right\}.
\]

**Covered Preempted market \(C_{IV}\).**

In this case we suppose \(ex ante\) that the market is covered with only the high quality platform serving the market. We identify the equilibrium prices for this market configuration and derive the best price responses...
of each platform in the usual way. The condition for a covered market where only the high-quality platform participates in the market is,

$$ (\gamma - a)(q_\alpha + q_\beta) - w_\alpha \geq \max \{0, (\gamma - a)(q_\alpha + q_\beta) - w_\beta \}. $$

The profit functions for platforms $\alpha$ and $\beta$ given $y_\alpha$ and $y_\beta$ are

$$ \pi^p_\alpha = \frac{8}{9} f^2 (y_\alpha - y_\beta) + w_\alpha, $$

$$ \pi^p_\beta = \frac{2}{9} f^2 (y_\alpha - y_\beta). $$

We note that in this configuration, given $w_\alpha$, platform $\beta$’s profit does not depend on $w_\alpha$. Thus any $w_\beta$ that meets the condition specified by (27) is a best response. Given $w_\beta$, it follows from the condition specified by (27) and the first order conditions of (28), that

$$ w_\alpha = \begin{cases} 
(\gamma - a)(q_\alpha y_\alpha + q_\beta y_\beta) & \text{if } w_\beta \geq (\gamma - a)(q_\alpha + q_\beta)y_\beta, \\
(\gamma - a)q_\alpha (y_\alpha - y_\beta) + w_\beta & \text{if } w_\beta < (\gamma - a)(q_\alpha + q_\beta)y_\beta.
\end{cases} $$

Thus the above characterizes the price equilibrium combinations for configuration $C_{IV}$.

A.3. Nash Equilibrium in the Price Subgame. In this section, we show the existence of pure strategy Nash equilibrium in the price-subgame. We look at the equilibrium price pairs derived in the pervious section and determine if they are best replies across all the configurations. We characterize the price subgame equilibria in terms of the tuple $(\gamma, y_\alpha, y_\beta)$ and give the conditions for their existence. Specifically, we give the conditions for these price equilibria to yield their corresponding market configurations.

We show that the uncovered market configuration, $(C_I)$, does not occur at a subgame price equilibrium. We then show that market configurations $C_{II}, C_{III}$ and $C_{IV}$ exist. In doing so, we determine the set of parametric values $(\gamma, y_\alpha, y_\beta, a)$ for which these different configurations exist and characterize the prices in each configuration using the same parameters.
Proof. We assume to arrive at a contradiction that their exists a pair of equilibrium prices \((w_{\alpha}^{ui}, w_{\beta}^{ui}) \in \mathcal{R}_I\) that are a pure strategy Nash equilibrium in the price subgame. To prove our Lemma we show that given a subgame \((\gamma, a, f, y_{\alpha}, y_{\beta})\) such that condition in (11) is met, the prices in the pair \((w_{\beta}^{ui}, w_{\alpha}^{ui})\) are not best reply pairs on the whole domain of strategies, i.e, there exists for at least one platform the incentive to deviate to a price that will yield a different configuration and higher profits. In particular, we show that \(w_{\beta}^{ui}\) does not beat all price strategies in the projection of \(\mathcal{R}_{II} \cup \mathcal{R}_{III} \cup \mathcal{R}_{IV}\) against \(w_{\alpha}^{ui}\).

As shown in section A.2 there are two possible characterizations for the equilibrium price pair that holds if configuration \(C_I\) is exogenously imposed. We show that prices satisfying both characterizations are not best reply pairs on the whole domain.

Case I. Equilibrium price pair \((w_{\alpha}^{ui}, w_{\beta}^{ui})\) in (9) and (10).

As previously discussed in section A.2, the above price characterizations yield configuration \(C_I\) only if the condition in (11) is met. We denote the profit for platform \(\beta\) under the price pair \((w_{\alpha}^{ui}, w_{\beta}^{ui})\) as \(\pi_{\beta}^{ui}\) and that under the pair \((w_{\alpha}^{ui}, w_{\beta}^{ui})\) as \(\pi_{\beta}^{ui}\). We also denote the difference between the two profits, \(\pi_{\beta}^{ui} - \pi_{\beta}^{ui}\), as \(d(\gamma)\). Let \(\gamma^*\) denote the upper bound value of \(\gamma\) such that configuration \(C_I\) is possible. We now show that there exists configurations with price pairs \((w_{\alpha}^{ui}, w_{\beta}^{ui})\) such that \(\pi_{\beta}^{ui} < \pi_{\beta}^{ui}\) for \(\gamma < \gamma^*\), which implies that these price characterization cannot be a subgame equilibrium. For this purpose we fix \(w_{\alpha}^{ui}\) and consider profits of platform \(\beta\) under configurations \(C_{III}\) and \(C_{IV}\).

Let \(\overline{w}_{\beta} = (\gamma - a)(q_{\alpha} + q_{\beta}) y_{\beta}\), then configuration \(C_{III}\) will arise whenever \(\gamma^* < \gamma < \gamma^*\), where \(\gamma^* = (a(4f(y_{\alpha} - y_{\beta}) + 3y_{\beta} - 6y_{\alpha})/(4f(y_{\alpha} - y_{\beta}) + 6y_{\alpha} + 3y_{\beta})\). The function \(d(\gamma)\) is convex since \(\partial^2 d(\gamma)/\partial^2 (\gamma) > 0\). Moreover, \(d(\gamma)\) has two roots:

\[
\gamma_1 = a \frac{4f(y_{\alpha} - y_{\beta}) + 18y_{\alpha} + 9y_{\beta}}{4f(y_{\alpha} - y_{\beta}) + 6y_{\alpha} + 3y_{\beta}}, \quad \text{and} \quad \gamma_2 = a.
\]
It follows that whenever \( \gamma_2 \leq \gamma < \gamma^* \) then \( d(\gamma) < 0 \). This implies that platform \( \beta \) would prefer to deviate to a covered market with a corner solution. However, this configuration is possible for all values of \( \gamma > a \) when \( \gamma^* < \gamma_2 \). And this occurs when \( y_\alpha/y_\beta \geq 5/2 \). So we now proceed to show that when \( y_\alpha/y_\beta < 5/2 \) that platform \( \beta \) would prefer to deviate to configuration \( C IV \) where all masses of CPs join it.

Let \( \overline{w}_\beta = (\gamma - a)(q_\alpha + q_\beta)y_\beta \) and consider the case when \( y_\alpha/y_\beta < 5/2 \). It follows that \( \gamma^* > a \), so for \( a < \gamma \leq \gamma^* \) a covered preempted market results. The difference \( d(\gamma) \) under this configuration is convex since \( \partial^2 d(\gamma)/\partial^2 \gamma > 0 \). The roots of \( d(\gamma) \) are \( r 1, r 2 \) and \( r 1 > \gamma^* \) whenever \( 1 < y_\alpha/y_\beta < 9/4 + 1 \). Since \( 5/2 < 9/4 + 1 \) for \( f \in (0, 1) \) platform \( \beta \) prefers to deviate to configuration \( C IV \) whenever \( y_\alpha/y_\beta < 5/2 \).

**Case II. Equilibrium price pair \( (w^{u\alpha}, w^{u\beta}) \) in (8).**

We show that if platform \( \beta \) picks the price \( \overline{w}_\beta = (\gamma - a)(q_\alpha + q_\beta)y_\beta \) then it makes a higher profit in the resulting configuration. We denote the profit of platform \( \beta \) under the price pair \( (w^{u\alpha}, w^{u\beta}) \) as \( \pi^{u\beta} \) and that under the price pair \( (w^{u\alpha}, \overline{w}_\beta) \) as \( \overline{\pi}_\beta \). Note that when platform \( \beta \) picks the price \( \overline{w}_\beta \) the market becomes covered and configuration \( C II \) emerges. The function \( \pi_\alpha - \pi_\alpha = d(w^{u\alpha}) \) is increasing in \( w^{u\alpha} \) since \( \partial d(w^{u\alpha})/\partial w^{u\alpha} > 0 \). Moreover, it has a single root at \( w^{u\alpha} = (\gamma - a)(y_\alpha q_\alpha + y_\beta q_\beta) \). Thus for any \( w^{u\alpha} > w^{u\alpha*} \) platform \( \beta \) would prefer to deviate to price \( \overline{w}_\beta \). Therefore an equilibrium price pair \( (w^{u\alpha}, \overline{w}_\beta) \) for which the characterization in (10) holds is not a SPE.

We now show that the equilibrium price pair characterized for configuration \( C II \) in Section A.2 is a subgame equilibrium price pair and give the conditions on \( \gamma, a, y_\beta \), and \( y_\alpha \) for this to hold.

**Lemma A.3.** Given a tuple \( (\gamma, a, f, y_\alpha, y_\beta) \), there exists a unique equilibrium price pair \( (w^{s\alpha}, w^{s\beta}) \in \mathcal{R}_{II} \) only if

\[
1 < \gamma < 2f(y_\alpha - y_\beta) + 30y_\alpha - 3y_\beta \frac{2f(y_\alpha - y_\beta) + 30y_\alpha - 3y_\beta}{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}.
\]

**Proof.** From section A.2 we know that the prices in the pair \( (w^{s\alpha}, w^{s\beta}) \) are unique and mutual best replies in the restricted domain \( \mathcal{R}_{II} \); which corresponds to the market configuration \( C II \). Therefore this price pair is our only candidate for the price equilibrium that falls in \( \mathcal{R}_{II} \). To show that the candidate price pair \( (w^{s\alpha}, w^{s\beta}) \) is a price subgame equilibrium, we need to show that the prices in these pair are also mutual best replies on the whole domain of strategies, i.e., given price \( w^{s\alpha} \), platform \( \beta \) does not have an incentive to change to price \( \overline{w}_\beta \) which will result in another configuration and a higher profit. Formally, we have to show that \( w^{s\beta} \) beats any strategy \( w^{\beta} \) in the projection \( R_I \cup R_{I II} \cup R_{I IV} \) against \( w^{s\alpha} \) and vice versa. We denote the equilibrium price candidate \( (w^{s\alpha}, w^{s\beta}) \) as \( (w^{s\alpha}, w^{s\beta}) \).

We first fix \( w^{s\beta} \) and show that platform \( \alpha \) has no incentive to deviate to any price \( \overline{w}_\alpha \) in any configuration. We denote the profit under the price pair \( (w^{s\alpha}, w^{s\beta}) \) as \( \pi^{s\alpha} \) and that under the pair \( (\overline{w}_\alpha, w^{s\beta}) \) as \( \overline{\pi}_\alpha \). We denote the difference \( \pi^{s\alpha} - \pi^{s\alpha} = d(\gamma) \).

1. **Platform \( \alpha \) has no incentive to deviate to configuration \( C I \).** We find platform \( \alpha ' \)'s best reply given \( w^{s\beta} \) under market configuration \( C I \) and show that the profit realized is less than that under configuration \( C II \) at price \( w^{s\alpha} \). Let \( \pi^{s\alpha} \) be the best reply of platform under configuration \( C I \). It is given as the solution to the following maximization problem,

\[
\max_{w_\alpha} \pi_\alpha(w_\alpha, w^{s\beta}) ,
\]

s.t. \( w_\alpha \leq \frac{w^{s\beta} q_\alpha y_\alpha + q_\beta y_\beta}{y_\beta q_\alpha + q_\beta} \).

\(12\)The root \( r 1 \) can be expressed as \( \alpha (4y^{s\beta}_\alpha f^2 + 4f^2 y^{s\alpha}_\alpha + 3f y_\alpha y_\beta + 216y_\alpha y_\beta - 8f^2 y_\alpha y_\beta + 108y^{s\beta}_\alpha + 3f y^{s\beta}_\alpha - 6f y^{s\beta}_\alpha)/f/(4f y^{s\beta}_\alpha + 6y^{s\beta}_\alpha - 3y_\alpha y_\beta - 8f y_\alpha y_\beta + 4f y^{s\beta}_\alpha - 3y^{s\beta}_\alpha). \)
The constraint in the above problem arises from the necessary conditions expressed in (6) for market configuration $C_I$ to hold. The profit function $\pi_\alpha$ is concave in $w_\alpha$, since $\frac{\partial^2 \pi_\alpha(w_\alpha)}{\partial^2 w_\alpha} < 0$. The unconstrained optimal solution to the above maximization problem is larger than the constraint. Therefore the constraint binds and it is the best reply.

We now compare the two profits under both configurations. After evaluating the difference $d(\gamma) = \pi_\alpha - \pi_\beta$, we obtain that $d(\gamma)$ is a convex function in $\gamma$, because, $\frac{\partial^2 d(\gamma)}{\partial^2 \gamma} > 0$. In addition, $d(\gamma) \geq 0$ since $d(\gamma)$ is a quadratic function in $\gamma$ with a single root at $(a(1 + f)y_\alpha + (2 + f)y_\beta)/(4(f - 1)y_\alpha + (f - 2)y_\beta)$. Therefore, given $w_\beta^*$ platform $\alpha$ has no incentive to deviate to a price $\pi_\alpha^*$ that would result in configuration $C_I$.

(2) Platform $\alpha$ cannot deviate to configuration $C_{II}$. Form section A.2 we know that $w_\alpha^*$ is defined only if the condition in (17) is satisfied. This implies that $w_\beta^*> (\gamma - a)(q_\alpha + q_\beta)y_\beta$. Therefore, it is not possible to have a covered market with content providers patronizing the two platforms when platform $\beta$'s price is fixed at $w_\beta^*$.

(3) Platform $\alpha$ has no incentive to deviate to configuration $C_{IV}$. We proceed in a similar manner to the first case. We find platform $\alpha$'s best reply given $w_\alpha^*$ under market configuration $C_{IV}$ and show that the profit realized is less than that under configuration $C_{II}$ at price $w_\alpha^*$. Let $\overline{w}_\alpha$ be the best reply of platform under configuration $C_{IV}$. It is given as the solution to the following maximization problem

$$\max \pi_\alpha(w_\alpha, w_\beta^*),$$

s.t. $w_\alpha \leq (\gamma - a)q_\alpha y_\alpha + q_\beta y_\beta$

Since $\pi_\alpha$ is linear and increasing in $w_\alpha$, the constraint binds and is the best response. Under this price $d(\gamma)$ is a convex function in $\gamma$, because $\frac{\partial^2 d(\gamma)}{\partial^2 \gamma} > 0$. The function $d(\gamma)$ has two roots $\gamma_1$ and $\gamma_2$, these have been defined in the proof of the previous Lemma. Since configuration $C_{II}$ is defined outside these two roots it follows that $d(\gamma)$ is positive. Therefore, platform $\alpha$ has no incentive to deviate to configuration $C_{IV}$.

We now fix $w_\alpha^*$ and show that platform $\beta$ has no incentive to deviate to any price $\overline{w}_\beta$ that will yield another configuration. We denote the profit of platform $\beta$ under the price pair $(w_\alpha^*, w_\beta^*)$ as $\pi_\beta^*$ and that under the pair $(\overline{w}_\beta, w_\alpha^*)$ as $\overline{\pi}_\beta$. We denote the difference $\pi_\beta^* - \overline{\pi}_\beta$ by $d(\gamma)$.

(1) Platform $\beta$ has no incentive to deviate to configuration $C_I$. We proceed in a similar fashion to the previous parts. We find platform $\beta$'s best reply given $w_\alpha^*$ under market configuration $C_I$ and show that the profit realized is less than that under configuration $C_{II}$ at price $w_\beta^*$. Let $\overline{w}_\beta$ be the best reply of platform under configuration $C_I$. It is given as the solution to the following maximization problem

$$\max \pi_\beta(w_\alpha^*, w_\beta),$$

s.t. $w_\beta \geq \frac{w_\alpha^* y_\beta (q_\alpha + q_\beta)}{q_\alpha y_\alpha + q_\beta y_\beta}$.

The constraint in the above problem arises from the necessary conditions expressed in (6) for market configuration $C_I$ to hold. The profit function $\pi_\beta$ is concave in $w_\beta$ since the $\frac{\partial^2 \pi_\beta(w_\beta)}{\partial^2 w_\beta} > 0$. Through computation one can show that the optimal solution is at the boundary since the constraint binds.

We now compare the two profits. The difference, $d(\gamma)$, is a convex function in $\gamma$, because, $\frac{\partial^2 d(\gamma)}{\partial^2 \gamma} > 0$. Moreover, this function is a quadratic function in $\gamma$ with a single root at $\gamma = \frac{a(4y_\alpha + 4fy_\beta + 2y_\beta + fy_\beta)}{4fy_\beta + fy_\beta + 4y_\alpha - 2y_\beta}$. Therefore, for all $\gamma$ the difference $d(\gamma) \geq 0$. Thus given $w_\alpha^*$, platform $\beta$ has no incentive to deviate to a price that results in $C_I$.

(2) Platform $\beta$ has no incentive to deviate to configuration $C_{III}$. We show that platform $\beta$ makes more profit under configuration $C_{II}$ than if it changed its price and deviated to configuration $C_{III}$. Let $\overline{w}_\beta$ be the best response price under configuration $C_{III}$ given $w_\alpha^*$. It is defined below,
\[ \bar{w}_\beta = \text{argmax } \pi_\beta(w_\alpha^*, w_\beta), \]
\[ \text{s.t. } w_\beta \leq (\gamma - a)(q_\alpha + q_\beta)y_\beta. \]  

(31)

The above profit function is concave in \( w_\beta \) since \( \frac{\partial^2 \pi_\beta}{\partial w_\beta^2} < 0 \). Moreover one can show through computation that the constraint in problem (31) binds at the optimum. We now compare profits under \( C_{11} \) and those resulting in \( C_{111} \) under the deviation price \( \bar{w}_\beta \). The difference in profits given by \( d(\gamma) \) is a convex function in \( \gamma \), because \( \frac{\partial^3 d(\gamma)}{\partial \gamma^3} > 0 \). In addition, the function \( d(\gamma) \) has a single root at \( \gamma = \frac{2f(y_\alpha - y_\beta) + 30y_\alpha - 3y_\beta}{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta} \), therefore \( d(\gamma) \geq 0 \). Thus given \( w_\alpha^* \), platform \( \beta \) has no incentive to deviate to a price that results in configuration \( C_{111} \).

(3) **Platform \( \beta \) has no incentive to deviate to configuration \( C_{1IV} \).** If platform \( \beta \) chooses to deviate to a configuration where all CPs subscribe to it, the best price it can offer is given by \( \bar{w}_\beta = w_\alpha^* + (k + a)q_\alpha(y_\beta - y_\alpha) \). Platform \( \beta \) has no incentive to deviate in this case. Therefore we consider cases where \( \bar{w}_\beta \) is positive. Proceeding in a similar manner to the previous cases we can show that \( d(\gamma) \geq 0 \) for \( \gamma > a \). This implies that platform \( \beta \) has no incentive to deviate to configuration \( C_{1IV} \).

We have shown that the equilibrium price pair \((w_\alpha^*, w_\beta^*)\) for which condition (17) holds is a pure strategy Nash Equilibrium in the price subgame. We next show that configuration \( C_{111} \) with a corner solution exists and give both the necessary and sufficient conditions under which this configuration exists.

**Lemma A.4.** Let Assumption 1 hold. Given a tuple \((\gamma, \alpha, f, y_\alpha, y_\beta)\), there exists a unique equilibrium price pair \((w_\alpha^*, w_\beta^*) \in \mathcal{R}_{111} \) such that \( w_\beta^* = (\gamma - a)(q_\alpha + q_\beta)y_\beta \) only if

\[ \frac{2f(y_\alpha - y_\beta) + 30y_\alpha - 3y_\beta}{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta} \leq \frac{\gamma}{a} \leq \min \left\{ \frac{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{2f(y_\alpha - y_\beta) + 6y_\alpha + 21y_\beta}, \frac{4f(y_\alpha - y_\beta) + 18y_\alpha - 9y_\beta}{4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta} \right\}. \]

Proof. In Section A.2 we determined that the prices, in the unique equilibrium pair \((w_\alpha^{ce}, w_\beta^{ce})\), are unique and mutual best replies in the restricted domain \( \mathcal{R}_{111} \) if a covered market configuration with a corner solution\(^{13}\) was assumed. Thus this price pair is our only price subgame equilibrium candidate.

Since our only candidate pair is \((w_\alpha^{ce}, w_\beta^{ce})\), we need to show that the prices in these pair are also mutual best replies on the whole domain of strategies, i.e., given price \( w_\alpha^{ce} \), platform \( \beta \) does not have an incentive to change to price \( w_\beta^{ce} \) which will result in another configuration and a higher profit. We show that \( w_\beta^{ce} \) beats any strategy \( \bar{w}_\beta \) in the projection \( R_I \cup R_{11} \cup R_{1IV} \) against \( w_\alpha^{ce} \) and vice versa. We denote \( w_\alpha^* = w_\alpha^{ce} \) and \( w_\beta^* = w_\beta^{ce} \).

We first fix \( w_\beta^* \) and show that given this price, platform \( \alpha \) has no incentive to deviate to a price that would result in configuration \( C_1, C_{11} \) or \( C_{1IV} \). We note that under the price \( w_\beta^* \) it is not possible to have the uncovered configurations \( C_I \) or \( C_{11} \) since all content providers have an incentive to participate. So we only look at the possibility of deviating to configuration \( C_{1IV} \). We denote the profit under the price pair \((w_\alpha^*, w_\beta^*)\) as \( \pi^*_\alpha \) and that under the pair \((\bar{w}_\alpha, w_\beta^*)\) as \( \bar{\pi}_\alpha \). We denote the difference \( \pi^*_\alpha - \bar{\pi}_\alpha \) as \( d(\gamma) \).

(1) **Platform \( \alpha \) has no incentive to deviate to configuration \( C_{1IV} \).** We find platform \( \alpha \)'s best reply given \( w_\alpha^* \) under market configuration \( C_{1IV} \) and show that the profit realized is less than that under configuration \( C_{111} \) at price \( w_\alpha^* \). Let \( \bar{w}_\alpha \) be the best reply of platform under configuration \( C_{1IV} \). It is given by

\[ \bar{w}_\alpha = \text{argmax } \pi_\alpha(w_\alpha, w_\beta^*), \]
\[ \text{s.t. } w_\alpha \leq (\gamma - a)(q_\alpha y_\alpha + q_\beta y_\beta). \]

\(^{13}\) A corner solution refers to the instance when \( \bar{w}_\alpha = (\gamma - a)(q_\alpha + q_\beta)y_\beta \).
We now fix \( w^*_\alpha \) and show that platform \( \beta \) has no incentive to deviate to any price \( \overline{w}_\beta \). We note that it is not possible for platform \( \beta \) to come up with prices which will result in configuration \( C_{IV} \) where all CP’s flock to platform \( \alpha \), because \( w^*_\alpha \) is defined only for \( \overline{\gamma} \leq \min \left\{ \frac{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{2f(y_\alpha - y_\beta) + 6y_\alpha + 21y_\beta}, \frac{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{2f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta} \right\} \), where as configuration \( C_{IV} \) results only if \( \overline{\gamma} > \frac{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{2f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta} \). We denote the profit of platform \( \beta \) under the price pair \((w^*_\alpha, w^*_\beta)\) as \( \pi^*_\beta \) and that under the pair \((\overline{w}_\beta, w^*_\alpha)\) as \( \pi^{*\beta}_\beta \). We denote the difference \( \pi^*_\beta - \pi^{*\beta}_\beta \) as \( d(\gamma) \).

1. **Platform \( \beta \) has no incentive to deviate to configuration \( C_I \).** We show that the best response given \( w^*_\alpha \), such that configuration \( C_I \) emerges, will yield a lower profit. Let \( \overline{w}_\beta \) denote the best response under \( C_I \) given \( w^*_\alpha \). It is given by,

\[
\overline{w}_\beta = \arg\max_{w_\beta} \pi_\beta(w^*_\alpha, w_\beta), \quad \text{s.t.} \quad w_\beta \geq \frac{w^*_\alpha(q_\alpha + q_\beta)y_\beta}{q_3y_\beta + q_\alpha y_\alpha}.
\]

For this configuration to occur we need the condition in (6) to be satisfied hence the constraint in the above maximization problem. Since \( \pi_\beta \) is independent of \( w_\beta \) we have the best response satisfying the constraint inequality i.e., \( \overline{w}_\beta \geq \frac{w^*_\alpha(q_\alpha + q_\beta)y_\beta}{q_3y_\beta + q_\alpha y_\alpha} \). The function \( d(\gamma) \) is a concave function in \( \overline{\gamma} \), because \( \frac{\partial^2 d(\gamma)}{\partial^2(\gamma)} < 0 \). Moreover, \( d(\gamma) \) has two roots at

\[
\gamma_1 = a \quad \text{and} \quad \gamma_2 = \frac{a(4f(y_\alpha - y_\beta) + 18y_\alpha - 9y_\beta)}{4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta}.
\]

Thus for all \( \gamma_1 \leq \overline{\gamma} \leq \gamma_2 \), we have \( d(\gamma) \geq 0 \). In Section A.2 the equilibrium pair \((w^*_{cc}, w^*_{cc})\) is defined only if \( \overline{\gamma} \in [\gamma_1, \gamma_2] \). Therefore platform \( \beta \) has no incentive to deviate to a price that results in \( C_I \).

2. **Platform \( \beta \) has no incentive to deviate to configuration \( C_{II} \).** This follows from the fact that the maximization problem given below has no solution.

\[
\overline{w}_\beta = \arg\max_{w_\beta} \pi_\beta(w^*_\alpha, w_\beta), \quad \text{s.t.} \quad w_\beta > (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta.
\]

We note that the supremum to the this problem is given by \( \overline{w}_\beta = (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta \). Therefore any price \( w_\beta \) satisfying the maximization constraint will yield a lower profit.

3. **Platform \( \beta \) has no incentive to deviate to configuration \( C_{IV} \).** If platform \( \beta \) chooses to deviate to a configuration where all CPs subscribe to it, the best price it can offer is denoted by \( \overline{w}_\beta \) and is given by,

\[
\overline{w}_\beta = \arg\max_{w_\beta} \pi_\beta(w^*_\alpha, w_\beta), \quad \text{s.t.} \quad w_\beta \leq w^*_\alpha + (\overline{\gamma} + a)q_\alpha(y_\beta - y_\alpha).
\]

\[14 \quad \frac{\partial^2 d(\gamma)}{\partial^2(\gamma)} > \left( 216\alpha(y_\alpha - y_\beta) \right)^{-1} (- (y_\alpha - y_\beta)^2 16 f^3 + (y_\alpha - y_\beta)^2 32 f^2 + (12y_\alpha y_\beta + 84y_\alpha^2 + 15y_\beta^2) f). \]

\[15 \quad \frac{\partial^2 d(\gamma)}{\partial^2(\gamma)} / \partial^2(\gamma) < \partial^2 d(\gamma) / \partial^2(\gamma) < \left( f(-12y_\alpha^2 - 108y_\alpha y_\beta - 33y_\beta^2 - 90y_\beta y_\alpha^2 - 48y_\beta^2 y_\alpha + 3f y_\alpha y_\beta^2 + 4f y_\alpha^2 + 41f y_\beta + 8y_\alpha^2 f^2 + 4y_\beta^2 f^2 - 12y_\alpha^2 f y_\beta) ) / (108\alpha(y_\alpha - y_\beta)(2y_\alpha + y_\beta)) \right). \]
The profit function is increasing in \( w_\beta \), therefore the constraint binds and we have \( \overline{w}_\beta = w_\alpha^* + (k + a)q_\alpha(y_\alpha - y_\beta) \).\(^\text{16}\) The difference \( d(\overline{\gamma}) \) between the profits under the price pair \((w_\alpha^*, w_\beta^*)\) in configuration \( C_{III} \) and that under price pair \((w_\alpha^*, \overline{w}_\beta)\) in configuration \( C_{IV} \) is concave whenever \( y_\alpha \leq y_\beta \frac{9 + f}{6 + f} \) and convex vice versa.\(^\text{17}\) Moreover, \( d(\overline{\gamma}) \) has two roots at

\[
\gamma_1 = \frac{a((3 - f)y_\beta + (12 + f)y_\alpha)}{((-9 - f)y_\beta + f y_\alpha)} \quad \text{and} \quad \gamma_2 = \frac{a((15 - 4f)y_\beta + (-6 + 4f)y_\alpha)}{((-4f + 3)y_\beta + (6 + 4f)y_\alpha)}.
\]

One can show that when \( y_\alpha \leq y_\beta \frac{9 + f}{6 + f} \) the interval in which configuration \( C_{III} \) is defined lies between the interval defined by the two roots. Since in this case \( d(\overline{\gamma}) \) is concave the difference is positive implying that platform \( \beta \) has no incentive to deviate. For the region in which configuration \( C_{III} \) is defined \( d(\overline{\gamma}) > 0 \) since in the previous cases we can show that \( d(\overline{\gamma}) \geq 0 \) for \( \overline{\gamma} > a \). This implies that platform \( \beta \) has no incentive to deviate. In the case when \( y_\alpha \geq y_\beta \frac{9 + f}{6 + f} \) the roots given by (32) above are negative. Since \( d(\overline{\gamma}) \) is convex and configuration \( C_{III} \) is defined only for positive \( \overline{\gamma} \) we have that platform \( \beta \) has no incentive to deviate.

\[ \square \]

We now show that configuration \( C_{III} \) with an interior solution exists and give both the necessary conditions under which this configuration exists.

**Lemma A.5.** Given a tuple \((\overline{\gamma}, a, f, y_\alpha, y_\beta)\), there exists a unique equilibrium price pair \((w_\alpha^*, w_\beta^*) \in \mathcal{R}_{III}\) such that \( w_\beta^* < (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta \) only if

\[
\frac{2f(y_\alpha - y_\beta) + 9y_\beta + 18y_\alpha}{2f(y_\alpha - y_\beta) + 6y_\alpha + 21y_\beta} < \overline{\gamma} < \frac{f + 18}{f}.
\]

**Proof.** We follow the same line of proof applied in the previous two lemmas. From section A.2, we know that the prices in the pair \((w_{\alpha}^c, w_{\beta}^c)\) are unique and mutual best replies in the restricted domain \( \mathcal{R}_{III} \); if a covered market configuration was assumed and an interior solution resulted.\(^\text{18}\) Thus this price pair is our only candidate for the price equilibrium pair that falls in \( \mathcal{R}_{III} \) (with an interior solution). Moreover, it is also shown in the same section that for \((w_{\alpha}^c, w_{\beta}^c)\) to be in \( \mathcal{R}_{III} \) it is necessary and sufficient that the condition expressed in (21) holds.

We now show that the prices in the equilibrium price pair \((w_{\alpha}^c, w_{\beta}^c)\) are also mutual best replies on the whole domain of strategies, i.e., given price \( w_{\alpha}^c \), platform \( \beta \) does not have an incentive to change to price \( \overline{w}_\beta \) which will result in another configuration and a higher profit, and vice versa. Formally, we show that \( w_{\beta}^c \) beats any strategy \( w_\beta \) in the projection \( \mathcal{R}_I \cup \mathcal{R}_{II} \cup \mathcal{R}_{IV} \) against \( w_{\alpha}^c \) and vice versa.

We first fix \( w_\beta^* = w_{\beta}^c \) and show that platform \( \alpha \) has no incentive to deviate to any price \( \overline{w}_\alpha \). We note that it is not possible for platform \( \alpha \) to come up with prices which will result in either configuration \( C_I \) or \( C_{III} \) because \( w_\alpha^* < (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\alpha \).\(^\text{19}\) We therefore check to see if platform \( \alpha \) deviates to a covered but preempted market, i.e, configuration \( C_{IV} \). We denote the profit of platform \( \alpha \) under the price pair \((w_\alpha^*, w_\beta^*)\) as \( \pi_\alpha^* \) and that under the pair \((\overline{w}_\alpha, w_\beta^*)\) as \( \overline{\pi}_\alpha \). We denote the difference \( \pi_\alpha^* - \overline{\pi}_\alpha \) as \( d(\overline{\gamma}) \).

(1) **Platform \( \alpha \) has no incentive to deviate to configuration \( C_{IV} \).** If platform \( \alpha \) chooses to deviate to a configuration where all CPs subscribe to it, the best price it can offer is denoted by \( \overline{w}_\alpha \), and it is given

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\(^{16}\)The constraint directly arises from the utility maximization by the CPs. In particular, all CPs have to prefer joining the low quality platforms including those with highest quality \((\overline{\gamma} + a)\).

\(^{17}\)\(\frac{\partial^2 d(\overline{\gamma})}{\partial^2 \overline{\gamma}} = ((33fy_\beta^3 - 27y_\beta^2 - 8y_\alpha f)^2 + 6y_\alpha^2f - 54y_\alpha fy_\beta - 39y_\alpha f^2y_\beta + 4y_\beta^2f^2)^2 + 4y_\beta^3f^2f) / (108(a(y_\alpha - y_\beta)))\).

\(^{18}\)An interior solution refers to the instance when \( w_\beta^c < (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta \).

\(^{19}\)The fact that \( w_\beta^c \) is an interior solution implies a covered market will result for any value \( \overline{w}_\alpha \).
by,

$$\bar{\omega}_\alpha = \text{argmax } \pi_\alpha (w_\alpha, w_\beta^*),$$

s.t. \( w_\alpha \leq q_\alpha (\gamma - a)(y_\alpha - y_\beta) + w_\beta^* \).

The constraint in the above maximization problem reflects the fact that all content providers should prefer platform \( \alpha \) to platform \( \beta \) for configuration \( C_{IV} \) to occur. Since \( \pi_\alpha \) is linear and increasing in \( w_\alpha, \bar{\omega}_\alpha = (\gamma - a)q_\alpha (y_\alpha - y_\beta) + w_\beta^* \). Under this price \( d(\gamma) \) is a convex function in \( \gamma \), because, \( \frac{d^2 d(\gamma)}{d\gamma^2} > 0 \). Moreover \( d(\gamma) \) has a single root at \( \gamma = a \frac{f + 18}{f + 9} y_\beta \). Thus for all values of \( \gamma \), the following inequality holds, \( d(\gamma) \geq 0 \). Consequently platform \( \alpha \) has no incentive to deviate to configuration \( C_{IV} \).

We now fix \( w_\alpha^* = w_\alpha^\ast \) and show that platform \( \beta \) has no incentive to deviate to any pair \( \bar{\omega}_\beta \) in any other configuration. We denote the profit of platform \( \beta \) under the price pair \( (w_\alpha^*, w_\beta^*) \) as \( \pi_\beta^* \) and that under the pair \( (\bar{\omega}_\beta, w_\alpha^*) \) as \( \pi_\beta^\ast \). We denote the difference \( \pi_\beta^* - \pi_\beta^\ast \) as \( d(\gamma) \).

1. **Platform \( \beta \) has no incentive to deviate to configuration \( C_I \).** We show that the best response given \( w_\alpha^\ast \), such that configuration \( C_I \) emerges, will yield a lower profit. Let \( \bar{\omega}_\beta \) denote the best response under \( C_I \) given \( w_\alpha^\ast \). It is given by,

$$\bar{\omega}_\beta = \text{argmax } \pi_\beta (w_\alpha^*, w_\beta),$$

s.t. \( w_\beta \geq \frac{y_\beta w_\alpha^*(q_\alpha + q_\beta)}{(q_\alpha y_\alpha + y_\beta q_\beta)} \).

For this configuration to occur the lowest quality content provider should not join platform \( \alpha \), which implies \( w_\alpha^* > (\gamma - a)(q_\alpha y_\alpha + (\beta + q_\alpha)) \). This implies that the configuration is possible only if \( \frac{\gamma}{\alpha} < \frac{7f(y_\alpha - y_\beta) + 36y_\alpha - 9y_\beta}{7f(y_\alpha - y_\beta) + 12y_\alpha + 15y_\beta} \). We denote this bound by \( \hat{\gamma} \). Moreover, from section A.2 we know that \( w_\alpha^* \) is defined only if \( \frac{\gamma}{\alpha} > \frac{2f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta}{2f(y_\alpha - y_\beta) + 6y_\alpha + 21y_\beta} \). We denote this upper bound by \( \hat{\gamma} \). Therefore, configuration \( C_{IV} \) can occur only if \( \hat{\gamma} < \gamma < \hat{\gamma} \). The function \( d(\gamma) \) is a convex function in \( \gamma \), because, \( \frac{d^2 d(\gamma)}{d\gamma^2} > 0 \). Moreover \( d(\gamma) \) has two roots at \( \gamma_1 \) and \( \gamma_2 \). These are given explicitly below,

$$\gamma_1 = \frac{a(Q(f, y_\beta) + \sqrt{(8f^2 + 216 + 96f)y_\alpha + 36(18f y_\beta^2 + 6f^2 y_\beta^3 - 6y_\alpha f^2 y_\beta + 36y_\alpha f y_\beta))}}{(36 - 30f + 67f^2)y_\beta + (8f^2 + 72 + 48f)y_\alpha},$$

$$\gamma_2 = \frac{a(Q(f, y_\beta) + \sqrt{(8f^2 + 216 + 96f)y_\alpha - 36(18f y_\beta^2 + 6f^2 y_\beta^3 - 6y_\alpha f^2 y_\beta + 36y_\alpha f y_\beta))}}{(36 - 30f + 67f^2)y_\beta + (8f^2 + 72 + 48f)y_\alpha},$$

where \( Q(f, y_\beta) = (67f^2 + 102f + 108)y_\beta \). Thus for \( \gamma \leq \gamma_2 \), we have \( d(\gamma) \geq 0 \). It is also the case that \( \gamma_2 \geq \hat{\gamma} \geq \hat{\gamma} \) when \( \frac{y_\alpha}{y_\beta} \leq \frac{f + 9}{f} \). Therefore for \( \frac{y_\alpha}{y_\beta} \leq \frac{f + 9}{f} \) platform \( \beta \) has no incentive to deviate. For \( \frac{y_\alpha}{y_\beta} > \frac{f + 9}{f}, \hat{\gamma} < \hat{\gamma} \) which implies that configuration \( C_{IV} \) is not possible. Thus given \( w_\alpha^* \), platform \( \beta \) has no incentive to deviate to a price that results in \( C_{IV} \).

2. **Platform \( \beta \) has no incentive to deviate to configuration \( C_{II} \).** For this configuration to occur the lowest quality content provider should not join platform \( \beta \), which implies \( w_\beta > (\gamma - a)(q_\alpha + q_\beta) y_\beta \). Therefore, platform \( \beta^\prime \)'s best price under this configuration is formally given by,

$$\bar{\omega}_\beta = \text{argmax } \pi_\beta (w_\alpha^*, w_\beta),$$

s.t. \( w_\beta > (\gamma - a)(q_\alpha + q_\beta)y_\beta \).

\[20 \frac{d^2 d(\gamma)}{d\gamma} \] = \[\frac{1}{486(6 + 5f)}(9y_\alpha - y_\beta)f\].

\[21 \frac{d^2 d(\gamma)}{d\gamma} \] = \[\frac{(y_\alpha - y_\beta)f((36 - 30f + 67f^2)y_\beta + (8f^2 + 72 + 48f)y_\alpha)}{486(y_\beta + 2y_\alpha)} \].
The profit function \( \pi_{\beta} \) is concave in \( w_{\beta} \). An interior solution to the above maximization problem exists only if \( w_{\beta} > (\bar{\gamma} - a)(q_{a} + q_{\beta})y_{\beta} \). One can show that this happens only if \( \bar{\gamma} < \hat{\gamma} \) where 
\[
\hat{\gamma} = \frac{(20f(y_{a} - y_{\beta}) + 9y_{a} + 18y_{\beta})}{(20f(y_{a} - y_{\beta}) - 3y_{a} + 30y_{\beta})}.
\]
But configuration \( C_{11} \) with an interior solution is only defined for \( \hat{\gamma} < \bar{\gamma} < a \frac{5f + 18}{5f + 6} \). Since \( \hat{\gamma} > \bar{\gamma} \), a maximum does not exist and the supremum of the profit function under this configuration is that given under the price \( \bar{\omega}_{\beta} = (\bar{\gamma} - a)(q_{a} + q_{\beta})y_{\beta} \). The function \( d(\bar{\gamma}) \), under this price, is a convex function of \( \bar{\gamma} \), because \( \frac{\partial^{2}d(\bar{\gamma})}{\partial \bar{\gamma}^{2}} > 0 \). Moreover, \( d(\bar{\gamma}) \) has a single root at \( \hat{\gamma} \). Therefore \( d(\bar{\gamma}) > 0 \) for all \( \bar{\gamma} < \frac{5f + 18}{5f + 6} \). This implies that given \( w_{\alpha}^{*} \), platform \( \beta \) has no incentive to deviate to a price that results in configuration \( C_{11} \).

(3) \textbf{Platform} \( \beta \text{ has no incentive to deviate to configuration } C_{1V} \text{ where all CPs migrate to platform } \alpha \).

For this configuration to occur the lowest quality content provider should not join platform \( \beta \) but platform \( \alpha \). This implies \( w_{\beta} \geq (\bar{\gamma} - a)(y_{\beta} - y_{a}) + w_{\alpha}^{*} \). Therefore, platform \( \beta \)'s best price under this configuration is formally given by,
\[
\bar{\omega}_{\beta} = \text{argmax} \quad \pi_{\beta}(w_{\alpha}^{*}, w_{\beta}),
\]
s.t. \( w_{\beta} \geq (\bar{\gamma} - a)(y_{\beta} - y_{a}) + w_{\alpha}^{*} \).

For this configuration to occur \( \bar{\gamma} \geq \hat{\gamma} \). The function \( d(\bar{\gamma}) \) is a convex function of \( \bar{\gamma} \), because \( \frac{\partial^{2}d(\bar{\gamma})}{\partial \bar{\gamma}^{2}} > 0 \). Moreover, \( d(\bar{\gamma}) \) has a single root at \( \frac{5f + 18}{5f + 6} \). Therefore \( d(\bar{\gamma}) \geq 0 \) for all \( \bar{\gamma} \), and in particular when \( \bar{\gamma} \geq \hat{\gamma} \). This implies that given \( w_{\alpha}^{*} \), platform \( \beta \) has no incentive to deviate to a price that results in configuration \( C_{1V} \).

(4) \textbf{Platform} \( \beta \text{ has no incentive to deviate to configuration } C_{1V} \text{ where all CPs migrate to platform } \beta \).

For this configuration to occur the highest quality content provider should join platform \( \beta \). This implies \( w_{\beta} \leq (\bar{\gamma} + a)(y_{\beta} - y_{a}) + w_{\alpha}^{*} \). Therefore, platform \( \beta \)'s best price under this configuration is formally given by,
\[
\bar{\omega}_{\beta} = \text{argmax} \quad \pi_{\beta}(w_{\alpha}^{*}, w_{\beta}),
\]
s.t. \( w_{\beta} \leq (\bar{\gamma} - a)(y_{\beta} - y_{a}) + w_{\alpha}^{*} \).

The function \( d(\bar{\gamma}) \) is a convex function of \( \bar{\gamma} \), because \( \frac{\partial^{2}d(\bar{\gamma})}{\partial \bar{\gamma}^{2}} > 0 \). Moreover, \( d(\bar{\gamma}) \) has a single root at \( \frac{5f + 18}{5f + 6} \). Therefore \( d(\bar{\gamma}) \geq 0 \) for all \( \bar{\gamma} \), and in particular when this configuration occurs. This implies that given \( w_{\alpha}^{*} \), platform \( \beta \) has no incentive to deviate to a price that results in configuration \( C_{1V} \).

We finally show that configuration \( C_{1V} \) exists. We give the necessary conditions for its existence together with the possible price characterizations in this configuration.

**Lemma A.6.** \textit{Given a tuple } \( (\bar{\gamma}, a, f, y_{a}, y_{\beta}) \), \textit{there exists an equilibrium price pair } \( (w_{\alpha}^{*}, w_{\beta}^{*}) \in \mathcal{R}_{1V} \text{ such that}

1. \( w_{\beta}^{*} > (\bar{\gamma} - a)y_{\beta}, \ w_{\alpha}^{*} = (\bar{\gamma} - a)(q_{a}y_{\beta} + q_{\beta}y_{a}) \) only if,

\[
\frac{\bar{\gamma}}{a} \geq \frac{4f(y_{a} - y_{\beta}) + 18y_{a} + 9y_{\beta}}{4f(y_{a} - y_{\beta}) + 6y_{a} + 3y_{\beta}} \quad \text{and} \quad y_{a} \geq \frac{f + 9}{f} y_{\beta}.
\]

2. \( \frac{\partial^{2}d(\bar{\gamma})}{\partial \bar{\gamma}^{2}} = \frac{((f + 3)(f + 3/2)^{2} - 63/4)^{2}y_{a}^{2} + ((f + 3)(f + 3/2)^{2} - 63/4)y_{a}y_{\beta} + 4f^{2}(f + 3)^{2}y_{\beta}^{2}}{486a(y_{a} - y_{\beta})}. \)

3. \( \text{If } \bar{\gamma} < \hat{\gamma} \text{ then we cannot have a covered market where all CP's patronize platform } \alpha. \)

4. \( \partial^{2}d(\bar{\gamma})/\partial^{2}\bar{\gamma} = (25f^{2} + 60f + 36)(y_{a} - y_{\beta})f/486a. \)
(2) \( w^*_\beta = (\gamma - a)y_\beta \), \( w^*_\alpha = (\gamma - a)(q_\beta y_\beta + q_\alpha y_\alpha) \) only if,
\[
\frac{\gamma}{a} \geq \frac{4f(y_\alpha - y_\beta) + 18y_\alpha - 9y_\beta}{4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta} \text{ and } y_\alpha \geq \frac{f + 9}{f} y_\beta.
\]
(3) \( w^*_\beta = (\gamma - a)y_\beta - c(\gamma - a)y_\beta \), \( w^*_\alpha = \frac{1}{3}(\gamma - a)(q_\beta y_\beta + q_\alpha y_\alpha) - c(\gamma - a)y_\beta \) only if,
\[
\frac{\gamma}{a} \geq \frac{4f(y_\alpha - y_\beta) + 18y_\alpha - 9y_\beta - 9y_\beta c}{4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta - 9y_\beta c} \text{ and } y_\alpha \geq \frac{10y_\beta - 9c_y_\beta}{f}.
\]
where \( 0 < c < 1 \) and \( y_\alpha \geq \frac{f + 9 - 9c}{f} y_\beta \).

(4) \( w^*_\beta = 0 \), \( w^*_\alpha = \frac{2}{3}(\gamma - a)(y_\alpha - y_\beta) \) only if,
\[
\frac{9 + 2f}{3 + 2f} \leq \frac{\gamma}{a} < \infty.
\]

Proof. From section A.2, the condition in (30) characterizes the equilibrium price pairs that exist if configuration \( C_{IV} \) is exogenously assumed. We show a subset of this characterization is a SPE for the range of values of \( \gamma \) stated in the Lemma.

Proving case 1 : \( \beta^*_\beta > (\gamma - a)(q_\alpha + q_\beta)y_\beta \), \( \beta^*_\alpha = (\gamma - a)(q_\beta y_\beta + q_\alpha y_\alpha) \).

Let \( (w^*_\alpha, w^*_\beta) \) be a price pair that satisfies the condition in 30 where \( w^*_\beta > (\gamma - a)(q_\alpha + q_\beta)y_\beta \). We fix \( w^*_\alpha \) and check whether platform \( \beta \) has an incentive to deviate to configuration \( C_{III} \). Note that this is the only configuration that platform \( \beta \) can deviate to; since \( w^*_\alpha = (\gamma - a)(q_\alpha y_\alpha + q_\beta y_\beta) \) the lowest quality content provider will join at least one platform. Thus platform \( \beta \) can only deviate to a covered market configuration.

We denote the profit of platform \( \beta \) under the price pair \( (w^*_\alpha, w^*_\beta) \) as \( \pi^*_\beta \) and that under the pair \( (\overline{w}_\beta, w^*_\alpha) \) as \( \pi^*_\beta \). We denote the profit difference \( \pi^*_\beta - \pi^*_\beta \) as \( d(\gamma) \). Platform \( \beta \) maximizes its profit function to find the best price \( \overline{w}_\beta \) that will yield configuration \( C_{III} \) given the tuple \( (y_\alpha, y_\beta, \gamma, a) \). The maximizing problem has a constraint which ensures that the price is less than the value gained by the lowest quality CP.

Let \( \overline{w}_\beta = \arg\max \pi_\beta(w^*_\alpha, w^*_\beta), \)
\[ s.t. 0 \leq w_\beta < (\gamma - a)(q_\alpha + q_\beta)y_\beta. \]
It follows that \( 0 \leq \overline{w}_\beta < (\gamma - a)(q_\alpha + q_\beta)y_\beta \) only if \( \gamma > a \) and \( \frac{y_\alpha}{y_\beta} < \frac{f + 9}{f} \). Consequently, market configuration \( C_{III} \) is possible with this price only if \( \frac{y_\alpha}{y_\beta} < \frac{f + 9}{f} \) since we assume in the problem formulation that \( \gamma > a \). Under the price \( \overline{w}_\beta \) the function \( d(\gamma) \) is a concave function of \( \gamma \), because \( \frac{\partial^2 d(\gamma)}{\partial^2 \gamma} < 0.25 \). Moreover, \( d(\gamma) \) has a single root \( a \). Therefore for all \( \gamma > a \) and \( y_\alpha < y_\beta \frac{f + 9}{f} \) platform \( \beta \) will deviate to configuration \( C_{III} \). This suggests that we potentially could have a preempted solution when \( y_\alpha \geq y_\beta \frac{f + 9}{f} \).

We fix \( w^*_\beta \) and check whether platform \( \alpha \) has an incentive to deviate to configuration \( C_I \) or \( C_{III} \) when \( \frac{f + 9}{f} \). We only consider those two configurations because configuration \( C_{III} \) is not possible given \( w^*_\beta > (\gamma - 1)y_\beta \). We denote the profit of platform \( \alpha \) under the price pair \( (w^*_\alpha, w^*_\beta) \) as \( \pi^*_\alpha \) and that under the pair \( (\overline{w}_\alpha, w^*_\beta) \) as \( \pi^*_\alpha \). We denote the difference \( \pi^*_\alpha - \pi^*_\alpha \) as \( d(\gamma) \).

Let \( \overline{w}_\alpha = \arg\max \pi_\alpha(w^*_\beta, w^*_\alpha), \)
\[ s.t. \ w^*_\alpha \leq w^*_\beta(y_\beta q_\beta + q_\alpha y_\alpha). \]
Let \( \overline{w}_\alpha \) be the interior solution. It follows that this solution exists whenever \( w^*_\beta \geq \frac{(q_\alpha + q_\beta)y_\alpha y_\beta}{q_\beta y_\beta q_\alpha + q_\alpha y_\alpha y_\beta} \). We denote this bound by \( w^*_\beta^* \). Therefore an interior solution \( \overline{w}_\alpha \) results in market configuration \( C_I \) only if
\[ \frac{\partial^2 d(\gamma)}{\partial^2 \gamma} = (-y_\alpha f + (9 + f)y_\beta)^2 f/(216a(y_\alpha - y_\beta)). \]
\( w^*_\beta \geq w^{**}_\beta > (\gamma - a)(q_\alpha + q_\beta)y_\beta \). One can show that this occurs only if \( \gamma < \frac{a(4f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta)}{(4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta)} \). We denote this bound as \( \gamma^* \). The function \( d(\gamma) \) under this price is a concave function of \( \gamma \), because \( \frac{\partial^2 d(\gamma)}{\partial \gamma^2} < 0 \). Moreover \( d(\gamma) \) has a single root at \( \gamma^* \). Therefore, for all \( \gamma < \gamma^* \) platform \( \beta \) will deviate to configuration \( C_1 \).

We now investigate the case when \( w^*_\beta \in (\gamma - a)(q_\alpha + q_\beta)y_\beta, w^{**}_\beta \). In this case \( \pi_\alpha = \frac{w^*_\beta(q_\alpha y_\beta + q_\alpha y_\alpha)}{y_\beta + y_\alpha} \).

We denote the difference \( \pi^*_\alpha - \pi_\alpha \) as \( d(w^*_\beta) \). This difference is convex in \( w^*_\beta \) since \( \frac{\partial^2 d(\gamma)}{\partial \gamma^2} = \frac{q_\alpha y_\alpha + q_\beta y_\beta}{y_\beta + y_\alpha} > 0 \). This difference has two roots at \( r_1 \) and \( r_2 \). Whenever \( w^*_\beta \in (r_1, r_2) \) platform \( \alpha \) has an incentive to deviate. It follows that this occurs whenever \( r_2 > r_1 \) which in turn results whenever \( \gamma < \gamma^* \). Therefore if \( w^*_\beta \in (\gamma - a)(q_\alpha + q_\beta)y_\beta, w^{**}_\beta \), which occurs only if \( \gamma < \gamma^* \) platform \( \alpha \) has an incentive to deviate. This implies that there is a potential for a preempted market if \( \gamma \geq \gamma^* \).

We now check whether there is a platform for deviate to configuration \( C_{II} \) or \( C_{II} \) where \( r_\alpha \in (0, 1) \). This can occur whenever \( w^*_\beta \in (\gamma - a)(q_\alpha + q_\beta)y_\beta, 2/9(-\gamma(2f + 3) + a(2f - 3))(y_\alpha - y_\beta)f \). Moreover, this interval is non-empty whenever \( \gamma < \frac{(4f(y_\alpha - y_\beta) + 18y_\alpha + 9y_\beta)}{(4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta)} \). This implies that platform \( \alpha \) would have an incentive to deviate to whenever the former applies since \( d(w^*_\beta) > 0 \) is positive in this range.

Putting all the above results together implies that configuration \( C_{IV} \) with the prize pair given above is possible only if \( \gamma > \gamma^* \).

**Proving case 2:** \( w^*_\beta = (\gamma - a)y_\beta, \ w^*_\alpha = \frac{1}{3}(\gamma - a)(q_\beta y_\beta + q_\alpha y_\alpha) \)

The first part of the proof where we fix \( w^*_\alpha \) and check whether platform \( \beta \) has an incentive to deviate to configuration \( C_{II} \) is exactly the same as in the previous case. We fix \( w^*_\beta \) and check whether platform \( \alpha \) has an incentive to deviate to configuration \( C_{II} \) when \( \frac{y_\alpha}{y_\beta} \geq \frac{f + 9}{f} \). We only consider these configuration because configuration \( C_1 \) and \( C_II \) are not possible given \( w^*_\beta = (\gamma - a)(q_\alpha + q_\beta)y_\beta \).

We denote the profit of platform \( \alpha \) under the price pair \( (w^*_\alpha, w^*_\beta) \) as \( \pi^*_\alpha \) and that under the pair \( (w_\alpha, w^*_\beta) \) as \( \pi_\alpha \). We denote the difference \( \pi^*_\alpha - \pi_\alpha \) as \( d(\gamma) \).

Let \( \pi^*_\alpha = \arg \max \pi_\alpha(w^*_\beta, w_\alpha) \), s.t. \( w_\alpha > (\gamma - a)(q_\beta y_\beta + q_\alpha y_\alpha) \).

It follows that \( \pi^*_\alpha \) exists whenever \( \gamma < \gamma^* \). It is also the case that function \( d(\gamma) \) is a concave function of \( \gamma \), because \( \frac{\partial^2 d(\gamma)}{\partial \gamma^2} < 0 \). Moreover, \( d(\gamma) \) has a single root \( \gamma^* \). Therefore for all \( \gamma \neq \gamma^* \) platform \( \beta \) will deviate to configuration \( C_{II} \). In particular, when \( \gamma < \gamma^* \) platform \( \alpha \) will always deviate since configuration \( C_{II} \) is defined for that range. This implies that a preempted market with prices \( (w^*_\alpha, w^*_\beta) \) occurs only if \( \gamma^* \leq \gamma \) and \( \frac{y_\alpha}{y_\beta} \geq \frac{f + 9}{f} \).

**Proving case 3:** \( w^*_\alpha = (\gamma - a)(q_\alpha y_\alpha + q_\beta y_\beta) - c(\gamma - a)y_\beta \) and \( w^*_\beta = (\gamma - a)(q_\alpha + q_\beta)y_\beta - c(\gamma - a)(q_\alpha + q_\beta)y_\beta \), where \( c \in [0, 1] \).

We fix \( w^*_\alpha \) and check whether platform \( \beta \) has an incentive to deviate to configuration \( C_{II} \). Note that this is the only configuration that platform \( \beta \) can deviate too since \( w^*_\alpha < (\gamma - a)(q_\alpha y_\alpha + q_\beta y_\beta) \) which implies that the lowest quality content provider will join at least one platform.

\[ \frac{\partial^2 d(\gamma)}{\partial \gamma^2} = \frac{(-30y_\alpha^2 + 48y_\beta^2 - 24y_\alpha y_\beta - 32y_\alpha + 30y_\alpha y_\beta + 9y_\beta^2 + 16f^2u_\alpha^2 - 24fy_\beta^2)}{(4f(y_\alpha - y_\beta) + 6y_\alpha + 3y_\beta)^2} \]

\[ \frac{\partial^2 d(\gamma)}{\partial \gamma^2} = \frac{(-1)(y_\alpha f - (0 - f)y_\alpha^2)^2f}{2160(y_\alpha - y_\beta)^2} \]
We denote the profit of platform $\beta$ under the price pair $(w^*_\alpha, w^*_\beta)$ as $\pi^*_\beta$ and that under the pair $(\overline{w}_\beta, w^*_\alpha)$ as $\overline{\pi}_\beta$. We denote the difference, $\pi^*_\beta - \overline{\pi}_\beta$ as $d(\overline{\gamma})$. Platform $\beta$ maximizes the following profit function to find the best price $\overline{w}_\beta$ that will yield configuration $C_{111}$ given the tuple $\left(y\alpha, y\beta, \overline{\gamma}\right)$.

Let $\overline{w}_\beta = \text{argmax} \; \pi_\beta(w^*_\alpha, w_\beta)$,
\[\text{s.t. } 0 < w_\beta < (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta(1 - c).\]

It follows that $\overline{w}_\beta \in (0, (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta(1 - c))$ whenever $y_\alpha/y_\beta \leq \frac{(9 - 9c + f)}{f}$. It is also the case that function $d(\overline{\gamma})$ is a concave function of $\overline{\gamma}$, because $\left(\frac{\partial^2 d(\overline{\gamma})}{\partial^2(\overline{\gamma})}\right) < 0$. Moreover, $d(\overline{\gamma})$ has a single root $a$. Therefore, for all $\overline{\gamma} > a$ and $y_\alpha/y_\beta < \frac{(9 - 9c + f)}{f}$ platform $\beta$ will deviate to configuration $C_{111}$. This suggests that we potentially could have a preempted solution when $\overline{\gamma} > a$ and $y_\alpha/y_\beta > \frac{(9 - 9c + f)}{f}$.

We now fix $w^*_\beta = (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta - c(\overline{\gamma} - a)y_\beta$ where $c \in [0, 1)$ and check whether platform $\alpha$ has an incentive to deviate to configuration $C_{111}$. We again consider only this configuration because configuration $C_I$ and $C_{11}$ are not possible given $w^*_\beta < (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta$.

Let $\overline{w}_\alpha = \text{argmax} \; \pi_\alpha(w^*_\beta, w_\alpha)$,
\[\text{s.t. } w_\alpha > (\overline{\gamma} - a)\frac{q_3y_\beta + q_\alpha y_\alpha - c(\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta}{3} = \frac{4f(y_\alpha - y_\beta) + 18y_\alpha - 9y_\beta c - 9y_\beta}{4f(y_\alpha - y_\beta) + 6y_\alpha - 9y_\beta c - 3y_\beta} > 0.\]

It follows that $\overline{w}_\alpha > (\overline{\gamma} - a)(q_\alpha y_\alpha + q_\beta y_\beta)$ and results in configuration $C_{111}$ only if $\frac{\overline{\gamma}}{a} = \frac{4f(y_\alpha - y_\beta) + 18y_\alpha - 9y_\beta c - 9y_\beta}{4f(y_\alpha - y_\beta) + 6y_\alpha - 9y_\beta c - 3y_\beta}$ and $\frac{y_\alpha}{y_\beta} \geq \frac{9 + f - 9c}{f}$.

Proving case 4: $w^*_\beta = 0$, $w^*_\alpha = q_\alpha(\overline{\gamma} - a)(y_\alpha - y_\beta)$.

We proceed in a similar way to that used in proving case 3. We fix $w^*_\beta$ and check whether platform $\beta$ has an incentive to deviate to configuration $C_{111}$. Note that this is the only configuration that platform $\beta$ can deviate too since $w^*_\alpha < (\overline{\gamma} - a)(q_\alpha y_\alpha + q_\beta y_\beta)$. This means that platform $\beta$ maximizes its profit function to find the best price $\overline{w}_\beta$ that will yield configuration $C_{111}$ given the tuple $\left(y\alpha, y\beta, \overline{\gamma}, a\right)$.

Let $\overline{w}_\beta = \text{argmax} \; \pi_\beta(w^*_\alpha, w_\beta)$,
\[\text{s.t. } w_\beta < 0.\]

However any price $w_\beta < 0$ is dominated by $w_\beta = 0$ thus platform $\beta$ has no incentive to deviate for any $\overline{\gamma}$.

We fix $w^*_\beta = 0$ and check whether platform $\alpha$ has an incentive to deviate to configuration $C_{111}$. We again consider these configuration because configuration $C_I$ and $C_{11}$ are not possible given $w^*_\beta < (\overline{\gamma} - a)(q_\alpha + q_\beta)y_\beta$.

Let $\overline{w}_\alpha = \text{argmax} \; \pi_\alpha(w^*_\beta, w_\alpha)$,
\[\text{s.t. } w_\alpha > q_\alpha(\overline{\gamma} - a)(y_\alpha - y_\beta).\]

It follows that $\overline{w}_\alpha > q_\alpha(\overline{\gamma} - a)(y_\alpha - y_\beta)$ only if $\frac{\overline{\gamma}}{a} < \frac{2f + 9}{2f + 3}$. This implies that when $\frac{\overline{\gamma}}{a} \geq \frac{2f + 9}{2f + 3}$ this price structure and market configuration are possible.

In this Appendix we have shown that there exists equilibrium price pairs that are Nash equilibrium in the price subgame. Moreover, we have shown the market configurations in which they occur and the conditions for them to occur. In particular, we have shown that each of the configurations $C_{111}$, $C_{111}$, and $C_{IV}$ exists.  

\[\frac{29 \frac{\partial^2 d(\overline{\gamma})}{\partial^2(\overline{\gamma})}}{3\frac{\partial^2 d(\overline{\gamma})}{\partial^2(\overline{\gamma})}} = \frac{-1}{216a} \frac{f(y_\alpha - y_\beta)(c - 1)^2}{a(y_\alpha - y_\beta)}.\]
A.3.1. CP Pricing Equilibrium for \( y_\alpha > y_\beta > 0 \). In this section, we provide results showing that given a tuple \((\gamma, a, f, y_\alpha, y_\beta)\) such that \( y_\alpha > y_\beta > 0 \), there exists a pure strategy price SPE pair \((w^*_\alpha, w^*_\beta)\). In addition, we prove that just one market configuration is feasible in the price SPE, and for market configurations \(C_{II} \) and \(C_{III} \), the price characterizations are unique. Specifically, we show the conditions under which particular market configurations arise as a function of the tuple \((\gamma, a, f, y_\alpha, y_\beta)\).

Our results show that the uncovered market configuration \(C_I \) does not occur at an SPE. In this configuration no CPs join the low-quality platform even though it has positive quality. We show that there exists a profitable price deviation by the low-quality platform that involves CPs joining this platform. On the other hand, we show that given a tuple \((\gamma, a, f, y_\alpha, y_\beta)\) one of the other configurations, \(C_{II} \), \(C_{III} \) or \(C_{IV} \), will emerge. In doing so, we determine the set of parametric values \((\gamma, a, f, y_\alpha, y_\beta)\) for which these different configurations exist.

We prove the existence of the price SPE constructively. To do that, we first identify candidate equilibrium price pairs in each possible market configuration (see Appendix A.2), and then check whether these price equilibrium pairs are indeed Nash equilibria of the price subgame (see Appendix A.3). We do so by verifying that the equilibrium price candidates are best replies on the whole domain of strategies: That is, not only are they best responses in their respective market configurations but also best replies if the other market configurations are taken into account.

We now present a theorem showing that for any tuple \((\gamma, a, f, y_\alpha, y_\beta)\) a price subgame Nash equilibrium exists, and only one market configuration is feasible. In addition, for market configurations \(C_{II} \) and \(C_{III} \), the price characterizations are unique.

**Theorem A.7.** Given a tuple \((\gamma, a, f, y_\alpha, y_\beta)\) that satisfies that \( y_\alpha > y_\beta > 0 \), there exists a Nash equilibrium pair \((w^*_\alpha, w^*_\beta)\) in the price subgame. Moreover, letting \( \tau_1 = \frac{2f(y_\alpha-y_\beta)+30y_\alpha-3y_\beta}{2f(y_\alpha-y_\beta)+18y_\alpha+9y_\beta} \), \( \tau_2 = \frac{2f(y_\alpha-y_\beta)+18y_\alpha+9y_\beta}{2f(y_\alpha-y_\beta)+6y_\alpha+21y_\beta} \), \( \tau_3 = \frac{4f(y_\alpha-y_\beta)+18y_\alpha-9y_\beta}{4f(y_\alpha-y_\beta)+6y_\alpha+9y_\beta} \), and \( \tau_4 = \frac{5f+18}{5f+6} \), the resulting market configuration is unique and the following holds:

1. If \( 1 < \frac{\gamma}{a} < \tau_1 \), then the equilibrium price pair is unique and \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{II}\).
2. If \( \frac{\gamma}{a} \leq \frac{\gamma}{a} < \min\{\tau_2, \tau_3\} \) then the equilibrium price pair is unique and \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{III}\).
3. If \( \tau_2 < \frac{\gamma}{a} < \tau_4 \) then the equilibrium price pair is unique and \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{III}\).
4. If \( \max\{\tau_3, \tau_4\} \leq \frac{\gamma}{a} \) then \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{IV}\).

**Proof.** Given a tuple \((\gamma, a, f, y_\alpha, y_\beta)\), we know from Lemma A.2 through A.6 that an equilibrium pair \((w^*_\alpha, w^*_\beta)\) exists. Moreover, cases 1, 2, and 3 directly follow from Lemma A.3 through A.5. In particular,

1. If \( 1 < \frac{\gamma}{a} < \frac{2f(y_\alpha-y_\beta)+30y_\alpha-3y_\beta}{2f(y_\alpha-y_\beta)+18y_\alpha+9y_\beta} \), then the equilibrium price pair is unique and \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{II}\).

This follows from Lemma A.3.

2. If \( \frac{2f(y_\alpha-y_\beta)+30y_\alpha-3y_\beta}{2f(y_\alpha-y_\beta)+18y_\alpha+9y_\beta} \leq \frac{\gamma}{a} < \min\left\{ \frac{2f(y_\alpha-y_\beta)+18y_\alpha+9y_\beta}{2f(y_\alpha-y_\beta)+6y_\alpha+21y_\beta}, \frac{4f(y_\alpha-y_\beta)+18y_\alpha-9y_\beta}{4f(y_\alpha-y_\beta)+6y_\alpha+9y_\beta} \right\} \) then the equilibrium price pair is unique and \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{III}\).

This follows from Lemma A.4.

3. If \( \tau_3 < \frac{\gamma}{a} < \frac{5f+18}{5f+6} \) then the equilibrium price pair is unique and \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{III}\).

This follows from Lemma A.5.

4. If \( \frac{\max\left\{ \frac{5f+18}{5f+6}, \frac{4f(y_\alpha-y_\beta)+18y_\alpha-9y_\beta}{4f(y_\alpha-y_\beta)+6y_\alpha+9y_\beta} \right\} \leq \frac{\gamma}{a} \leq \infty \) then \((w^*_\alpha, w^*_\beta) \in \mathcal{R}_{IV}\).

This follows from Lemma A.6.

\[ \square \]

Figure 8 shows the resulting market configurations for different values of the quality investment ratio \( I = y_\alpha/y_\beta \), the inverse of a scaled coefficient of variation, \( \gamma/a \) and a fixed mass \( f \) of consumers. For a fixed \( I \), as \( \gamma/a \) increases the covered market is more likely. At the extreme, when \( \gamma/a \) is high, CPs tend to be close to each other with respect to quality. Hence, a decision made by a CP will be mirrored by other CPs and a covered market is likely. On the other hand, for a fixed and low value of \( \gamma/a \), as the investment
A.3.2. Proof of Theorem 3.1. We first show that without loss of generality we can assume platform $\beta$ chooses a price $w_\beta \geq 0$. This will enable us to show that no content provider joins platform $\beta$ and consequently enable us to rule out the existence of configuration $C_{II}$ and $C_{III}$.

**Lemma A.8.** Platform $\beta$ charges $w_\beta \geq 0$.

**Proof.** We first show that $w_\beta \geq 0$ dominates any price $w_\beta < 0$. Assume that $w_\beta < 0$ and $r_\beta > 0$, then platform $\beta$ makes negative revenue on the content provider side. By raising its price to $w_\beta = 0$ it increases its total revenue. This is because the revenue from the content provider side becomes non-negative and the profits on the consumer side increase: This happens across all configurations because $r_\alpha$ is non-decreasing in $w_\beta$.

Since by Lemma A.8, $w_\beta \geq 0$, it follows that any content provider $j$ joining platform $\beta$ will get utility $v_j \leq 0$. Therefore, no content provider has incentive to join platform $\beta$. This implies that market configurations $C_{II}$ and $C_{III}$ where content providers patronize both platforms do not exist. We now show that there trivially exists pure strategy subgame equilibrium price pairs when one platform has zero investment. We show that these prices result in configurations $C_I$ and $C_{IV}$ and give the conditions on $\gamma$ for these to occur. We now proceed to prove Theorem 3.1.

**Proof.** We first derive the demand function $r_\alpha(w_\alpha)$. Given $y_\beta = 0, y_\alpha > 0$ and Lemma A.8, the content provider decisions are as if only one platform is on offer. Therefore, the demand addressed to platform $\alpha$ is equal to the mass of content providers with content quality $\gamma_j$ such that $\gamma_j y_\alpha q_\alpha \geq w_\alpha$ and is given by

$$r_\alpha(w_\alpha) = \frac{1}{2a} \left( \frac{1}{\gamma} + a - \frac{w_\alpha}{q_\alpha y_\alpha} \right).$$
The value \( \pi_\alpha \) that maximizes platform alpha’s profit problem for platform \( \alpha \) is represented as,

\[
\pi_\alpha = \arg\max \pi_\alpha(w_\alpha),
\]

subject to \( w_\alpha \geq (y_\alpha q_\alpha)(\gamma - a) \).

The profit function does not depend on \( w_\beta \) so platform \( \alpha \) maximizes the above function with respect to \( w_\alpha \) and ensuring that \( w_\alpha \geq y_\alpha q_\alpha(\gamma - a) \). This last constraint reflects the fact that when the constraint binds all content providers are on board; a price lower than this yields no more content providers and results in a loss of revenue. The interior solution for the above maximization is \( w_\alpha^* = \frac{y_\alpha}{2}f y_\alpha(2f(a - \gamma) + 3(\gamma + a)) \), and occurs whenever \( 1 < \gamma < (9 + 2f)/(3 + 2f) \). In this case since \( w_\alpha^* > q_\alpha y_\alpha(\gamma - a) \) the resulting configuration is \( C_I \). The constraint binds when \( \gamma \geq (9 + 2f)/(3 + 2f) \). In this instance the resulting configuration is \( C_{IV} \) since all content providers join platform \( \alpha \).

A.4. Best reply in the domain \( [0, y_h] \). In this section we show that if a platform is restricted to be the low quality platform (i.e., if the other platform invests in a quality larger than zero) it prefers not to invest.

In order to avoid confusion when platform \( \beta \) is the high quality firm we will change notation as follows; we label the high(low) quality platform as \( h(l) \) and the quality associated with it as \( y_h(l) \). Given \( y_h \), we will compute firm \( l \)'s best reply. We will show that the profit for the low quality firm is decreasing in \( y_l \) across all configurations which are possible given \( (\gamma, y_h, a) \). This will help us infer that the low quality platform chooses 0 as its best response. Since the choice of \( y_l \) by the low quality firm determines the market configuration we define the critical limits for which the various configurations exist given \( y_h \).

- Market is uncovered, with positive masses of consumers on both platforms, in the in the price subgame whenever,

\[
y_l < 2y_h \frac{a(f + 15) - (f + 9\gamma)}{(9 - 2f)\gamma + a(2f + 3)}.
\]

- Market is covered and a corner solution applies in the price subgame whenever,

\[
y_l \in \left[ 2y_h \frac{a(f + 15) - (f + 9\gamma)}{(9 - 2f)\gamma + a(2f + 3)}, 2y_h \frac{a(f + 9) - (f + 3)\gamma}{(21 - 2f)\gamma - (9 - 2f)a} \right],
\]

if \( 1 < \frac{\gamma}{a} \leq \frac{f + 15}{f + 9} \),

\[
y_l \in \left[ 0, 2y_h \frac{(f + 9) a - (f + 3)\gamma}{(21 - 2f)\gamma - (9 - 2f)a} \right],
\]

if \( \frac{f + 15}{f + 9} < \frac{\gamma}{a} \leq \frac{5f + 18}{5f + 6} \),

\[
y_l \in \left[ 0, 2y_h \frac{(2f + 9) a - (2f + 3)\gamma}{(3 - 4f)\gamma + (9 + 4f)a} \right],
\]

if \( \frac{5f + 18}{5f + 6} < \frac{\gamma}{a} \leq \frac{2f + 9}{3f + 6} \).

- Market is covered and an interior solution applies in the price subgame whenever,

\[
y_l \in \left( 2y_h \frac{-(f + 3)\gamma + (f + 9)a}{(21 - 2f)\gamma - (9 - 2f)a}, y_h \right),
\]

if \( 1 < \frac{\gamma}{a} < \frac{5f + 18}{5f + 6} \).

- Market is preempted whenever,

\[
y_l \in \left[ 2y_h \frac{(2f + 9)a - (2f + 3)\gamma}{(3 - 4f)\gamma + (9 + 4f)a}, y_h \right],
\]

if \( \frac{5f + 18}{5f + 6} \leq \frac{\gamma}{a} < \frac{9 + 2f}{3 + 2f} \).

\[
\frac{\gamma}{a} \geq \frac{9 + 2f}{3 + 2f}.
\]

We now show that given the tuple \( (\gamma, y_h, a) \) the profit function \( \pi_l \) is decreasing in every configuration that it is defined.
Lemma A.9. Given $(y_l, \tau, a), f \geq 3/5$ and $y_l \in [0, y_h)$, the profit function $\pi_l(y_l, y_h)$ is decreasing in $y_l$ for all market configurations for which it is defined.

Proof. We show that for each configuration the revenue function $r_l = \pi_l(y_l, y_h) + c(y_l)$ is decreasing in $y_l$.

Uncovered Configuration $C_1$: Let the revenue function in this configuration be defined by $r_{ul}$, one can by show that for $1 < \frac{\tau}{a} < \frac{f+15}{f+9}$ and $y_l$ in the set defined in (35), $\frac{\partial r_{ul}}{\partial y_l} < 0$. Hence the revenue function in this configuration is decreasing in $y_l$.

Covered Configuration with interior solution $C_{II}$: Let the revenue function in this configuration be defined by $r_{ci}$. One can also show that for $1 < \frac{\tau}{a} < \frac{2f+18}{2f+3}$ and $y_l$ in the set defined in (39), the derivative of the above function, $\frac{\partial r_{ci}}{\partial y_l} < 0$. Therefore the profit function is decreasing in $y_l$ when $y_l$ lies in the set specified in (39).

Covered Configuration with corner solution $C_{III}$: Let the revenue function in this configuration be defined by $r_{cc}$. One can show that for $1 < \frac{\tau}{a} < \frac{2f+9}{2f+3}$ and $y_l$ in the sets defined in (36), (37) and (38) the derivative of the above function, $\frac{\partial r_{cc}}{\partial y_l} < 0$.

Pre-empted Configuration $C_{IV}$: Let the revenue function in this configuration be defined by $r_p$. The derivative of the above function, $\frac{\partial r_p}{\partial y_l} = -2f/9f^2\tau$. The above derivative is negative therefore the profit function is decreasing in $y_l$ when this configuration is defined.

We now proceed to present a lemma that shows the best response of platform $\beta$ given the tuple $(y_\alpha, \tau, a, f)$ and the domain $[0, y_a)$. Let $j, i \in \{\alpha, \beta\}$ and $B_i(y_l)$ be the set of $y^*_i \in [0, \infty]$ such that

$$y^*_i \in \arg \max_{y_l \in [0, \infty]} \pi_i(y_l, y_j).$$

Then, the best response of platform $\beta$ given $y_\alpha$ is denoted by $B_\beta(y_\alpha)$. This lemma essentially states that if a platform is restricted to be the low quality platform, it prefers not to invest.

Lemma A.10. Given the tuple $(y_\alpha, \tau, a, f), f \geq 3/5$ and the domain $[0, y_h)$, then $B_\beta(y_\alpha) = 0$.

Proof. We show that given $(y_h, \tau, a)$ platform $l$ will prefer not to invest. Specifically, we show that the profit function $\pi_l$ is continuous in $y_l$ over the domain $[0, y_h)$. This coupled with Lemma A.9 implies that platform $l$ picks the lowest quality as the Lemma claims. We split the domain in which $\frac{\tau}{a}$ lies into four sections depending on the number and type of market configurations that are possible. We show that in each section the profit function is continuous.

Case I. $1 < \frac{\tau}{a} < \frac{f+15}{f+9}$

Four market configurations are possible when $1 < \frac{\tau}{a} < \frac{f+15}{f+9}$; these are uncovered ($C_1$), uncovered ($C_{II}$) with both platforms participating, covered with a corner solution and covered with an interior solution (both of which are in configuration $C_{III}$). Given a $\frac{\tau}{a}$ in the above range, the domain $[0, y_h)$ in which $y_l$ lies can be partitioned into three sets, each of which corresponds to one of the latter three market configurations. These partitions are captured in (35), (36) and (39). By Lemma A.9, we know that profits are decreasing in $y_l$ for each partition. We will first show that the value of the profit function in the partition defined in (35), is larger than any profit attained in the partition defined in (36). Similarly, we show that any profit attained when $y_l$ lies in the partition defined by the constraint in (39) is not greater than that attained when $y_l$ lies in the partition specified by (36). Lastly we show that the profit of a platform in configuration $C_{IV}$ tends to that in configuration $C_1$ as $y_l \to 0$ and is in fact equal at the limit.

To show the first result we compare the infimum value of the profit function in the uncovered configuration to the highest possible profit attained when platform $l$ chooses $y_l$ such that a covered market with a corner solution results (i.e, $y_l$ is in the set specified in (36)). Let $y^*_{l \to g}(y_l, y_h) = \pi^*_{l \to g}(y_l, y_h)$ (Since $\pi^*_{l \to g}$ is right continuous, the limit exists). Since
\(\pi^u(y_l, y_h) > \pi^u(y^*_l, y^*_h)\) when \(y_l\) satisfies the inequality in (35), it also follows from Lemma A.9 that \(\pi^u(y_l, y_h) > \pi^i_\gamma(y^*_l, y^*_h)\) when \(\gamma\) lies in the set specified in (36).

To show the second result, we compare the lowest value of the profit function in the covered configuration with a corner solution to the supremum profit value attained when platform \(l\) chooses \(y_l\) such that a covered market with an interior solution results. The interval over which the covered configuration with an interior solution, \(C_{III}\), is defined is open. Let \(y^i_l = 2y_h\frac{-(3\bar{\gamma}+(f+9)\alpha)}{(21-2f)\gamma-(9-2f)\alpha}\), we define the supremum of \(\pi^i_c(y_l, y_h)\) over the range in which this configuration is defined as \(\pi^i_c(y^i_l, y_h)\). We note that \(\pi^i_c\) is the infimum of the interval over which this configuration is defined, therefore \(\lim_{y_l \to y^i_l} \pi^i_c(y_l, y_h) = \pi^i_c(y^i_l, y_h)\)

since \(\pi^i_c(y_l, y_h)\) is left continuous. By plugging in \(y_l = y^i_l\) into \(\pi^i_c(y_l, y_h)\) we note that \(\pi^i_c(y^i_l, y_h) = \pi^i_c(y^i_l, y_h)\). Therefore, it follows from Lemma A.9, that \(\pi^i_c(y_l, y_h) > \pi^i_c(\tilde{y}, y_h)\) when \(y_l\) satisfies the constraint in (36) and \(\tilde{y}\) satisfies the constraint in (39).

Finally, since \(\pi^u(y_l, y_h)\) is left continuous by plugging \(y_l = 0\) to the function \(\pi^u(y_l, y_h)\) we show that \(\lim_{y_l \to 0} \pi^u_l(y_l, y_h) = \pi^u_l(0, y_h)\). Where \(\pi^u_l(0, y_h)\) is the profit function when configuration \(C_l\) is defined.

**Case II.** \(\frac{5f+18}{4f+6} \leq \frac{\bar{\gamma}}{a} < \frac{5f+18}{2f+9}\).

In this instance three market configurations are possible depending on the value of \(y_h\) and \(y_l\); these are uncovered \(C_l\), covered with a corner solution, \(C_{III}\), and covered with an interior solution, \(C_{II}\). Given a \(\frac{\bar{\gamma}}{a}\) in the above range, the domain \((0, y_h)\) in which \(y_l\) lies can be partitioned into two sets each of which corresponds to the latter two of the three market configurations. These partitions are captured in (36) and (39).

We proceed in a similar manner as we did for the previous case. By Lemma A.9 we know that profits are decreasing in \(y_l\) for each partition. We claim that any profit attained in the partitions defined in (39) is less than that attained by the minimum profit in the partition defined in (36). The proof is exactly the same as that described in case I. We then show that as \(y_l \to 0\) the profit of a platform in configuration \(C_{III}\) with a corner solution approaches that of the profit in configuration \(C_l\) and in the limit when \(y_l = 0\) they are equal. Since \(\pi^i_c(y_l, y_h)\) is left continuous by plugging \(y_l = 0\) to the function \(\pi^i_c(y_l, y_h)\) one can show that \(\lim_{y_l \to 0} \pi^i_c(y_l, y_h) = \pi^i_c(0, y_h)\). Where \(\pi^i_c(0, y_h)\) is the profit function when configuration \(C_l\) is defined.

**Case III.** \(\frac{5f+18}{4f+6} \leq \frac{\bar{\gamma}}{a} < \frac{2f+9}{2f+9}\).

In this section we need only show that the profit function is continuous across configuration \(C_{III}\) with a corner solution and a pre-empted market configuration \(C_{IV}\). Indeed these are the only two configurations possible when \(y_l > 0\). (We already showed in the previous case that \(\lim_{y_l \to 0} \pi^i_c(y_l, y_h) = \pi^i_{ui}(0, y_h)\)). To show this result we compare the infimum value of the profit function in the covered configuration with a corner solution to the profit value attained when platform \(l\) chooses a \(y_l\) such that a pre-empted market results. Note the interval over which the covered configuration with a corner solution, \(C_{III}\), is defined is open on its upper limit. Let \(y^p_l = 2y_h\frac{(2f+9)\alpha-9+4f\bar{\gamma}}{(3-4f)\gamma+(9+4f)\alpha}\), we define the infimum of \(\pi^i_c(y_l, y_h)\) over the range in which this configuration is defined as \(\pi^i_c(y^p_l, y_h)\). Since \(\pi^i_c(y, y_h)\) is right continuous \(\lim_{y_l \to y^p_l} \pi^i_c(y_l, y_h) = \pi^i_c(y^p_l, y_h)\). Note \(y^p_l\) is the infimum of the range. By plugging in \(y_l = y^p_l\) into the profit functions under a covered market (with a corner solution) and a pre-empted market, we find that \(\pi^i_c(y^p_l, y_h) = \pi^i(y^p_l, y_h)\). This implies that the profit function is continuous across these two market configurations at this point.

**Case IV.** \(\frac{2f+9}{2f+9} \leq \frac{\bar{\gamma}}{a} < \infty\).

When \(\bar{\gamma}\) falls in the above range only market configuration \(C_{IV}\) is possible when \(y_l > 0\). We showed in Lemma A.9 that the profit function \(\pi_l(y_l, y_h)\) is decreasing in \(y_l\) in this configuration. So we only need show that \(\lim_{y_l \to 0} \pi^i_l(y_l, y_h) = \pi^i_{ui}(y_l, y_h)\), where \(\pi^i_{ui}(0, y_h)\) is the profit function in \(C_l\). Since \(\pi^i_l(y, y_h)\) is left continuous \(\lim_{y_l \to 0} \pi^i_l(y_l, y_h) = \pi^i_l(0, y_h)\). Via simple algebra, one can show that \(\pi^i_l(0, y_h) = \pi^i_{ui}(0, y_h)\) which shows that \(\pi_l\) is continuous. \(\square\)

Before proving Theorem 3.2, we first show that a symmetric equilibrium is not feasible. Let \(j, i \in \{\alpha, \beta\}\) and \(B_j(y_j)\) be the set of \(y_j^* \in [0, \infty]\) such that \(\pi^i_l(y_l, y_{-l}) = \arg\max_{y_l \in [0, \infty]} \pi_l(y_l, y_{-l})\).
Lemma A.11. If $y_j \in [0, \infty]$ then $y_j \notin B_i(y_j)$.

Proof. We show that given $y_j$, platform $i$ never chooses $y_i = y_j \geq 0$ and therefore a symmetric equilibrium is not possible. A symmetric argument applies for the other platform. Assume $y_j \in B_i(y_j)$ so that $y_i = y_j > 0$, then both platforms would make zero profits because of Bertrand competition on both sides of the market. We now check if platforms would prefer $y_i = y_j = 0$ and show that there exists a profitable deviation for platform $i$. There are two cases to consider, when $\gamma/a < (2f + 9)/(2f + 3)$ and $\gamma/a \geq (2f + 9)/(2f + 3)$. The arguments for both cases are very similar so we present only one.

Case $I$. $\gamma/a < \frac{2f + 9}{2f + 3}$.

Let platform $i$ increase its quality by a small $\epsilon > 0$; platform $i$ becomes the high quality platform. Results from Theorem 3.1 imply that the resulting equilibrium profit $\pi_i$ for the high quality platform given the subgame $(\gamma, a, \epsilon, y_j)$ can be expressed as follows using a Taylor series expansion, $\pi_i(\epsilon, 0) = Q(\gamma, f, a)\epsilon - \frac{1}{2}I'(0)\epsilon - o(\epsilon^2)$, where $Q(\gamma, f, a)$ is a positive number. There exists an $\epsilon^*$ such that for all $\epsilon \in (0, \epsilon^*)$ the above quantity is positive. Thus platform $i$ would prefer to set quality $\epsilon \in (0, \epsilon^*)$ instead of 0.

We now proceed to find the sets in which the best replies lie given each platform’s investment level. Given quality choice $y_\alpha$, platform $\beta$ can choose a best reply that depends on whether it acts as a high-quality or a low-quality platform. In the former case it chooses a reply in the domain $(y_\alpha, \infty)$ and in the latter case it chooses a reply in the domain $[0, y_\alpha)$. Lemma A.10 shows that if platform $\alpha$ invests in a positive quality and platform $\beta$ acts as a low quality platform, then platform $\beta$ prefers not to invest. By symmetry a similar claim exists for platform $\alpha$ given platform $\beta$’s quality choice. We now proceed to prove Theorem 3.2.

A.5. Proof of Theorem 3.2.

Proof. Given that platform $\beta$ invests in $y_\beta > 0$ platform $\alpha$ can choose to be a low quality or a high quality platform. The best response for platform $\alpha$ given it acts as a low-quality (high-quality) platform is given by $B_\alpha(y_\beta) = 0$ ($B_\beta(y_\beta) \in (y_\beta, \infty)$). The former follows from Lemma A.10 and the latter from Lemma A.11. The overall best response is the maximum of these two best responses, i.e., the value for which the profit function is highest. Let $\bar{y} = \{y_\alpha | y_\alpha > y_\beta\}$. Given $y_\beta > 0$ then $B_\alpha(y_\beta) \in \{0 \cup \bar{y}\}$. If $y_\beta = 0$ then the best response is given by $y^* = \{y_\alpha | I'(y_\alpha) = r'_\alpha(y_\alpha)\}$, where $r_\alpha(y_\alpha)$ is the revenue made by the high quality platform. Note that given the tuple $(\gamma, \alpha, f)$, $r_\alpha(y_\alpha)$ is a linear function in $y_\alpha$. It follows that $y^*$ is a singleton since the profit function is concave in $y_\alpha$. Therefore given $y_\beta = 0$, $B_\alpha(y_\beta) = y^*$. Since the explicit form of the revenue function depends on whether $\gamma/a < (9 + 2f)/(3 + 2f)$ or $\gamma/a \geq (9 + 2f)/(3 + 2f)$ we have two implicit characterizations of this singleton. The sets in which platform $\alpha$’s best response lies is similar by symmetry. Consequently, the only points of intersection are $[y^*, 0]$ and $[0, y^*]$. 

Let $r(y, y^*)$ denote the revenue function of the platform that is responding to an investment level of $y^*$ by the other platform. When $\gamma/a < \frac{2f + 11}{2f + 6}$ holds and the platform acts as a high quality platform (i.e., chooses $y > y^*$), the choice of $y$ may result in three possible revenue functions. This revenue function is made up of a concatenation of three other revenue functions. These are,

$$r_{ci}(y, y^*) \text{ if } y \in \left( y^*, y^* + \frac{1}{2} \frac{(21 - 2f)(\gamma + 2f/9)a}{(f + 9)a - (f - 3)\gamma} \right)$$
$$r_{cc}(y, y^*) \text{ if } y \in \left( y^* + \frac{1}{2} \frac{(21 - 2f)(\gamma + 2f/9)a}{(f + 9)a - (f - 3)\gamma}, y^* + \frac{1}{2} \frac{(9 - 2f)(\gamma + 2f/3)a}{(f + 15)a - (f + 9)\gamma} \right)$$
$$r_{ui}(y, y^*) \text{ if } y \in \left( y^* + \frac{1}{2} \frac{(9 - 2f)(\gamma + 2f/3)a}{(f + 15)a - (f + 9)\gamma}, \infty \right)$$

The restrictions over which these functions are defined are derived from the market configurations in Theorem A.7. The first refers to the revenue function when the market is covered with an interior solution, the second refers to the revenue function when the market is covered with a corner solution, and the last refers to when the market is uncovered with masses present in both configurations. Before proving the existence of an SPE, we provide the following lemma that gives an upper bound function for the revenue function.
Lemma A.12. If \( y \in \left[ y^* \frac{(21-2f)\gamma^2 + (2f-9)a}{(f+9)a-(f-3)\gamma^2}, \infty \right] \) then \( r^{ci}(y, y^*) \geq r(y, y^*) \).

Proof. Note that \( r(y, y^*) \) is continuous since \( \lim_{y \to y^*} r^{ci}(y, y^*) = r^{cc}(y, y^*) \) and \( \lim_{y \to y^*} r^{cc}(y, y^*) = r^{ui}(\hat{y}, y^*) \) where \( \hat{y} = y^* \frac{(21-2f)\gamma^2 + (2f-9)a}{(f+9)a-(f-3)\gamma^2} \) and \( \hat{y} = y^* \frac{(9-2f)\gamma^2 + (2f+3)a}{(f+15)a-(f+9)\gamma^2} \). The revenue functions \( r^{cc}(y, y^*) \) and \( r^{ui}(y, y^*) \) are increasing in \( y \) since the derivatives are positive. Moreover, the former is convex in \( y \), while the latter is concave in \( y \). The difference \( r^{ci}(\hat{y}, y^*) - r^{cc}(\hat{y}, y^*) \) is positive whenever \( 1 < \frac{\gamma}{a} < \frac{f+15}{f+9} \). Furthermore, \( r^{ci}(\hat{y}, y^*) - r^{ui}(\hat{y}, y^*) \) and \( r^{ui}(\hat{y}, y^*) - r^{cc}(\hat{y}, y^*) \) are also positive in this interval. This coupled with the fact that the revenue functions are increasing implies that \( r^{ci}(y, y^*) > r^{cc}(y, y^*) \) in the domain of \( y \) where \( r^{cc}(y, y^*) \) is defined. Moreover, the concavity of \( r^{ui}(y, y^*) \) also implies that \( r^{ci}(y, y^*) > r^{ui}(y, y^*) \) in the domain of \( y \) where the latter is defined. Therefore, whenever \( 1 < \frac{\gamma}{a} < \frac{f+15}{f+9} \), then \( r^{ci}(y, y^*) \geq r(y, y^*) \). In the case where \( \frac{f+15}{f+9} < \frac{\gamma}{a} < \frac{18+5f}{6+5f} \), only two market configurations exist, a covered market with an interior solution and an uncovered market with a corner solution. The difference between the revenue functions over the region where the covered market with a corner solution is defined is concave in \( y \) and increasing. This follows because i) the second derivative of the difference, \( r^{ci}(y, y^*) - r^{cc}(y, y^*) \), is negative whenever \( y > y^* \); ii) the difference is increasing in \( y \) since the derivative is positive in this range; iii) for \( y > \hat{y} \), the range in which the covered market with corner solution is defined, the difference is positive.

A.6. Proof of Theorem 3.3.

Proof. We show that for a \( c \geq 1 \) and \( f \) large enough the pair \((y^*, 0)\), as defined in the Theorem statement is a SPE. Some of the expressions involved are too large to put in the paper. Where this is the case we state the importance of the results for the proof. In Theorem 3.2, we showed that the pair \((0, y^*)\) is a candidate equilibrium pair. In particular \( y^* \) is the best response of one platform given the other platform chooses not to invest. We proceed to show that for a quadratic investment function when one platform invests in \( y^* \) the other opts not to invest concluding that a SPE exists for this investment function. To analyze this response we partition the space in which \( \gamma/a \) lies into three regions corresponding to the types of market configurations that exists in each of the region. Note that the revenue function, which equals profit plus investment cost, is of a different form in each of these regions, hence the different analysis.

Case I. \( 1 < \frac{\gamma}{a} < \frac{5f+18}{6f+6} \).

We find a differentiable upper-bound of \( r(y, y^*) \) and show that the best response when the platform acts as a high-quality platform, under this function, is dominated by the best response when it acts as a low-quality platform. Let this upper-bound be denoted by \( r^{est} \). Lemma A.12 shows that \( r^{ci}(y, y^*) \) over the domain \( y > y^* \) is an upper-bound of \( r(y, y^*) \). So we find the best response under this function and compare it with the best response when the platform acts as a low-quality platform and opts not to invest. Let the maximum profit value when the platform acts as a high-quality platform under the upper-bound revenue be denoted by \( \pi^{est}(y^{est}, y^*) \), where

\[
y^{est} = \arg\max \pi^{est}(y, y^*) ,
\]

s.t. \( y > y^* \).

(42)

Let the maximum profit value when the platform acts as a low-quality platform be denoted by \( \pi^{low} \). From Lemma A.10 this occurs at \( y = 0 \). One can show that \( \pi^{low} > \pi^{est} \).
**Case II.** \( \frac{5f+18}{3f+6} \leq \frac{7}{a} < \frac{9+3f}{a+6} \):

In this region two market configurations are possible if the platform picks \( y > y^* \). These are a preempted market and a covered market with a corner solution. Since in the preempted market several price equilibria exist we pick the one that yields the highest price and use that to calculate an upper-bound for the profit function. We denote it by \( \pi^\text{est} \). We compare this solution against \( \pi^\text{low} \), which is the profit of the platform when it choose to be the low-quality platform. In a similar manner to the first case we show \( \pi^\text{low} > \pi^\text{est} \).

In particular \( \pi^\text{est} - \pi^\text{low} = -2/27f^2(3a - 4f\gamma - 3\gamma)(a - \gamma + f\gamma) < 0 \) whenever \( f > 1 - a/k \).

**Case III.** \( \frac{5f+18}{3f+6} < \frac{7}{a} \):

In this interval, when the platform decides to act as a high quality platform, only the pre-empted market exists. Moreover, there are multiple price equilibria. So we use the price equilibria that yields the highest possible profit and use it to derive an upper-bound for the profit function. The analysis then proceeds in exactly the same manner as that in case II because the upper-bound for the profit function when the platform chooses to act as a high-quality is the same.

\[ \square \]

**APPENDIX B. TECHNICAL DETAILS FOR THE NON-NEUTRAL MODEL**

In this section we characterize the sets of prices that yield the relations defined in Section 4.1.2.

**B.1. Sets of Prices that yield relations (i),(ii) and (iii) as defined in section (4.1.2).** We now define sets \( \mathcal{W}_{R(i)} \), \( \mathcal{W}_{R(ii)} \) and \( \mathcal{W}_{R(iii)} \) that contain price pairs \((w_\alpha, w_\beta)\) that may yield relations (i), (ii) and (iii), as defined in sections 4.1.2, respectively. First we define set \( \mathcal{W}_{R(i)} \) by solving for the price pairs for which \( F_i(y_\alpha, \cdot) > F_i(y_\beta, \cdot) \). We characterize this set below:

\[
\mathcal{W}_{R(i)} = \begin{cases}
\{(w_\alpha, w_\beta) | w_\alpha < \overline{w}_\alpha, w_\beta > \overline{w}_\beta \text{ and } w_\alpha < q_\alpha fA + q_\alpha/q_\beta w_\beta \} & \text{if } y_\alpha \in \Lambda, \\
\{(w_\alpha, w_\beta) | w_\alpha < \overline{w}_\alpha, w_\beta > \overline{w}_\beta \} & \text{if } y_\alpha \in \Delta, \\
\{(w_\alpha, w_\beta) | w_\alpha \geq 0, w_\beta \geq 0 \} & \text{if } y_\alpha \in \Theta.
\end{cases}
\]

Here, \((q_\alpha, q_\beta) = (2/3, 1/3), A = (y_\alpha - y_\beta)(\gamma + a)/(\gamma - a), \Lambda = \left(y_\beta, y_\beta f(\gamma + a)^2/(\gamma - a)^2\right), \Delta = \left[y_\beta f(\gamma + a)^2/(\gamma - a)^2, 2y_\beta \frac{\gamma}{\gamma + a}\right], \Theta = \left[2y_\beta \frac{\gamma}{\gamma + a}, \infty\right]. \)

If \( y_\alpha \in \Lambda \) then \( \overline{w}_\alpha = q_\alpha fA + q_\alpha(\gamma - a)f y_\beta \), and \( \overline{w}_\beta = (\gamma + a)f y_\alpha - f A q_\beta \). If \( y_\alpha \in \Delta \) then \( \overline{w}_\alpha = (\gamma + a)/3f y_\alpha \) and \( \overline{w}_\beta = (\gamma - a)/3f y_\beta \). If \( y_\alpha \in \Theta \) and \( y_\alpha \in \Delta \) or \( y_\alpha \in \Theta \) then \( \overline{w}_\alpha = (\gamma + a)/3f y_\alpha \) and \( \overline{w}_\beta = (\gamma - a)/3f y_\beta \).

We similarly characterize set \( \mathcal{W}_{R(ii)} \) by solving for price pairs for which \( F_i(y_\alpha, \cdot) > F_i(y_\beta, \cdot) \):

\[
\mathcal{W}_{R(ii)} = \begin{cases}
\{(w_\alpha, w_\beta) | w_\alpha > \overline{w}_\alpha, w_\beta = \overline{w}_\beta \text{ and } w_\alpha > q_\alpha fA + q_\alpha/q_\beta w_\beta \} & \text{if } y_\alpha \in \Lambda, \\
\{(w_\alpha, w_\beta) | w_\alpha > \overline{w}_\alpha, w_\beta < \overline{w}_\beta \} & \text{if } y_\alpha \in \Delta, \\
\{(w_\alpha, w_\beta) \in \emptyset \} & \text{if } y_\alpha \in \Theta.
\end{cases}
\]

Here, \((q_\alpha, q_\beta) = (1/3, 2/3), A, \Lambda, \Delta, \text{ and } \Theta \) are as previously defined. In addition, if \( y_\alpha \in \Lambda \) then \( \overline{w}_\alpha = q_\alpha fA + q_\alpha(\gamma - a)f y_\beta \), and \( \overline{w}_\beta = (\gamma + a)f y_\alpha - f A q_\beta \). If \( y_\alpha \in \Delta \) then \( \overline{w}_\alpha = (\gamma + a)/3f y_\alpha \) and \( \overline{w}_\beta = (\gamma - a)/3f y_\beta \). We similarly characterize set \( \mathcal{W}_{R(iii)} \) by solving for price pairs for which \( F_i(y_\alpha, \cdot) = F_i(y_\beta, \cdot) \):

\[
\mathcal{W}_{R(iii)} = \begin{cases}
\{(w_\alpha, w_\beta) | w_\alpha = w_\alpha, w_\beta = w_\beta \} & \text{if } w_\beta \leq (\gamma - a)/2f y_\beta \text{ and } y_\alpha \in \Lambda, \\
\{(w_\alpha, w_\beta) | w_\alpha = w_\alpha, w_\beta = w_\beta \} & \text{if } w_\alpha \geq (\gamma + a)/2f y_\beta \text{ and } y_\alpha \in \Lambda, \\
\{(w_\alpha, w_\beta) | w_\alpha = q_\alpha fA + q_\alpha/q_\beta w_\beta \} & \text{if } w_\beta \geq (\gamma - a)/2f y_\beta \text{ and } y_\alpha \in \Lambda, \\
\{(w_\alpha, w_\beta) | w_\alpha \geq w_\alpha, w_\beta \leq w_\beta \} & \text{if } y_\alpha \in \Delta \cup 2y_\beta \frac{\gamma}{\gamma + a}, \\
\{(w_\alpha, w_\beta) \in \emptyset \} & \text{if } y_\alpha \in \Theta/2y_\beta \frac{\gamma}{\gamma + a}.
\end{cases}
\]
Here, \((q_\alpha, q_\beta) = (1/2, 1/2)\), \(A, \Lambda, \Delta, \) and \(\Theta\) are as previously defined. In addition, if \(y_\alpha \in \Lambda\) then \(w_\alpha = q_\alpha fA + q_\alpha(\gamma - a)fy_3\), and \(w_\beta = (\gamma + a)fy_3 - fAq_\beta\). If \(y_\alpha \in \Delta\) then \(w_\alpha = (\gamma + a)1/2fy_3\) and \(w_\beta = (\gamma - a)1/2fy_3\).

We note that if a price pair lies on the intersection of any of the sets \(W_{R(i)}, W_{R(ii)}, \) and \(W_{R(iii)}\), then more than one equilibrium allocation exists.

**B.2. Relation (ii) does not hold on the equilibrium path.** In this section we show that relation (ii) defined in 4.1.2 does not lie on the equilibrium path.

We first present a lemma showing that if a price subgame results in multiple CP allocation equilibria, such that relations (i) and (ii) hold, the CP allocation equilibrium for which relation (i) holds yields the highest profit for platform \(\alpha\). Let \(\pi_\alpha(w_\alpha, w_\beta) (\hat{p}_\alpha(w_\alpha, w_\beta))\) denote platform \(\alpha\)'s profit whenever relation (i) ((ii)) holds.

**Lemma B.1.** Given a price pair \((w_\alpha, w_\beta)\) such that the CP allocations which yield relations (i) and (ii) can occur, then \(\pi_\alpha(w_\alpha, w_\beta) > \hat{\pi}_\alpha(w_\alpha, w_\beta)\).

**Proof.** Let \(\bar{\gamma}_\alpha, \bar{\gamma}_\beta (\hat{\gamma}_\alpha, \hat{\gamma}_\beta)\) denote the CP demand when relation (i) ((ii)) holds. It follows that \(\bar{\gamma}_\alpha \geq \hat{\gamma}_\alpha\) and \(\bar{\gamma}_\beta \leq \hat{\gamma}_\beta\) since \(\bar{\gamma}_\alpha > \hat{\gamma}_\alpha\) and \(\bar{\gamma}_\beta < \hat{\gamma}_\beta\). Moreover, \(\bar{\gamma}_\alpha \bar{\gamma}_\alpha > \hat{\gamma}_\alpha \hat{\gamma}_\alpha\). To see this note that,

\[
\bar{\gamma}_\alpha = q_\alpha ((\gamma + a) + (\gamma - a)\bar{\gamma}_\alpha - y_3 ((\gamma + a) + (\gamma - a)\bar{\gamma}_\beta))
\]

\[
\hat{\gamma}_\alpha = q_\alpha ((\gamma + a) + (\gamma - a)\hat{\gamma}_\alpha - y_3 ((\gamma + a) + (\gamma - a)\hat{\gamma}_\beta)).
\]

Therefore, \(\pi_\alpha(w_\alpha, w_\beta) = \bar{\gamma}_\alpha w_\alpha + \bar{\gamma}_\alpha w_\beta > \hat{\gamma}_\alpha w_\alpha + \hat{\gamma}_\alpha w_\beta = \hat{\pi}_\alpha(w_\alpha, w_\beta)\). \(\square\)

We now show in the next lemma that if an SPE exists then the CP allocation that holds in the equilibrium path does not yield relation (ii).

**Lemma B.2.** If \((w_\alpha^*, w_\beta^*)\) is an SPE of the quality subgame then the CP allocation on the equilibrium path does not yield relation (ii).

**Proof.** From Appendix B.1, there are three cases to consider; \(y_\alpha \in \Lambda, y_\alpha \in \Delta\) and \(y_\alpha \in \Theta\). In the last case, a CP allocation equilibrium that yields relation (ii) does not exist. Therefore, we only consider the first two cases.

**(a):** \(y_\alpha \in \Lambda\)

Let \(\hat{\gamma}_\alpha (w_\alpha^*, w_\beta^*) = \hat{\gamma}_\alpha w_\alpha^* + p_\alpha^* \hat{\gamma}_\alpha\) denote the revenue under the equilibrium price pair \((w_\alpha^*, w_\beta^*)\). Here, \(\hat{\gamma}_\alpha (\hat{\gamma}_\alpha)\) refers to the mass of CPs (consumers) at equilibrium and \(p_\alpha^*\) is the price offered to the consumer at equilibrium. Let \(\pi_\alpha = w_\alpha^* - \epsilon\), where \(\epsilon > 0\). Observe that since \(y_\alpha \in \Lambda\) and \((w_\alpha^*, w_\beta^*) \in W_{R(i)}\), it is always possible to choose \(\epsilon\) such that \((\pi_\alpha, w_\beta^*) \in W_{R(ii)}\). Let \(\pi_\alpha(w_\alpha, w_\beta)\) be the profit function for platform \(\alpha\) generated by choosing the CP allocation equilibrium which yields relation (ii), whenever more than one CP allocation equilibrium is possible. This profit function is quadratic and concave in \(w_\alpha\) over the interval \(I = (\max\{w_\alpha, 1/3fA + 1/2w_\beta\}, (\gamma - a)1/3fy_3\})\. Here, \(w_\alpha\) is as defined in appendix B.1, where \((q_\alpha, q_\beta) = (1/3f, 2/3fy_3)\). The unrestricted maximum \(w_\alpha^* = \arg\max \pi(w_\alpha, w_\beta) < \max\{w_\alpha, (\gamma - a)1/3fy_3\}\). Therefore, \(\pi(w_\alpha, w_\beta)\) is decreasing in the interval \(I\). If we pick an \(\epsilon\) small enough, then \(\hat{\gamma}_\alpha \in I\). This implies that \(\pi(w_\alpha, w_\beta) > \pi_\alpha(w_\alpha^*, w_\beta^*)\). Note also that at price \((\pi_\alpha, w_\beta^*)\), the revenue arising under the CP allocation equilibrium which yields relation (i) is higher, see lemma B.1. Therefore price \(\pi_\alpha\) dominates price \(w_\alpha^*\) and platform \(\alpha\) has an incentive to deviate.

**(b):** \(y_\alpha \in \Delta\)

Let \(\overline{\pi}_\alpha = \overline{w}_\alpha - \epsilon\), where \(\epsilon > 0\) such that \((\overline{w}_\alpha, w_\beta^*) \in W_{R(i)} \cap W_{R(ii)}\). Here, \(w_\alpha\) is as defined in appendix B.1, with \((q_\alpha, q_\beta) = (1/3f, 2/3fy_3)\). Denote the revenue under the equilibrium price pair \((w_\alpha^*, w_\beta^*)\) by \(\hat{\gamma}_\alpha (w_\alpha^*, w_\beta^*) = \hat{\gamma}_\alpha w_\alpha^* + p_\alpha^* \hat{\gamma}_\alpha\). At the price pair \((\overline{w}_\alpha, w_\beta^*)\) only one CP allocation
equilibrium is possible; the one that yields relation (i). Let \(\pi_0(\pi_0, w_0^*) = \pi_0 w_0^* + \pi_0, q_0\) represent the revenue at this price.

We next show that \(\pi_0(\pi_0, w_0^*) > \pi_0(w_0^*, w_0^*)\). First we note that revenue made on the CP side by platform \(\alpha\) is higher under the new price since \(\pi_0 > \pi_0 = 0\). This follows from the fact that \(w_0 < w_0 < (\gamma + a)/2 f y_0\). Therefore, a positive mass of CPs will patronize the platform since they gain positive utility upon joining. Revenue on the consumer side is also higher under this new deviation. To see this, note that since relation (i) holds, \(\pi_0 < q_0\). This implies \(\pi_0 < \pi_0\) which further implies that \(\pi_0 > p_0^*\). Observe that \(\pi_0 > q_0\) (since the CP allocation equilibrium yields relation (i)), \(y_0 \geq y_0\) and

\[
\pi_0 = q_0(y_0(\gamma + a) + (\gamma - a)\pi_0) - y_0((\gamma + a) + (\gamma - a)\pi_0),
\]

\[
p_0^* = q_0(y_0(\gamma + a) + (\gamma - a)\pi_0) - y_0((\gamma + a) + (\gamma - a)\pi_0).
\]

\[\square\]

### B.3. Proof of Theorem 4.1

In this Appendix, we show that given the tuple \((\pi_0, a, f, y_0, y_0)\) such that \(y_0 > y_0\) a unique SPE exists in the CP price game. We define the baseline CP price game as the game induced by selecting the CP allocation equilibrium which yields relation (i), whenever multiple equilibria exist. We then show that this game has a unique SPE. 

In addition, we show that all reduced extensive form games, for which an SPE exist, have this same unique SPE. Thus without loss of generality we may only consider the baseline CP price subgame. Recall we are considering reduced extensive form games in which the CP allocation equilibria chosen, whenever multiple equilibria exist, are those that yield either relation (i) or (iii).

**Proof.** The proof involves the following two steps.

**Step 1** Baseline-CP price game has a unique SPE

Given a tuple \((\pi_0, a, f, y_0, y_0)\) such that \(y_0 > y_0\) and price \(y_0\), we denote platform \(\alpha\)’s profit function by \(\pi_0(w_0, w_0)\). This profit function is quadratic in \(w_0\) over the range \(I = [(\pi_0 - a)/2 f y_0, \min(\pi_0 + a)/2 f y_0, 2 f y_0 + 2 w_0)]\). It is linear and decreasing for \(w_0 < (\pi_0 - a)/2 f y_0\). Let \(w_0^* = \arg \max \pi_0(w_0, w_0)\) be the unrestricted maximum of the quadratic function. This value is given by \(w_0^* = \frac{1}{t}((3 - 2 f)\pi_0 + (3 + 2 f) a) f y_0\). Whenever \(y_0 > y_0\) then \(w_0^* < \min((\pi_0 + a)/2 f y_0, 2 f y_0 + 2 w_0), \) where \(w_0 \geq (\pi_0 - a)/3 f y_0\). Therefore, given \(w_0\), the best response is given by \(w_0^* = \max\{w_0^*, (\pi_0 - a)/2 f y_0\}\). Given another \(w_0^*\), the profit function \(\pi_0(w_0, w_0) = \pi_0(w_0, w_0) + k(w_0, w_0^*)\) over the range \([0, 2 f y_0 + 2 ((\pi_0 - a)/3 f y_0)]\). Therefore \(w_0^* = w_0^* = \arg \max \pi_0(w_0, w_0)\) and the best response \(w_0^*\) is also the same. Thus given any \(w_0\) the best response is a constant \(w_0^*\). We can similarly show that given any \(w_0\), the best response by platform \(\beta\) is given by \(w_0^* = \max\{w_0^*, (\pi_0 - a)/3 f y_0\}\) where \(w_0^* = \frac{1}{t}((3 + 2 f)\pi_0 + (3 - f) a) f y_0\). Thus the pair \((w_0^*, w_0^*)\) form a unique SPE.

If \(w_0^* \leq (\pi_0 - a)/2 f y_0\) then all CPs will connect to platform \(\alpha\). Following some algebra the former holds when \(\pi_0 \geq \frac{9 f}{4 + 2 f}\). On the other hand, only a fraction of the CPs join the platform whenever \(\pi_0 < \frac{9 f}{4 + 2 f}\).

If \(w_0^* \leq (\pi_0 - a)/3 f y_0\) then all CPs will connect to platform \(\beta\). Following some algebra, one can show the former holds when \(\pi_0 \geq \frac{9 f}{4 - f}\). Therefore, only a fraction of the CPs join the platform whenever \(\pi_0 < \frac{9 f}{4 - f}\).

We define the following sets of prices which we use to characterize market configurations that hold at the SPE.

\[
\mathcal{R}_0^* = \{(w_0, w_0) r_0(w_0, w_0) < 1, r_0(w_0, w_0) < 1\},
\]

\[
\mathcal{R}_0^* = \{(w_0, w_0) r_0(w_0, w_0) < 1, r_0(w_0, w_0) = 1; \text{ or } r_0(w_0, w_0) = 1, r_0(w_0, w_0) < 1\},
\]

\[
\mathcal{R}_0^* = \{(w_0, w_0) r_0(w_0, w_0) = 1, r_0(w_0, w_0) = 1\}.
\]

The set \(\mathcal{R}_0^*\) consists of prices \((w_0, w_0)\) such that only a fraction of the CPs in the market subscribe to the platforms. Set \(\mathcal{R}_0^*\) consists of prices \((w_0, w_0)\) such that the market is covered; all CPs patronize either
platform \( \alpha \) or \( \beta \) but not both. Lastly set \( R_{II}^{n} \) consists of a pair of prices such that the market is covered with all CPs patronizing both platforms. We summarize results of the previous paragraph below.

a): If \( 1 < \frac{\gamma}{a} < \frac{9+2f}{3+2f} \), then \((w_{\alpha}^{*}, w_{\beta}^{*}) \in R_{I}^{n}\).

b): If \( \frac{9+2f}{3+2f} < \frac{\gamma}{a} < \frac{9-f}{3-f} \) then \((w_{\alpha}^{*}, w_{\beta}^{*}) \in R_{II}^{n}\).

c): If \( \frac{9-f}{3-f} \leq \frac{\gamma}{a} < \infty \) then \((w_{\alpha}^{*}, w_{\beta}^{*}) \in R_{II}^{n}\).

Step. 2 CP price games that have a SPE, have the same SPE as the baseline CP price game. Given a price game and any pair \((w_{\beta}, w_{\beta}') \in \mathbb{R}^{+}\), the profit value \(\pi_{\alpha}(w_{\alpha}, w_{\beta}) = \pi_{\alpha}(w_{\alpha}, w_{\beta}') + k(w_{\beta}, w_{\beta}') \) for all \(w_{\alpha} \in [0, 2/3faA + 2((7 - a)/3f\beta y)]\) except possibly at \(w_{\alpha} = 1/2fA + w_{\beta}'\) and \(w_{\alpha} = 1/2fA + w_{\beta}\). Therefore the best response for platform \( \alpha \) given platform \( \beta \) charges \( w_{\beta}' \) is given by \( w_{\alpha}^{*} \) as defined in the previous step if \( w_{\alpha}^{*} \neq 1/2fA + w_{\beta}' \). In the case \( w_{\alpha}^{*} = 1/2fA + w_{\beta}' \) then a best response does not exist.

Similarly, we can show that the best response given any \( w_{\alpha} \) is given by \( w_{\beta}^{*} \), as defined in step 1, if a best response exists. Thus if the best responses intersect they only do so at the price pair \((w_{\alpha}^{*}, w_{\beta}^{*})\).

B.4. The case when \( y_{\alpha} = y_{\beta} \). Lemma 2.2 implies that neither relation (i) or (ii) hold at the SPE when \( y_{\alpha} = y_{\beta} \). Therefore, if an SPE exists it must be that relation (iii) holds. We bound the maximum revenue value that can result in instances for which an SPE exists. Given a tuple \((\gamma, a, f)\), we show later in the investment stage that this pair is not an SPE because either platform has an incentive to deviate.

We now provide an upper-bound for the revenue gained by the platforms if an SPE exists. Revenue for both platforms is derived only from the CP side. This follows because only relation (iii) can hold in equilibrium; due to Bertrand competition, platforms earn no revenue from the consumer side. As discussed in section 4.1.3, the allocation of consumers is evenly divided when relation (iii) holds, i.e. \( q_{\alpha} = q_{\beta} = 1/2f \). Moreover, if relation (iii) holds then \( r_{\alpha} = r_{\beta} \) which further implies \( w_{\alpha} = w_{\beta} \). Let revenue for platform \( \alpha \) and \( \beta \) be represented by \( \pi_{\alpha}(w_{\alpha}^{*}, w_{\beta}^{*}) \) and \( \pi_{\beta}(w_{\alpha}^{*}, w_{\beta}^{*}) \) respectively at the SPE price \((w_{\alpha}^{*}, w_{\beta}^{*})\).

Then \( \pi_{\alpha}(w_{\alpha}^{*}, w_{\beta}^{*}) = \pi_{\beta}(w_{\alpha}^{*}, w_{\beta}^{*}) \) where \( w_{\alpha}^{*} \in ((\gamma - a)q_{\alpha}y_{\alpha}, (\gamma + a)q_{\alpha}y_{\alpha}) \). We consider only prices in this range because other prices are dominated and will not be picked in equilibrium. Consider the function \( \pi(w_{\alpha}) = r_{\alpha}w_{\alpha} \) where \( r_{\alpha} = 1/2a(\gamma + a - w_{\alpha}/q_{\alpha}y_{\alpha}) \). This function is concave and quadratic in \( w_{\alpha} \) and at the value \( w_{\alpha}^{*} \) we have \( \pi(w_{\alpha}^{*}) = \pi_{\alpha}(w_{\alpha}^{*}, w_{\beta}^{*}) \). Let,

\[
\hat{w}_{\alpha} = \text{argmax} \pi(w_{\alpha}),
\]

s.t. \( w_{\alpha} \in ((\gamma - a)(q_{\alpha}y_{\alpha}), (\gamma + a)(q_{\alpha}y_{\alpha})) \).

If an SPE exists platform \( \alpha \)'s profits are bounded by \( \pi(\hat{w}_{\alpha}) \). Since platform \( \beta \)'s profits are the same as \( \alpha \)'s they are also bounded from above by the same value.

B.5. Proof of Theorem 4.2. Given a quality choice \( y_{\beta} \), we derive platform’s \( \alpha \) best response and vice versa. We then find the intersection points that form the SPE. We will give Lemmas that define the best responses and then we will be able to infer the SPE’s from these responses.

Lemma B.3. Let \( R_{1} \) hold. Then

\[
B_{i}(y_{\beta}) = \begin{cases} 
y^{*}(\gamma, a, f, c) & \text{if } y_{\beta} < \gamma, \\
0 & \text{if } y_{\beta} \geq \gamma,
\end{cases}
\]

where

\[
y^{*}(\gamma, a, f, c) = \frac{a^{2}f}{216c} \left( \frac{(2f + 3)^{2}}{a^{3}} \gamma^{2} + \frac{90 - 8(f - 3)^{2}}{a^{2}} \gamma + \frac{(2f + 9)^{2} - 36}{a} \right),
\]

\[
y(\gamma, a, f, c) = \frac{48\gamma f a + 36f a^{2} + 12f \gamma^{2} + 4f^{2} \gamma^{2} - 8f^{2} \gamma a + 4f^{2} a^{2} + 9\gamma^{2} + 18\gamma a + 9a^{2})^{2}}{-432ac(-60\gamma a - 45a^{2} - 15\gamma^{2} + 2f a^{2} + 2f^{2} \gamma^{2} - 4\gamma f a)},
\]
Proof. We first find the best response given \( y_\beta = 0 \). Let \( y^* = \text{argmax } \pi_\alpha(y_\alpha, y_\beta), \text{s.t. } y_\alpha \in \mathbb{R}_+ \). In region \( \mathcal{R}_1 \) the market is uncovered as shown in Appendix B.3. The profit function \( \pi_\alpha(y_\alpha, y_\beta) \) is quadratic and concave in \( y_\alpha \) in this market configuration. The best response, \( B_\alpha(y_\beta) = y^*(\cdot) \) exists since \( \pi_\alpha(y_\alpha, y_\beta) \) is coercive and its value is that given in the Lemma statement.

Next, we find the best response when \( y_\beta > 0 \). The price equilibria that holds depends on whether platform \( \alpha \) acts as the high-quality or the low-quality platform. Indeed, given \( y_\beta > 0 \) platform \( \alpha' \)'s choice of investment \( y_\alpha \) will determine which of the following three relations defined in section 4.1.2 will hold on the equilibrium path.

If \( y_\alpha \) is higher (lower) than \( y_\beta \) then relation (i)/(ii) will hold. When \( y_\alpha = y_\beta \), relation (ii) may hold if an SPE exists. Therefore, we partition the domain \([0, \infty)\) depending on whether platform \( \alpha \) acts a high quality or low quality platform. These partitions are defined as, \( I_1 = [0, y_\beta), I_2 = (y_\beta, \infty) \). In interval \( I_1 \) (\( I_2 \)) only (ii)/(i) holds. In contrast, at the point \( y_\alpha = y_\beta \) relation (iii) holds if an SPE exists. In order to find the best reply given \( y_\beta \) we proceed as follows. We find the best response of platform \( \alpha \) in partitions \( I_1 \) and \( I_2 \). We pick the best reply among these choices and show it dominates the maximum possible choice given \( y_\alpha = y_\beta \) as calculated in the previous appendix.

\[
y^*_\alpha_1 = \text{argmax } \hat{\pi}_\alpha(y_\alpha, y_\beta) \\
\text{s.t. } y_\alpha \in I_1.
\]

Where \( \hat{\pi}_\alpha \) is the profit function in the interval \( I_1 \). In a like manner we denote the best reply in interval \( I_2 \) by \( y^*_\alpha_2 \). Formally,

\[
y^*_\alpha_2 = \text{argmax } \pi_\alpha(y_\alpha, y_\beta) \\
\text{s.t. } y_\alpha \in I_2.
\]

Where \( \pi_\alpha \) is the profit function in the interval \( I_2 \). The profit function \( \hat{\pi}_\alpha \) is concave in \( y_\alpha \). Let \( y^*_\alpha_1 \) be the unrestricted solution. This value is less than zero in region \( \mathcal{R}_1 \). Therefore, the lower constraint in the maximization problem 43 binds and we have \( y^*_\alpha_1 = 0 \). On the other hand, the unrestricted maximization of problem 44 yields \( y^*_\alpha_2 = y^*(\gamma, \alpha, f, c) \) as defined in the statement of the Lemma.

Next we compare the profit values at the solutions in both intervals. Let \( \pi^*_\alpha = \pi_\alpha(y_\alpha, y_\beta)|_{y_\alpha=y^*_\alpha_2} \) and \( \hat{\pi}^*_\alpha = \hat{\pi}_\alpha(y_\alpha, y_\beta)|_{y_\alpha=0} \). One can show that the difference \( \pi^*_\alpha - \hat{\pi}^*_\alpha \) is decreasing in \( y_\beta \). Moreover, the two are equal at \( y_\beta = \gamma \); the value presented in the Lemma statement. Hence whenever \( y_\beta < \gamma \), \( y^*_\alpha_2 \) dominates \( y^*_\alpha_1 \) and vice versa. In addition, \( y^*_\alpha_2 > y_\beta \) whenever \( y_\beta < \gamma \). Therefore this solution lies in the interior of \( I_2 \).

To complete this proof we now show that the profit attainable when \( y_\alpha = y_\beta \) is less than \( \max\{\hat{\pi}^*_\alpha, \pi^*_\alpha\} \). We denote the upper-bound profit value when \( y_\alpha = y_\beta \) by \( \overline{\pi}^*_\alpha \). It suffices to show that this value is less than \( \pi^*_\alpha \) when \( y_\beta > \gamma \) and less than \( \hat{\pi}^*_\alpha \) when \( y_\beta < \gamma \). We first show the former, i.e, the difference \( \pi^*_\alpha - \overline{\pi}^*_\alpha \) is positive whenever \( y_\beta > \gamma \). This difference\(^{35}\) is convex and quadratic in \( y_\beta \). Moreover it has roots at 0 and \( \gamma \). Therefore the difference is positive whenever \( y_\beta > \gamma \). Next we show \( \hat{\pi}^*_\alpha - \pi^*_\alpha \) is positive. The difference is quadratic and convex in \( y_\beta \). Moreover, the roots are imaginary\(^{36}\) thus we infer that the difference is positive.

\[\square\]

A similar analysis follows for platform \( \beta \). The best responses of the platforms intersect at points \((y^*, 0)\) and \((0, y^*)\). Consequently this points form an SPE, see Figure 9.

We next state a number of Lemmas that yield the other results in Theorem 4.2. We omit the proofs because they are very similar. Where there’s significant divergence we add comments.

\[^{35}y_\alpha - \overline{\pi} = y_\alpha(48f^2\gamma a + 36f^2a^2 + 12f^2\gamma^2 + 8f^2\gamma^2 - 16f^2\gamma a + 8f^2a^2 - 27f\gamma^2 - 54f\gamma a - 27f a^2 + 432cy3\alpha) / 432a.\]

\[^{36}\text{root} = -27a^2 - 16f^2a^2 + 144fa^2 + 32f^2\gamma a + 48f\gamma^2 + 192fa - 16f^2\gamma^2 + 27\gamma^2 + 547a \pm \sqrt{-3(\gamma + a)(96fa + 21a + 21\gamma + 32f(\gamma)(32f^2a + 9a - 64f^2\gamma a + 18\gamma a + 32f \gamma^2 + 9\gamma^2)) f/(864ca)}\].
Lemma B.4. Let Assumption 1, \( f > \frac{1}{3} \) and \( R_2 \) hold. Then

\[
B_i(y) = \begin{cases} 
  y_h^*(\gamma, a, f, c) & \text{if } y < \gamma, \\
  y_l^*(\gamma, a, f, c) & \text{if } y \geq \gamma.
\end{cases}
\]

where

\[
y_h^*(\gamma, a, f, c) = \frac{a^2f}{216c} \left( \frac{(2f+3)^2f^2}{a^3} - \frac{90-8(f-3)^2}{a^2} \gamma + \frac{(2f+9)^2-36}{a} \right),
\]

\[
y_l^*(\gamma, a, f, c) = \frac{a^2f}{432c} \left( \frac{f-3)^2}{a^3} \gamma^2 - \frac{-90+2(f+6)^2}{a^2} \gamma + \frac{(f-9)^2-72}{a} \right).
\]

Lemma B.5. Let Assumption 1 and \( R_3 \) hold. Then

\[
B_i(y) = \begin{cases} 
  y^*(\gamma, a, f, c) & \text{if } y < \gamma, \\
  0 & \text{if } y \geq \gamma.
\end{cases}
\]

where

\[
y_h^*(\gamma, a, f, c) = \frac{f}{9c} \left( 4f\gamma^2 + 3\gamma - 3a \right),
\]

\[
y_l^*(\gamma, a, f, c) = 0,
\]

\[
\gamma^*(\gamma, a, f, c) = \frac{(-4f\gamma - 3\gamma + 3a)^2a}{c(18\gamma^2 + 9a^2 + 3\gamma^2 - f\gamma^2 + 2f\gamma a - f^2a^2)}.
\]

Lemma B.6. Let Assumption 1 and \( R_4 \) hold. Then

\[
B_i(y) = \begin{cases} 
  y_h^*(\gamma, a, f, c) & \text{if } y < \gamma, \\
  y_l^*(\gamma, a, f, c) & \text{if } y \geq \gamma.
\end{cases}
\]

where

\[
y_h^*(\gamma, a, f, c) = \frac{f}{9c} \left( 4f\gamma^2 + 3\gamma - 3a \right),
\]

\[
y_l^*(\gamma, a, f, c) = \frac{af}{432c} \left( \frac{(f-3)^2}{a^3} \gamma^2 - \frac{-90+2(f+6)^2}{a^2} \gamma + \frac{(f-9)^2-72}{a} \right).
\]
Lemma B.7. Let $f > 0.47$ and Assumption 1 and $\mathcal{R}_5$ hold. Then

$$B_\alpha(y_\beta) = \begin{cases} y_h^*(\gamma, a, f, c) & \text{if } y_\beta < \gamma, \\ y_l^*(\gamma, a, f, c) & \text{if } y_\beta \geq \gamma, \end{cases}$$

where

$$y_h^*(\gamma, a, f, c) = \frac{f}{9c} (4f\gamma + 3\gamma^2 - 3a),$$

$$y_l^*(\gamma, a, f, c) = \frac{f}{18c} (-2f\gamma + 3\gamma^2 - 3a),$$

$$\bar{y}(\gamma, a, f, c) = \frac{1}{120\gamma^2} (20f^2\gamma^2 + 36f\gamma^2 - 36f a + 9\gamma^2 - 18\gamma a + 9a^2).$$

Proof: We first find the best response given $y_\beta = 0$. Similar to the proof of Lemma B.3 we let $y^* = \arg\max \pi_\alpha(y_\alpha, y_\beta)$, s.t. $y_\alpha \in \mathbb{R}^+$. In region $\mathcal{R}_5$ the market is covered as shown in Appendix B.3. The profit function $\pi_\alpha(y_\alpha, y_\beta)$ is quadratic and concave in $y_\alpha$ in this market configuration. The best response, $B_\alpha(y_\beta) = y^*(\cdot)$ exists since $\pi_\alpha(y_\alpha, y_\beta)$ is coercive and the solution is that given by $y_h^*(\gamma, a, f, c)$ in the Lemma statement.

Next we find the best response when $y_\beta > 0$. Given $y_\beta > 0$ a platform $\alpha$ decides to have a quality that is the same as platform $\beta$ or to be either the high or low quality platform. The choice made will determine which of the three relations defined in section 4.1.2 will hold. If $y_\alpha > (\leq) y_\beta$ then relation (i)(ii) results. If $y_\alpha = y_\beta$ and an SPE exists then relation (iii) holds. Therefore we partition the domain $y_\alpha$ into two regions that do not include the point $y_\beta$. These partitions are defined as $I_1 = [0, y_\beta], I_2 = (y_\beta, \infty)$.

In order to find the best reply we proceed as follows. We find the best response of platform $\alpha$ in partitions $I_1$ and $I_2$. We pick the best reply among these choices and show it dominates the choice $y_\alpha = y_\beta$. Let

$$y_{\alpha1}^* = \arg\max_{y_\alpha \in I_1} \pi_\alpha(y_\alpha, y_\beta)$$

$$y_{\alpha2}^* = \arg\max_{y_\alpha \in I_2} \pi_\alpha(y_\alpha, y_\beta)$$

Here, $\pi_\alpha$ is the profit function in the interval $I_1$. In a like manner we denote the best reply in interval $I_2$ by $y_{\alpha2}^*$. Formally,

$$y_{\alpha1}^* = \arg\max_{y_\alpha \in I_1} \pi_\alpha(y_\alpha, y_\beta)$$

$$y_{\alpha2}^* = \arg\max_{y_\alpha \in I_2} \pi_\alpha(y_\alpha, y_\beta)$$

Here, $\pi_\alpha$ is the profit function in the interval $I_2$. The profit function $\hat{\pi}_\alpha$ is concave in $y_\alpha$. Let $y_{\alpha1}^*$ be the unrestricted solution of problem 45. Its value is given by $y_{\alpha1}^* = y_l^*(\gamma, a, f, c)$. This value is greater than zero in region $\mathcal{R}_5$. The unrestricted maximization of problem 46 yields $y_{\alpha2}^* = y_h^*(\gamma, a, f, c)$ as defined in the statement of the Lemma.

Next we compare the profit values at these solutions. Let $\pi_\alpha^* = \pi_\alpha(y_\alpha, y_\beta)|y_\alpha = y_{\alpha1}^*$ and $\hat{\pi}_\alpha^* = \hat{\pi}_\alpha(y_\alpha, y_\beta)|y_\alpha = y_{\alpha1}^*$. One can show that the difference $\pi_\alpha^* - \hat{\pi}_\alpha^*$ is decreasing in $y_\beta$. Hence whenever $y_\beta < \gamma$, $y_{\alpha2}^*$ dominates $y_{\alpha1}^*$ and vice versa. Moreover, the two are equal at $y_\beta = \gamma$, this is the value presented in the Lemma statement. In addition, one can show that $\gamma \in (y_{\alpha1}^*, y_{\alpha2}^*)$ whenever $f > 3(2\sqrt{19} - 1)/50 \approx 0.47$.

To complete this proof we now show that the highest profit attainable when $y_\alpha = y_\beta$, and a SPE in the quality subgame exists, is less than $\max\{\pi_\alpha^*, \hat{\pi}_\alpha^*\}$. We denote the upper-bound profit value when $y_\alpha = y_\beta$ by $\overline{\pi}_\alpha$. It suffices to show that this value is less than $\pi_\alpha^*$. From previous Appendix it follows that $\pi_\alpha^* < \overline{\pi}_\alpha$. To see this note that when $y_\alpha = y_\beta$, and an SPE results, platform $\alpha$ makes no revenue on the consumer side.

The best responses for platform $\beta$ are similarly derived. These responses intersect at the investment pairs $(y_h^*, y_l^*)$ and $(y_l^*, y_h^*)$. Thus these form the SPE as stated in the Theorem, see Figure 10.
The best reply responses of both platforms in region $R_1$. The intersection points, $(y^*, 0)$ and $(0, y^*)$, give the equilibrium investment levels in region $R_1$.

**APPENDIX C. SOCIAL WELFARE, CP AND CONSUMER SURPLUS COMPARISON**

**C.1. Social Welfare Comparison.** In this section we provide comparison of social welfare at the SPE under both models. First, we characterize the difference between welfare of the non-neutral and neutral regime in terms of the following exogenous parameters $(\gamma, a, f, c)$. We show that this difference is non-negative and that in general, the non-neutral regime is favored to the neutral regime. Since the SPE for both models have been characterized for $f > 3/5$ the comparisons are also based for the same range of $f$.

We denote the difference between the non-neutral and neutral welfare by $dw$. The welfare functions at the SPE’s have different forms in both models depending on whether the market is covered or uncovered. The restrictions on the tuple $(\gamma, a, f, c)$ that define the limits in which the CP market is uncovered and covered coincide in both models. So we compare the welfare between the two regimes for each of the regions defined in Section 4.1.5. After some algebra, the difference in welfare for the different regions are given below:

<table>
<thead>
<tr>
<th>Regions</th>
<th>Difference in welfare $dw$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$ and $R_3$</td>
<td>$dw = 0$</td>
</tr>
<tr>
<td>$R_2$ and $R_4$</td>
<td>$dw = \frac{(\gamma - a)^2 f^2 - 6(\gamma + a)(\gamma + 3a) f + 9(\gamma + a)^2)(-(\gamma - a)^2 f^2 + (60a^2 - (\gamma - 7a)^2) f + 6(\gamma + a)(2\gamma + 3))f^2}{93412ca^2}$</td>
</tr>
<tr>
<td>$R_5$</td>
<td>$dw = \frac{1}{324c} f^2(3a + 5\gamma + 2\gamma f)(3\gamma - 3a - 2\gamma f)$</td>
</tr>
</tbody>
</table>

For $f \geq 3/5$ the value $dw$ is positive in regions $R_2$, $R_4$ and $R_5$. In these regions, the low-quality platform makes an investment that increases the gross value of CPs and Consumer surplus compared to their values in the neutral regime. On the other hand, in regions $R_1$ and $R_3$ the welfare in both regimes is the same because the investment levels are the same.

**C.2. CP Surplus Comparison.** In this section we compare the CP surplus in both regimes. Let $decp$ denote the difference in CP surplus between the two regimes. The following table shows this difference in the regions defined in Section 4.1.5.
C.3. Consumer surplus Comparison. In this subsection we compare the consumer surplus in both regimes. Let $dc$ denote the difference in consumer surplus between the two regimes. The following table shows the consumer surplus difference in the regions defined in Section 4.1.5.

<table>
<thead>
<tr>
<th>Regions</th>
<th>Difference in welfare $dc$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{R}_1$ and $\mathcal{R}_3$</td>
<td>$dc = 0$</td>
</tr>
<tr>
<td>$\mathcal{R}_2$ and $\mathcal{R}_4$</td>
<td>$dc = \frac{f^2(-6(\tau+a)(\tau+3a)f+(\tau-a)^2f^2+9(\tau+a)^2))/(3(\tau+a)-f(\tau-a))^2}{186624ca^2}$</td>
</tr>
<tr>
<td>$\mathcal{R}_5$</td>
<td>$dc = \frac{1}{34a}f^2(\tau(3 - 2f) - 3a)/(\gamma)$</td>
</tr>
</tbody>
</table>

In regions $\mathcal{R}_1$ and $\mathcal{R}_3$ consumer surplus is the same under both regimes because both platforms invest in the same qualities. However, in regions $\mathcal{R}_2$, $\mathcal{R}_4$ and $\mathcal{R}_5$ the value $dc$ is positive.

C.4. Platform profits Comparison. In this subsection we compare aggregate and individual platform profits in both regimes. Let $dtp$ denote the difference in aggregate platform profit between the two regimes. The following table shows this difference in the regions defined in Section 4.1.5.

<table>
<thead>
<tr>
<th>Regions</th>
<th>Difference in aggregate $dtp$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{R}_1$ and $\mathcal{R}_3$</td>
<td>$dtp = 0$</td>
</tr>
<tr>
<td>$\mathcal{R}_2$ and $\mathcal{R}_4$</td>
<td>$dtp = \frac{f^2(-6(\tau+a)(\tau+3a)f+(\tau-a)^2f^2+9(\tau+a)^2)/(3(\tau+a)-f(\tau-a))^2)}{186624ca^2}$</td>
</tr>
<tr>
<td>$\mathcal{R}_5$</td>
<td>$dtp = -\frac{1}{108c}f^2(\tau(3 - 2f) - 3a)/(\gamma)(k(6f - 1) + a)$.</td>
</tr>
</tbody>
</table>

In regions $\mathcal{R}_1$ and $\mathcal{R}_3$ the profits of the platforms are the same in both the neutral and non-neutral regimes. This follows because the investments across platforms are equal in both regimes. In regions $\mathcal{R}_2$, $\mathcal{R}_4$ and $\mathcal{R}_5$ the aggregate profit in the neutral regime is higher than that in the non-neutral regime since $dtp$ is positive.
Next we show that profit of the high-quality platform is larger in the neutral regime in regions $R_2$, $R_4$ and $R_5$. Let $dtph$ be the difference between the high-quality platform’s profit in both regimes.

<table>
<thead>
<tr>
<th>Regions</th>
<th>Difference in profits $dtph$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$ and $R_3$</td>
<td>$dtph = 0$</td>
</tr>
<tr>
<td>$R_2$ and $R_4$</td>
<td>$dtph = \frac{f^3(-6(\gamma+a)(\gamma+3a)f+(\gamma-a)^2f^2+9(\gamma+a)^2)(-3(k+a)(k+3a)f+17(k-a)^2+9(k+a)^2f)}{11664a^2}$</td>
</tr>
<tr>
<td>$R_5$</td>
<td>$dtph = -\frac{4}{81}f^3(\gamma(3 - 2f) - 3a)$</td>
</tr>
</tbody>
</table>

Given the restrictions that define regions $R_2$, $R_4$ and $R_5$, $dtph$ is positive whenever $f > 3/5$.

Next we show that in general the low-quality platform prefers the non-neutral regime. Let the difference of profit between the low-quality platform in the neutral and non-neutral regimes be $dtpl$. The table below shows this quantity in the different regions.

<table>
<thead>
<tr>
<th>Regions</th>
<th>Difference in profits $dtpl$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$ and $R_3$</td>
<td>$dtpl = 0$</td>
</tr>
<tr>
<td>$R_2$ and $R_4$</td>
<td>$dtpl = \frac{1}{186624a^2}f^2((-6(\gamma+a)(\gamma+3a)f+(\gamma-a)^2f^2+9(\gamma+a)^2))^2$</td>
</tr>
<tr>
<td>$R_5$</td>
<td>$dtpl = \frac{1}{324}f^2\gamma f^3(\gamma(3 - 2f) - 3a)^2$</td>
</tr>
</tbody>
</table>

The value $dtpl$ is non-negative. In regions $R_1$ and $R_3$ the profits are the same because the investments across both platforms are the same. In contrast, the low-quality platform’s profits in regions $R_2$, $R_4$ and $R_5$ are superior in the non-neutral regime because $dtpl$ is positive.