Research Article

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An optimal control of a risk-sensitive problem for backward doubly stochastic differential equations with applications

<https://doi.org/10.1515/rose-2020-2024>

Received May 30, 2018; accepted October 25, 2019

Abstract: In this paper, we are concerned with an optimal control problem where the system is driven by a backward doubly stochastic differential equation with risk-sensitive performance functional. We generalized the result of Chala [A. Chala, Pontryagin's risk-sensitive stochastic maximum principle for backward stochastic differential equations with application, *Bull. Braz. Math. Soc. (N.S.)* **48** (2017), no. 3, 399–411] to a backward doubly stochastic differential equation by using the same contribution of Djehiche, Tembine and Tempone in [B. Djehiche, H. Tembine and R. Tempone, A stochastic maximum principle for risk-sensitive mean-field type control, *IEEE Trans. Automat. Control* **60** (2015), no. 10, 2640–2649]. We use the risk-neutral model for which an optimal solution exists as a preliminary step. This is an extension of an initial control system in this type of problem, where an admissible controls set is convex. We establish necessary as well as sufficient optimality conditions for the risk-sensitive performance functional control problem. We illustrate the paper by giving two different examples for a linear quadratic system, and a numerical application as second example.

Keywords: Backward doubly stochastic differential equation, risk-sensitive, optimal control, stochastic maximum principle, variational principle, logarithmic transformation

MSC 2010: 93E20, 60H30, 60G20

Communicated by: Vyacheslav L. Girko

1 Introduction

In this paper, we aim, using Pontryagin's maximum principle, to prove necessary as well as sufficient optimality conditions, for a risk-sensitive control problem associated with dynamics driven by a backward doubly stochastic differential equation (BDSDE in short). We solve the problem by using the approach developed by Djehiche, Tembine and Tempone in [5]; see also [2]. Our contribution can be summarized as follows: Djehiche et al. [5] have established a stochastic maximum principle for a class of risk-sensitive mean-field type control problems, where the distribution enters only through the mean of a state process, which means that the drift, diffusion, and terminal cost functions depend on the state, the control and the means of the state process. Their work extends the results of Lim and Zhou [10] to risk-sensitive control problems for dynamics that are non-Markovian and without mean-field term.

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Nonlinear BDSDEs have been introduced by Pardoux and Peng [12]. They have considered a new kind of BSDE that is a class of BDSDEs with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic Itô integral dW_t and a backward stochastic Itô integral $\overleftarrow{dB}_t$. More precisely, they dealt with the following BDSDE:

$$\begin{cases} -dY_t = f(t, Y_t, Z_t) dt + g(t, Y_t, Z_t) \overleftarrow{dB}_t - Z_t dW_t, \\ Y_T = \xi. \end{cases} \quad (1.1)$$

Here we regard (1.1) as a backward equation in two different aspects. The first is with respect to B for which the time variable is reversed. The second is with respect to W , which is forward in time while the boundary condition is given at the terminal instead of the initial time. They proved that if f and g are uniform Lipschitz, then (1.1) has a unique solution (Y_t, Z_t) in the interval $[0, T]$, for any Z_t being square integrable, and the terminal condition ξ is a $\mathcal{F}_T^W$ -measurable and square integrable random variable. They also showed that BDSDEs can produce a probabilistic representation for solutions to some quasi-linear stochastic partial differential equations. Since this is the first existence and uniqueness result, many papers have been devoted to existence and/or uniqueness results under weaker assumptions; see [12] and the references therein. Further, the control processes are assumed to be adapted to a subfiltration of the filtration generated by both forward and backward filtration. In the risk-sensitive control problem, the system is governed by the nonlinear BDSDE

$$\begin{cases} dy_t = -f(t, y_t, z_t, v_t) dt - g(t, y_t, z_t, v_t) \overleftarrow{dB}_t + z_t dW_t, \\ y_T = \xi, \end{cases}$$

where f, g are given maps, $B = (B_t)_{t \geq 0}$ and $W = (W_t)_{t \geq 0}$ are two mutually independent standard Brownian motion processes, z_t is square integrable and the terminal condition ξ is a $\mathcal{F}_T^W$ -measurable and square integrable random variable. The control variable $v = (v_t)$ is called admissible control. We denote by $\mathcal{U}$ the class of all admissible controls.

We define the criterion to be minimized, with initial risk-sensitive functional cost, as follows:

$$J^\theta(v) = \mathbb{E} \left[\exp \theta \left\{ \Psi(y_0) + \int_0^T l(t, y_t, z_t, v_t) dt \right\} \right],$$

where Ψ and l are given maps and (y_t, z_t) is the trajectory of the system controlled by v_t .

A control u is called optimal if it solves $J^\theta(u) = \inf_{v \in \mathcal{U}} J^\theta(v)$.

The existence of an optimal solution for this problem has been solved to achieve the objective of this paper, and we have established necessary as well as sufficient optimality conditions for this model. In this paper, we generalize the results obtained by Chala in [2] to the BDSDE. The idea here is to reformulate in the first step the risk-sensitive control problem in terms of an augmented state process and terminal payoff problem. An intermediate stochastic maximum principle (SMP in short) is then obtained by applying the SMP of [1, 8] for a loss functional without running cost; for the same particular cases see [7]. Then we transform the intermediate adjoint processes to a simpler form, using the fact that the set of controls is convex. Then we establish necessary as well as sufficient optimality conditions by using the Logarithmic transform established by El Karoui and Hamadène [6]. More precisely, the method of Lim and Zhou [10] shows in fact that it is enough to use a generic square integrable martingale to transform the pair (p_1, q_1) into the adjoint process $(\tilde{p}_1(t), 0)$, where the process $\tilde{p}_1(t)$ is still a square integrable martingale, which would mean that $\tilde{p}_1(t) = \tilde{p}_1(T)$, and is equal to the constant $\mathbb{E}[\tilde{p}_1(T)]$. But this generic martingale need not be related to the adjoint process $\vec{p}(t)$ as in Lim and Zhou [10]. Instead, it will be part of the adjoint equation associated with the risk-sensitive SMP (see Proposition 4.2 below).

We note that necessary as well as sufficient optimality conditions for risk-sensitive controls, where the systems are governed by a stochastic differential equation (SDE in short), have been studied by Lim and Zhou [10]. We also note that necessary as well as sufficient optimality conditions for stochastic controls, where the systems are governed by a nonlinear forward stochastic differential equation with jumps, have been studied by Shi and Wu in [14] in the case where the set of admissible controls is convex, and in [15] in

the general case with an application to finance. Furthermore, the systems governed by a mean-field SDE have been studied by Djehiche et al. in [5]. For some general case, we can note the paper of Khallout and Chala [9] with the fully coupled forward backward stochastic differential equation, and where the admissible control set could be convex.

Our goal in this paper is to establish a set of necessary as well as sufficient optimality conditions for BDSDE with risk-sensitive performance cost. The backward doubly problem under consideration is not simple extension from the mathematical point of view, but also provides interesting models in many applications such as mathematical economics or optimal control. The proof of our main result is based on the spike variation method based on Theorem 3.2 as a preliminary step for the risk-neutral control problem.

The paper is organized as follows: In Section 2, we give the precise formulation of the problem; we introduce the risk-sensitive model, formulate the problem and give the various assumptions used throughout this paper. In Section 3, we shall study our system of BDSDE, and give the relation between the risk-neutral and risk-sensitive SMP. In Section 4, we give the new method of the transformation of the adjoint processes; the aim of this result will be shown in that section. We give our first main result, the necessary optimality conditions for risk-sensitive control problem under additional hypothesis, and the second result will be given in the following section. In Section 5, we derive our second main result of this paper, which are sufficient optimality conditions for risk-sensitive controls. In Section 6, we improve the quality of the paper by giving two applications to a linear quadratic stochastic control problem, the method which used the Riccati equations is applied in a second example, and we give a numerical example. Section 7 consists of a conclusion and an outlook.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in which one-dimensional Brownian motions $W = (W_t : 0 \leq t \leq T)$ and $B = (B_t : 0 \leq t \leq T)$ are defined, where W and B are two mutually independent standard Brownian motion processes. Let $\mathcal{N}$ denote the class of $\mathbb{P}$ -null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define $\mathcal{F}_t^{(W,B)} = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, where for any process $\{L_t\}_{t \in [0,T]}$ one has $\mathcal{F}_{s,t}^L = \sigma\{L_r - L_s : s \leq r \leq t\} \vee \mathcal{N}$ and $\mathcal{F}_t^L = \mathcal{F}_{0,t}^L$.

Note that the collection $\{\mathcal{F}_t^{(W,B)} : t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a filtration. We may define the subfiltration $(\mathcal{G}_t)_{t \in [0,T]}$ such that $\mathcal{G}_t \subset \mathcal{F}_t^{(W,B)}$ for all $t \in [0, T]$.

Let $\mathcal{M}^2([0, T]; \mathbb{R})$ denote the set of one-dimensional jointly measurable random processes $\{\varphi_t : t \in [0, T]\}$ which satisfy the following conditions:

- (i) $\|\varphi\|_{\mathcal{M}^2} = \mathbb{E}[\int_0^T |\varphi_t|^2 dt] < \infty$.
- (ii) φ_t is $\mathcal{F}_t^{(W,B)}$ -measurable for any $t \in [0, T]$.

Similarly, we denote by $\mathcal{S}^2([0, T]; \mathbb{R})$ the set of one-dimensional continuous random processes which satisfy the following conditions:

- (i) $\|\varphi\|_{\mathcal{S}^2} = \mathbb{E}[\sup_{t \in [0,T]} |\varphi_t|^2] < \infty$.
- (ii) φ_t is $\mathcal{F}_t^{(W,B)}$ -measurable for any $t \in [0, T]$.

Let T be a positive real number and let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}$.

Definition 2.1. An admissible control v is a measurable process with values in $\mathcal{A}$ such that $\mathbb{E} \int_0^T |v_t|^2 dt < \infty$. We denote by $\mathcal{U}$ the set of all admissible controls.

For any $v \in \mathcal{U}$, we consider the following BDSDE system:

$$\begin{cases} dy_t = -f(t, y_t, z_t, v_t) dt - g(t, y_t, z_t, v_t) \overleftarrow{dB}_t + z_t dW_t, \\ y_T = \xi, \end{cases} \quad (2.1)$$

where $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \rightarrow \mathbb{R}$, $g : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \rightarrow \mathbb{R}$ are jointly measurable and such that for any $(y, z, v) \in \mathbb{R} \times \mathbb{R} \times \mathcal{A}$ one has $f(\cdot, y, z, v) \in \mathcal{M}^2([0, T] \times \mathbb{R})$, $g(\cdot, y, z, v) \in \mathcal{M}^2([0, T] \times \mathbb{R})$, and z_t is square integrable and the terminal condition ξ is a $\mathcal{F}_T^W$ -measurable and square integrable random variable.

Note that the integral with respect to $(B_t)_{t \in [0, T]}$ is a “backward” Itô integral, while the integral with respect to $(W_t)_{t \in [0, T]}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô–Skorohod integral; for more details we refer to [11].

We define the criterion to be minimized, with initial risk-sensitive performance functional cost, as

$$J^\theta(v) = \mathbb{E} \left[\exp \theta \left\{ \Psi(y_0) + \int_0^T l(t, y_t, z_t, v_t) dt \right\} \right], \quad (2.2)$$

where $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ and $l : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \rightarrow \mathbb{R}$ are jointly measurable and θ is the risk-sensitive index.

The optimal control problem is to minimize the functional J^θ over $\mathcal{U}$ if $u \in \mathcal{U}$ is an optimal control (solution), that is,

$$J^\theta(u) = \inf_{v \in \mathcal{U}} J^\theta(v). \quad (2.3)$$

We assume the following.

Assumption 2.2. There exist constants $c > 0$ and $0 \leq \lambda < 1$ such that for any $(w, t) \in \Omega \times [0, T]$ and

$$(y_1, z_1, v_1), (y_2, z_2, v_2) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A}$$

we have

$$\begin{aligned} |f(t, y_1, z_1, v_1) - f(t, y_2, z_2, v_2)|^2 &\leq c(|y_1 - y_2|^2 + |z_1 - z_2|^2 + |v_1 - v_2|^2), \\ |g(t, y_1, z_1, v_1) - g(t, y_2, z_2, v_2)|^2 &\leq c(|y_1 - y_2|^2 + |v_1 - v_2|^2) + \lambda |z_1 - z_2|^2. \end{aligned}$$

Theorem 2.3. For any given admissible control v , we suppose that Assumption 2.2 holds. Then the BDSDE (2.1) has a unique solution $(y_t, z_t) \in \mathcal{M}^2([0, T]; \mathbb{R}) \times \mathcal{S}^2([0, T]; \mathbb{R})$.

Proof. See [12, Theorem 1.1, p. 212]. □

A control that solves problem (2.1)–(2.3) is called optimal. Our objective is to establish risk-sensitive necessary and sufficient optimality conditions, satisfied by a given optimal control, in the form of risk-sensitive SMP.

We also assume the following.

Assumption 2.4. (i) f, g, l and Ψ are continuously differentiable with respect to (y, z, v) .
(ii) All derivatives of f, g and l are bounded by $C(1 + |y| + |z| + |v|)$.
(iii) The derivative of Ψ is bounded by $C(1 + |y|)$.

Under the above Assumptions 2.2 and 2.4, for each $v \in \mathcal{U}$, equation (2.1) has a unique strong solution, and the cost function J^θ is well defined from $\mathcal{U}$ into $\mathbb{R}$.

For more details the reader can see paper of Han, Peng and Wu in [8].

Remark 2.5. We use the Euclidean norm $|\cdot|$ in $\mathbb{R}$, $\top$ is a matrix transpose and Tr is the trace of a matrix. All the equalities and inequalities mentioned in this paper are in the sense of $dt \times d\mathbb{P}$ almost surely on $[0, T] \times \Omega$.

3 Risk-sensitive stochastic maximum principle of backward doubly type control

The proof of our risk-sensitive stochastic maximum principle necessitates a certain auxiliary state process x_t , which is the solution of the following forward SDE:

$$dx_t = l(t, y_t, z_t, v_t) dt, \quad x_0 = 0.$$

Our control problem of (2.1)–(2.3) is equivalent to

$$\left\{ \begin{array}{l} \inf_{v \in \mathcal{U}} \mathbb{E}[\exp \theta \{\Psi(y_0) + x_T\}] \quad \text{subject to} \\ dx_t = l(t, y_t, z_t, v_t) dt, \\ dy_t = -f(t, y_t, z_t, v_t) dt - g(t, y_t, z_t, v_t) \overleftarrow{dB}_t + z_t dW_t, \\ x_0 = 0, y_T = \xi. \end{array} \right. \quad (3.1)$$

We require the following condition:

$$A_T^\theta := \exp \theta \left\{ \Psi(y_0) + \int_0^T l(t, y_t, z_t, v_t) dt \right\}.$$

If we put $\Theta_T = \Psi(y_0) + \int_0^T l(t, y_t, z_t, v_t) dt$, then the risk-sensitive loss functional is given by

$$\mathcal{H}(\theta, v) := \frac{1}{\theta} \log \left[\mathbb{E} \left(\exp \theta \left\{ \Psi(y_0) + \int_0^T l(t, y_t, z_t, v_t) dt \right\} \right) \right] = \frac{1}{\theta} \log [\mathbb{E}(\exp \theta \Theta_T)].$$

When the risk-sensitive index θ is small, the loss functional $\mathcal{H}(\theta, u)$ can be expanded as

$$\mathbb{E}(\Theta_T) + \frac{\theta}{2} \text{Var}(\Theta_T) + O(\theta^2),$$

where $\text{Var}(\Theta_T)$ denotes the variance of Θ_T . If $\theta < 0$, the variance of Θ_T , as a measure of risk, improves the performance $\mathcal{H}(\theta, v)$, in which case the optimizer is called risk seeker. But when $\theta > 0$, the variance of Θ_T worsens the performance $\mathcal{H}(\theta, v)$, in which case the optimizer is called risk averse. The risk-neutral loss functional $\mathbb{E}(\Theta_T)$ can be seen as a limit of the risk-sensitive functional $\mathcal{H}(\theta, v)$ when $\theta \rightarrow 0$; for more details see [4].

Next, let us introduce the following notations.

Notation 3.1. We set $X := \begin{pmatrix} x \\ y \end{pmatrix}$, $M_t := \begin{pmatrix} W_t \\ B_t \end{pmatrix}$, $d\vec{p}(t) := \begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix}$,

$$F(t, y_t, z_t, v_t) := \begin{pmatrix} l(t, y_t, z_t, v_t) \\ -f(t, y_t, z_t, v_t) \end{pmatrix} \quad \text{and} \quad G(t, y_t, z_t, v_t) := \begin{pmatrix} 0 \\ z_t - g(t, y_t, z_t, v_t) \end{pmatrix}.$$

With these notations the system (3.1) can be rewritten in the following compact SDE form:

$$\begin{cases} dX_t = F(t, y_t, z_t, v_t) dt + G(t, y_t, z_t, v_t) dM_t, \\ X \begin{pmatrix} 0 \\ T \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}. \end{cases} \quad (3.2)$$

For convenience, we will use the following notations throughout this paper: For $\phi \in \{f, g, l\}$ we define

$$\begin{cases} \phi(t) = \phi(t, y_t, z_t, v_t), \\ \partial \phi(t) = \phi(t, y_t, z_t, v_t) - \phi(t, y_t^u, z_t^u, u_t), \\ \phi_\zeta(t) = \frac{\partial \phi}{\partial \zeta}(t, y_t, z_t, v_t), \quad \zeta = y, z, v, \\ \tilde{H}^\theta(t) = \tilde{H}^\theta(t, x_t, y_t, z_t, v_t, \vec{p}(t), \vec{q}(t)), \\ \partial \tilde{H}^\theta(t) = \tilde{H}^\theta(t, x_t, y_t, z_t, v_t, \vec{p}(t), \vec{q}(t)) - \tilde{H}^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t)), \\ \tilde{H}_\zeta^\theta(t) = \frac{\partial \tilde{H}^\theta}{\partial \zeta}(t, x_t, y_t, z_t, v_t, \vec{p}(t), \vec{q}(t)), \quad \zeta = y, z, v, \end{cases}$$

and

$$\begin{cases} H^\theta(t) = H^\theta(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)), \\ \partial H^\theta(t) = H^\theta(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)) - H^\theta(t, y_t^u, z_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)), \\ H_\zeta^\theta(t) = \frac{\partial H^\theta}{\partial \zeta}(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)), \quad \zeta = y, z, v, \end{cases}$$

where v_t is an admissible control from $\mathcal{U}$.

We suppose that Assumptions 2.2 and 2.4 hold. We may combine the SMP for a risk-neutral controlled differential equation of backward doubly stochastic type from [1, 8] with the result of Yong [16] and with aug-

mented state dynamics (x, y, z) to derive the adjoint equation. There exist two unique $\mathcal{G}_t$ -adapted pairs of processes (p_1, q_1) and (p_2, q_2) which solve the following system of backward SDEs:

$$\begin{cases} d\vec{p}(t) = -A(t) dt + R(t) dM_t, \\ \begin{pmatrix} p_1(T) \\ p_2(0) \end{pmatrix} = \theta A_T^\theta \begin{pmatrix} 1 \\ -\Psi_y(y_0^u) \end{pmatrix}, \end{cases} \quad (3.3)$$

with

$$\mathbb{E} \left[\sum_{i=1}^2 \sup_{t \in [0, T]} |p_i(t)|^2 + \sum_{i=1}^2 \int_0^T |q_i(t)|^2 dt \right] < \infty,$$

where

$$A(t) = \begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$

and

$$R(t) = \begin{pmatrix} [c]ccq_1(t) & 0 \\ p_3(t) & -q_2(t) \end{pmatrix}$$

such that

$$p_3(t) = -\text{Tr} \left[\begin{pmatrix} l_z(t) & -f_z(t) \\ 0 & -g_z(t) \end{pmatrix} \begin{pmatrix} p_1(t) & q_1(t) \\ p_2(t) & q_2(t) \end{pmatrix} \right].$$

Let $\tilde{H}^\theta$ be the Hamiltonian associated with the optimal state dynamics (x^u, y^u, z^u) , and let the two pairs of adjoint process $((p_1, q_1), (p_2, q_2))$ be given by

$$\tilde{H}^\theta(t) := \tilde{H}^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t)) = l(t)p_1(t) - f(t)p_2(t) - g(t)q_2(t). \quad (3.4)$$

Theorem 3.2. *We suppose that Assumptions 2.2 and 2.4 hold. If (x^u, y^u, z^u) is an optimal solution of the risk-neutral control problem (3.2), then there exist two pairs of $\mathcal{G}_t$ -adapted processes $((p_1, q_1), (p_2, q_2))$ that satisfy (3.3) such that*

$$\tilde{H}_v^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(u_t - v_t) \leq 0 \quad (3.5)$$

for all $u \in \mathcal{A}$, almost every $t \in [0, T]$ and $\mathbb{P}$ -almost surely, where $\partial \tilde{H}^\theta(t)$ is defined in Notation 3.1.

4 Finding the new adjoint equations and necessary optimality conditions

As mentioned, Theorem (3.2) is a good SMP for the risk-neutral control problem of forward-backward doubly type. We follow the same approach used in [2, 9], and suggest a transformation of the adjoint processes (p_1, q_1) and (p_2, q_2) in such a way that we can omit the first component (p_1, q_1) in (3.3) and express the SMP in terms of only one adjoint process which we denote by $(\tilde{p}_2, \tilde{q}_2)$.

By noting that $dp_1(t) = q_1(t) dW_t$ and $p_1(T) = \theta A_T^\theta$, the explicit solution of this backward SDE is

$$p_1(t) = \theta \mathbb{E}[A_T^\theta | \mathcal{G}_t] = \theta V_t^\theta, \quad (4.1)$$

where $V_t^\theta := \mathbb{E}[A_T^\theta | \mathcal{G}_t]$, $0 \leq t \leq T$.

In view of (4.1), it would be natural to choose a transformation of $(\vec{p}, \vec{q})$ into an adjoint process $(\tilde{p}, \tilde{q})$, where

$$\tilde{p}_1(t) = \frac{1}{\theta V_t^\theta} p_1(t) = 1.$$

We consider the following transform:

$$\tilde{p}(t) = \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} := \frac{1}{\theta V_t^\theta} \vec{p}(t), \quad 0 \leq t \leq T. \quad (4.2)$$

By using (3.3) and (4.2), we have

$$\tilde{p}(\cdot) := \begin{pmatrix} \tilde{p}_1(T) \\ \tilde{p}_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -\Psi_Y(y_0^u) \end{pmatrix}.$$

The following properties of the generic martingale V^θ are essential in order to investigate the properties of these new process $(\tilde{p}(t), \tilde{q}(t))$: The process Λ^θ is the first component of the $\mathcal{G}_t$ -adapted pair of processes (Λ^θ, N) which is the unique solution to the following quadratic backward SDE:

$$\begin{cases} d\Lambda_t^\theta = -\left(l(t) + \frac{\theta}{2}|N(t)|^2\right) dt + N(t)dM_t, \\ \Lambda_T^\theta = \Psi(y_0^u), \end{cases} \quad (4.3)$$

where $\mathbb{E}[\int_0^T |N(t)|^2 dt] < \infty$.

Next, we will state and prove the necessary optimality conditions for the system driven by a BDSDE with a risk-sensitive performance functional type. To this end, let us summarize and prove some lemmas that we will use thereafter.

Lemma 4.1. *Suppose that Assumption 2.4 holds. Then*

$$\mathbb{E}\left[\sup_{t \in [0, T]} |\Lambda_t^\theta|\right] \leq C_T. \quad (4.4)$$

In particular, V^θ solves the following linear backward SDE:

$$dV_t^\theta = \theta N(t)V_t^\theta dM_t, \quad V_T^\theta = A_T^\theta. \quad (4.5)$$

Hence, the process defined on $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t \geq 0}, \mathbb{P})$ by L_t^θ , where

$$L_t^\theta := \frac{V_t^\theta}{V_0^\theta} = \exp\left(\theta \int_0^t N(s)dM_s - \frac{\theta^2}{2} \int_0^t |N(s)|^2 ds\right), \quad 0 \leq t \leq T, \quad (4.6)$$

is a uniformly bounded $\mathcal{G}_t$ -martingale.

Proof. First, we prove (4.4). By Assumption 2.4, l and Ψ are bounded by a constant $C > 0$. We have

$$0 < e^{-(1+T)C\theta} \leq A_T^\theta \leq e^{(1+T)C\theta}. \quad (4.7)$$

Therefore, V^θ is a uniformly bounded $\mathcal{G}_t$ -martingale satisfying

$$0 < e^{-(1+T)C\theta} \leq V_t^\theta \leq e^{(1+T)C\theta}, \quad 0 \leq t \leq T. \quad (4.8)$$

The sufficient conditions of the logarithmic transform established in [6, Proposition 3.1] can be applied in the martingale V^θ as follows:

$$V_t^\theta = \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right), \quad 0 \leq t \leq T,$$

and $V_0^\theta = \exp(\theta \Lambda_0^\theta) = \mathbb{E}[A_T^\theta]$.

It is very easy to see from (4.8) and the boundedness of l that

$$\mathbb{E}\left[\sup_{t \in [0, T]} |\Lambda_t^\theta|\right] \leq C_T,$$

where C_T is a positive constant that depends only on T and the boundedness of l and Ψ .

Second, we find the explicit form of (4.5). Using the second Itô's formula on

$$V_t^\theta = \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right),$$

we get

$$dV_t^\theta = \theta N(t) V_t^\theta dM_t.$$

Now, we can prove (4.6) by starting from the integral form of (4.5) such that

$$dV_t^\theta = \theta N(t) V_t^\theta dM_t, \quad V_T^\theta = A_T^\theta.$$

On the other hand, we have

$$V_t^\theta = \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right).$$

Using expression (4.3), we can write

$$V_t^\theta = \exp\left(\theta \int_0^t N(s) dM_s - \frac{\theta^2}{2} \int_0^t |N(s)|^2 ds + \theta \Lambda_0^\theta\right).$$

Then

$$L_t^\theta := \frac{V_t^\theta}{V_0^\theta} = \exp\left(\theta \int_0^t N(s) dM_s - \frac{\theta^2}{2} \int_0^t |N(s)|^2 ds\right), \quad 0 \leq t \leq T.$$

In view of (4.4), the above equality is a uniformly bounded $\mathcal{G}_t$ -martingale.

For more details of this proof the reader can visit the papers of Chala [2, 3]. $\square$

Proposition 4.2. *The main risk-sensitive of second adjoint equation for $(\tilde{p}_2, \tilde{q}_2)$ and (V^θ, N) becomes*

$$\begin{cases} d\tilde{p}_2(t) = -H_y^\theta(t) dt - H_z^\theta(t) dW_t^\theta - \{\tilde{q}_2(t) + \theta N_2(t) \tilde{p}_2(t)\} \overleftarrow{dB}_t^\theta, \\ dV_t^\theta = \theta N(t) V_t^\theta dM_t, \\ \tilde{p}_2(T) = -\Psi_y(y_0), \quad V_T^\theta = A_T^\theta. \end{cases} \quad (4.9)$$

The solution $(\tilde{p}, \tilde{q}, V^\theta, N)$ of system (4.9) is unique such that

$$\mathbb{E}\left[\sup_{t \in [0, T]} |\tilde{p}(t)|^2 + \sup_{t \in [0, T]} |V^\theta(t)|^2 + \int_0^T (|\tilde{q}(t)|^2 + |N(t)|^2) dt\right] < \infty, \quad (4.10)$$

where

$$H^\theta(t) := H^\theta(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)) = l(t) - (f(t) - \theta N_1(t) z_t) \tilde{p}_2(t) - g(t) \tilde{q}_2(t). \quad (4.11)$$

Proof. We wish to identify the processes $\tilde{\alpha}$ and $\tilde{\beta}$ such that

$$d\tilde{p}(t) = -\tilde{\alpha}(t) dt + \tilde{\beta}(t) dM_t, \quad (4.12)$$

where

$$\tilde{\beta}(t) := \begin{pmatrix} \tilde{\beta}_1(t) \\ \tilde{\beta}_2(t) \end{pmatrix} := \begin{pmatrix} \tilde{\beta}_{11}(t) & \tilde{\beta}_{12}(t) \\ \tilde{\beta}_{21}(t) & \tilde{\beta}_{22}(t) \end{pmatrix}.$$

By applying Itô's formula to the process $\vec{p}(t) = \theta V_t^\theta \tilde{p}(t)$ and using the expression of V^θ in (4.5), we obtain

$$\begin{aligned} d\tilde{p}(t) = & -\frac{1}{\theta V_t^\theta} \left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \right] dt \\ & - \theta N(t) \tilde{\beta}(t) dt + \frac{1}{\theta V_t^\theta} \begin{pmatrix} q_1(t) & 0 \\ p_3(t) & -q_2(t) \end{pmatrix} dM_t - \theta \tilde{p}(t) N(t)^\top dM_t. \end{aligned}$$

By combining the above equation with (4.12) and using the relation

$$\tilde{p}(t) = \frac{1}{\theta V_t^\theta} \vec{p}(t),$$

the drift coefficient $\tilde{\alpha}(t)$ can be written as

$$\tilde{\alpha}(t) = \begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} + \theta N(t) \tilde{\beta}(t),$$

and the diffusion term of the process $\tilde{p}(t)$ as

$$\tilde{\beta}(t) = \begin{pmatrix} \tilde{q}_1(t) & 0 \\ \tilde{p}_3(t) & -\tilde{q}_2(t) \end{pmatrix} - \theta \tilde{p}(t) N(t)^\top. \quad (4.13)$$

Finally, we obtain

$$d\tilde{p}(t) = - \left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right] dt + \tilde{\beta}(t) [dM_t - \theta N(t) dt].$$

It is easily verified that

$$d\tilde{p}_1(t) = \tilde{\beta}_1(t) [dM_t - \theta N(t) dt], \quad \tilde{p}_1(T) = 1.$$

In view of (4.6), we may use Girsanov's theorem [4] to claim that

$$d\tilde{p}_1(t) = \tilde{\beta}_1(t) dM_t^\theta, \quad \mathbb{P}^\theta\text{-a.s.}, \quad \tilde{p}_1(T) = 1,$$

where $dM_t^\theta = dM_t - \theta N(t) dt$. By using Notation 3.1, dM_t^θ can be written as

$$dM_t^\theta = \begin{pmatrix} dW_t \\ dB_t \end{pmatrix} - \theta \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} dt,$$

which is a $\mathbb{P}^\theta$ -Brownian motion, where

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} \Big|_{\mathcal{F}_t^{(W,B)}} := L_t^\theta = \exp \left(\theta \int_0^t N(s) dM_s - \frac{\theta^2}{2} \int_0^t |N(s)|^2 ds \right), \quad 0 \leq t \leq T.$$

But according to (4.6) and (4.7), the probability measures $\mathbb{P}^\theta$ and $\mathbb{P}$ are in fact equivalent. Hence, noting that

$$\tilde{p}_1(t) := \frac{1}{\theta V_t^\theta} p_1(t)$$

is square integrable, we get that

$$\tilde{p}_1(t) = \mathbb{E}^{\mathbb{P}^\theta} [\tilde{p}_1(T) \mid \mathcal{G}_t] = 1.$$

Thus, its quadratic variation is

$$\int_0^T |\tilde{\beta}_1(t)|^2 dt = 0.$$

This implies that, for almost every $0 \leq t \leq T$, $\tilde{\beta}_1(t) = 0$, $\mathbb{P}^\theta$ -a.s. and $\mathbb{P}$ -a.s. we have

$$d\tilde{p}(t) = - \left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right] dt + \tilde{\beta}(t) dM_t^\theta. \quad (4.14)$$

Now replacing (4.13) in (4.14), we obtain

$$d\tilde{p}(t) = - \left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right] dt + \begin{pmatrix} \tilde{\beta}_{11}(t) & \tilde{\beta}_{12}(t) \\ \tilde{\beta}_{21}(t) & \tilde{\beta}_{22}(t) \end{pmatrix} \begin{pmatrix} dW_t^\theta \\ \overleftarrow{dB}_t^\theta \end{pmatrix},$$

where

$$\begin{aligned} \tilde{\beta}_{11}(t) &= \tilde{q}_1(t) - \theta N_1(t) \tilde{p}_1(t), & \tilde{\beta}_{12}(t) &= -\theta N_2(t) \tilde{p}_1(t), & \tilde{\beta}_{21}(t) &= \tilde{p}_3(t) - \theta N_1(t) \tilde{p}_2(t), \\ \tilde{\beta}_{22}(t) &= -\tilde{q}_2(t) - \theta N_2(t) \tilde{p}_2(t), & \tilde{p}_3(t) &= -\text{Tr} \left[\begin{pmatrix} l_z(t) & -f_z(t) \\ 0 & -g_z(t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) & \tilde{q}_1(t) \\ \tilde{p}_2(t) & \tilde{q}_2(t) \end{pmatrix} \right]. \end{aligned}$$

Therefore, the second component $\tilde{p}_2(t)$ given in (4.14) has the form

$$d\tilde{p}_2(t) = -\{l_y(t)\tilde{p}_1(t) - f_y(t)\tilde{p}_2(t) - g_y(t)\tilde{q}_2(t)\} dt \\ - \{l_z(t)\tilde{p}_1(t) - (f_z(t) - \theta N_1(t))\tilde{p}_2(t) - g_z(t)\tilde{q}_2(t)\} dW_t^\theta - \{\tilde{q}_2(t) + \theta N_2(t)\tilde{p}_2(t)\} \overleftarrow{dB}_t^\theta,$$

and the main risk-sensitive second adjoint equation for $(\tilde{p}_2, \tilde{q}_2)$ and (V^θ, N) becomes

$$\begin{cases} d\tilde{p}_2(t) = -H_y^\theta(t) dt - H_z^\theta(t) dW_t^\theta - \{\tilde{q}_2(t) + \theta N_2(t)\tilde{p}_2(t)\} \overleftarrow{dB}_t^\theta, \\ dV_t^\theta = \theta N(t) V_t^\theta dM_t, \\ \tilde{p}_2(T) = -\Psi_y(y_0), \quad V_T^\theta = A_T^\theta. \end{cases}$$

The solution $(\tilde{p}, \tilde{q}, V^\theta, N)$ of the system (4.9) is unique such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{p}(t)|^2 + \sup_{t \in [0, T]} |V^\theta(t)|^2 + \int_0^T (|\tilde{q}(t)|^2 + |N(t)|^2) dt \right] < \infty,$$

where

$$H^\theta(t) := H^\theta(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)) = l(t) - (f(t) - \theta N_1(t)z_t)\tilde{p}_2(t) - g(t)\tilde{q}_2(t).$$

The proof of this proposition is completed. $\square$

Now, we can state the necessary optimality conditions.

Theorem 4.3 (Necessary optimality conditions for risk-sensitive). *Suppose that Assumption 2.2 and Assumption 2.4 hold. If (y^u, z^u, u) is an optimal solution of the risk-sensitive control problem (2.1)–(2.3). Then there exist two pairs of $\mathcal{G}_t$ -adapted processes (V^θ, N) , $(\tilde{p}, \tilde{q})$ which satisfy (4.9) and (4.10) such that*

$$H_v^\theta(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t))(u_t - v_t) \leq 0 \quad (4.15)$$

for all $u \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}$ -almost surely.

Proof. We arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes $(\tilde{p}_2, \tilde{q}_2)$ and (V^θ, N) which solve (4.9), where the Hamiltonian $\tilde{H}^\theta$ associated with (3.1), given by (3.4) satisfies

$$\tilde{H}^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t)) = \{\theta V_t^\theta\} H^\theta(t, y_t^u, z_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)), \quad (4.16)$$

and H^θ is the risk-sensitive Hamiltonian given by (4.11). Hence, since $V^\theta > 0$, the variational inequality (3.5) translates into

$$H_v^\theta(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t))(u_t - v_t) \leq 0$$

for all $u \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}$ -almost surely. This finishes the proof of Theorem 4.3. $\square$

5 Sufficient optimality conditions for risk-sensitive performance cost

In this section, we study when the necessary optimality conditions (3.5) become sufficient.

Theorem 5.1 (Sufficient optimality conditions for risk-sensitive). *Assume that the functions Ψ and*

$$(x_t, y_t, z_t, v_t) \rightarrow \tilde{H}^\theta(t, x_t, y_t, z_t, v_t, \vec{p}(t), \vec{q}(t))$$

are convex and that for any $v_t \in \mathcal{U}$, the variable $y_T = \xi$ is a one-dimensional $\mathcal{F}_T^W$ -measurable random variable such that $\mathbb{E}|\xi|^2 < \infty$. Then u is an optimal solution of the control problem (2.1)–(2.3) if it satisfies (3.5).

Proof. Let u be an arbitrary element of $\mathcal{U}$ (candidate to be optimal). For any $v \in \mathcal{U}$, we have

$$J^\theta(v) - J^\theta(u) = \mathbb{E}[\exp(\theta\{\Psi(y_0) + x_T\})] - \mathbb{E}[\exp(\theta\{\Psi(y_0^u) + x_T^u\})].$$

By applying the Taylor's expansion and the convexity of Ψ , we get

$$J^\theta(v) - J^\theta(u) \geq \mathbb{E}[\theta \exp(\theta\{\Psi(y_0^u) + x_T^u\})(x_T - x_T^u)] + \mathbb{E}[\theta \exp(\theta\{\Psi(y_0^u) + x_T^u\})\Psi_y(y_0^u)(y_0 - y_0^u)].$$

It follows from (3.3), that $p_1(T) = \theta A_T^\theta$ and $p_2(0) = -\theta A_T^\theta \Psi_y(y_0^u)$. Then we have

$$J^\theta(v) - J^\theta(u) \geq \mathbb{E}[p_1(T)(x_T - x_T^u)] - \mathbb{E}[p_2(0)(y_0 - y_0^u)]. \quad (5.1)$$

Applying Itô's formula to $p_1(t)(x_t - x_t^u)$ and $p_2(t)(y_t - y_t^u)$ leads to

$$\mathbb{E}[p_1(T)(x_T - x_T^u)] = \mathbb{E}\left[\int_0^T (l(t, y_t, z_t, v_t) - l(t, y_t^u, z_t^u, u_t))p_1(t) dt\right]$$

and

$$\begin{aligned} -\mathbb{E}[p_2(0)(y_0 - y_0^u)] &= -\mathbb{E}\left[\int_0^T \tilde{H}_y^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(y_t - y_t^u) dt\right] \\ &\quad - \mathbb{E}\left[\int_0^T \tilde{H}_z^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(z_t - z_t^u) dt\right] \\ &\quad - \mathbb{E}\left[\int_0^T (f(t, y_t, z_t, v_t) - f(t, y_t^u, z_t^u, u_t))p_2(t) dt\right] \\ &\quad - \mathbb{E}\left[\int_0^T (g(t, y_t, z_t, v_t) - g(t, y_t^u, z_t^u, u_t))q_2(t) dt\right]. \end{aligned}$$

Putting the two above formulas into (5.1), we get

$$\begin{aligned} J^\theta(v) - J^\theta(u) &\geq \mathbb{E}\left[\int_0^T \tilde{H}^\theta(t, x_t, y_t, z_t, v_t, \vec{p}(t), \vec{q}(t)) dt\right] - \mathbb{E}\left[\int_0^T \tilde{H}^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t)) dt\right] \\ &\quad - \mathbb{E}\left[\int_0^T \tilde{H}_y^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(y_t - y_t^u) dt\right] \\ &\quad - \mathbb{E}\left[\int_0^T \tilde{H}_z^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(z_t - z_t^u) dt\right]. \end{aligned} \quad (5.2)$$

Since the Hamiltonian $\tilde{H}^\theta$ is convex with respect to (y, z, v) , we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \tilde{H}^\theta(t, x_t, y_t, z_t, v_t, \vec{p}(t), \vec{q}(t)) dt\right] - \mathbb{E}\left[\int_0^T \tilde{H}^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t)) dt\right] \\ &\geq \mathbb{E}\left[\int_0^T \tilde{H}_y^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(y_t - y_t^u) dt\right] \\ &\quad + \mathbb{E}\left[\int_0^T \tilde{H}_z^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(z_t - z_t^u) dt\right] \\ &\quad + \mathbb{E}\left[\int_0^T \tilde{H}_v^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(v_t - u_t) dt\right]. \end{aligned}$$

Then, by using the above inequality in (5.2), we obtain

$$J^\theta(v) - J^\theta(u) \geq \mathbb{E} \left[\int_0^T \tilde{H}_v^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \vec{p}(t), \vec{q}(t))(v_t - u_t) dt \right] \geq 0.$$

In virtue of the necessary optimality conditions (3.5), the last inequality implies that $J^\theta(v) - J^\theta(u) \geq 0$. Thus the theorem is proved. $\square$

Remark 5.2. In virtue of (4.16) there is a relationship between the Hamiltonian with respect to risk-neutral and the Hamiltonian with respect to risk-sensitive. In fact, we have

$$J^\theta(v) - J^\theta(u) \geq \mathbb{E} \left[\int_0^T \theta V_t^\theta H_v^\theta(t, y_t^u, z_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t))(v_t - u_t) dt \right] \geq 0.$$

We know that $\theta V_t^\theta > 0$. Then the above inequality can be rewritten as

$$J^\theta(v) - J^\theta(u) \geq \mathbb{E} \left[\int_0^T H_v^\theta(t, y_t^u, z_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t))(v_t - u_t) dt \right] \geq 0.$$

In virtue of the necessary optimality conditions (4.15), the last inequality implies that $J^\theta(v) - J^\theta(u) \geq 0$.

6 Applications: A linear quadratic risk-sensitive control problem

We illustrate the paper by giving two different examples for a linear quadratic system, and a numerical application as second example.

6.1 Example 1

We provide a concrete example of a risk-sensitive backward doubly stochastic LQ problem, give the explicit optimal control and validate our major theoretical results in Theorem 5.1 (risk-sensitive sufficient optimality conditions). First, let the control domain be $\mathcal{A} = [-1, 1]$. Consider the following linear quadratic risk-sensitive control problem

$$\begin{cases} \inf_{v \in \mathcal{U}} \mathbb{E} \left[\exp \theta \left\{ \frac{1}{2} \int_0^T v_t^2 dt + \frac{1}{2} (y_0)^2 \right\} \right] & \text{subject to} \\ dy_t = -(Ay_t + Bz_t + Cv_t + D) dt - (A'y_t + B'z_t + C'v_t + D') \overleftarrow{dB}_t + z_t dW_t, \\ y_T = \xi, \end{cases} \quad (6.1)$$

where A, B, C, D, A', B', C' and D' are positive real constants.

Let (y_t, z_t) be a solution of (6.1) associated with v_t . Then there exist two unique $\mathcal{G}_t$ -adapted pairs of processes $(p_1, p_2), (q_1, q_2)$ of the following forward-backward doubly SDE system (called adjoint equation), according to equation (3.3):

$$\begin{cases} dp_1(t) = q_1(t) dW_t, \\ dp_2(t) = [Ap_2(t) + A'q_2(t)] dt + [Bp_2(t) + B'q_2(t)] dW_t - q_2(t) \overleftarrow{dB}_t, \\ p_1(T) = \theta A_T^\theta, \quad p_2(0) = -\theta y_0 A_T^\theta, \end{cases} \quad (6.2)$$

where

$$A_T^\theta := \exp \theta \left\{ \frac{1}{2} \int_0^T v_t^2 dt + (y_0)^2 \right\}.$$

We give the Hamiltonian $\tilde{H}^\theta$ defined by

$$\begin{aligned}\tilde{H}^\theta(t) &:= \tilde{H}^\theta(t, x_t, y_t, z_t, v_t, \vec{p}(t), \vec{q}(t)) \\ &= \frac{1}{2}v_t^2 p_1(t) - (Ay_t + Bz_t + Cv_t + D)p_2(t) - (A'y_t + B'z_t + C'v_t + D')q_2(t).\end{aligned}$$

We have $\tilde{H}_v^\theta(t) = [v_t p_1(t) - Cp_2(t) - C'q_2(t)]$. Maximizing the Hamiltonian yields

$$u_t = \frac{1}{p_1^u(t)}(Cp_2^u(t) + C'q_2^u(t)). \quad (6.3)$$

We only need to prove that u_t is an optimal control of (6.1).

Theorem 6.1 (Risk-sensitive sufficient optimality conditions for a linear quadratic control problem). *Assume that $\theta > 0$ and suppose that u_t satisfies (6.3), where $(\vec{p}, \vec{q})$ satisfy (6.2). Then u_t is the unique optimal control of the above BDSDE of the linear quadratic problem (6.1).*

Proof. From the definition of the functional cost J^θ we have

$$J^\theta(v_t) - J^\theta(u_t) = \mathbb{E} \left[\exp \theta \left\{ \frac{1}{2} \int_0^T v_t^2 dt + \frac{1}{2} (y_0)^2 \right\} \right] - \mathbb{E} \left[\exp \theta \left\{ \frac{1}{2} \int_0^T u_t^2 dt + \frac{1}{2} (y_0)^2 \right\} \right].$$

We put $x_T = \frac{1}{2} \int_0^T v_t^2 dt$ and, by applying the Taylor expansion, we have

$$J^\theta(v_t) - J^\theta(u_t) = \mathbb{E} \left[\theta \exp \theta \left\{ x_T^u + \frac{1}{2} (y_0^u)^2 \right\} (x_T - x_T^u) \right] + \mathbb{E} \left[\theta y_0^u \exp \theta \left\{ x_T^u + \frac{1}{2} (y_0^u)^2 \right\} (y_0 - y_0^u) \right].$$

It follows from (3.3) that $p_1^u(T) = \theta A_T^\theta$ and $p_2^u(0) = -\theta y_0^u A_T^\theta$. Then we have

$$J^\theta(v_t) - J^\theta(u_t) = \mathbb{E} \left[p_1^u(T)(x_T - x_T^u) \right] - \mathbb{E} \left[p_2^u(0)(y_0 - y_0^u) \right]. \quad (6.4)$$

By applying Itô's formula to $p_1^u(t)(x_t - x_t^u)$ and $p_2^u(t)(y_t - y_t^u)$ and combining them with (6.4), we get

$$\begin{aligned}J^\theta(v_t) - J^\theta(u_t) &= \mathbb{E} \left[\int_0^T v_t(v_t - u_t)p_1^u(t) dt \right] + \mathbb{E} \left[\int_0^T u_t(v_t - u_t)p_1^u(t) dt \right] \\ &\quad - \mathbb{E} \left[\int_0^T C(v_t - u_t)p_2^u(t) dt \right] - \mathbb{E} \left[\int_0^T C'(v_t - u_t)q_2^u(t) dt \right].\end{aligned}$$

Because of $\theta > 0$, we have $(v_t - u_t) > 0$. Thus we get the following result:

$$J^\theta(v_t) - J^\theta(u_t) \geq \mathbb{E} \left[\int_0^T u_t(v_t - u_t)p_1^u(t) dt \right] - \mathbb{E} \left[\int_0^T C(v_t - u_t)p_2^u(t) dt \right] - \mathbb{E} \left[\int_0^T C'(v_t - u_t)q_2^u(t) dt \right].$$

Then

$$J^\theta(v_t) - J^\theta(u_t) \geq \mathbb{E} \left[\int_0^T (u_t p_1^u(t) - Cp_2^u(t) - C'q_2^u(t))(v_t - u_t) dt \right].$$

By replacing u_t with its value in (6.3), we obtain

$$J^\theta(v_t) \geq J^\theta(u_t),$$

i.e. u_t is optimal. This proof is finished $\square$

6.2 Example 2

In this section, we apply the risk-sensitive maximum principles obtained in the previous section (Theorem 4.3) to deal with the linear-quadratic risk-sensitive stochastic optimal control problem (2.1)–(2.3)

mentioned in Section 2. Our state dynamics is

$$\begin{cases} dy_t = -(y_t + v_t) dt - \sigma v_t \overleftarrow{dB}_t + z_t dW_t, \\ y_T = \xi, \end{cases} \quad (6.5)$$

and our functional cost is the following expected exponential-of-integral form:

$$J^\theta(v) = \mathbb{E} \left[\exp \theta \left\{ \int_0^T l(t, y_t, z_t, v_t) dt \right\} \right], \quad (6.6)$$

where

$$\theta > 0, \quad \theta \neq 1, \quad l(t, y_t, z_t, v_t) = \frac{1}{2}(v_t^2 + y_t^2).$$

We want to minimize (6.6) subject to (6.5) by choosing v over $\mathcal{U}$. Hence, we may apply Theorem 4.3 to solve our linear-quadratic risk-sensitive stochastic optimal control problem (6.5), (6.6). The Hamiltonian function (4.11) is defined by

$$\begin{aligned} H^\theta(t) &:= H^\theta(t, y_t, z_t, v_t, \tilde{p}_2(t), \tilde{q}_2(t), V^\theta(t), N(t)) \\ &= \frac{1}{2}(v_t^2 + y_t^2) - (y_t + v_t - \theta N_1(t)z_t)\tilde{p}_2(t) - \sigma v_t \tilde{q}_2(t). \end{aligned} \quad (6.7)$$

Let (y_t^u, z_t^u, u_t) be an optimal solution. The adjoint equation (4.9) can be written by

$$\begin{cases} d\tilde{p}_2(t) = [-y_t + \tilde{p}_2(t)] dt - \theta N_1(t)\tilde{p}_2(t) dW_t^\theta - [\tilde{q}_2(t) + \theta N_2(t)\tilde{p}_2(t)] \overleftarrow{dB}_t^\theta, \\ \tilde{p}_2(T) = 0. \end{cases} \quad (6.8)$$

Minimizing the Hamiltonian (6.7), we obtain

$$u_t = \tilde{p}_2^u(t) + \sigma \tilde{q}_2^u(t). \quad (6.9)$$

By substituting (6.9) into the BDSDE (6.5), we have

$$\begin{cases} dy_t^u = -[y_t^u + \tilde{p}_2^u(t) + \sigma \tilde{q}_2^u(t)] dt - [\sigma(\tilde{p}_2^u(t) + \sigma \tilde{q}_2^u(t))] \overleftarrow{dB}_t + z_t^u dW_t, \\ y_T^u = \xi. \end{cases}$$

Similarly, by substituting equation (6.9) into the BDSDE (6.8) and replacing $dW_t^\theta = dW_t - \theta N_1(t) dt$ and $\overleftarrow{dB}_t^\theta = \overleftarrow{dB}_t - \theta N_2(t) dt$, we get

$$\begin{cases} d\tilde{p}_2^u(t) = [-y_t^u + (1 + \theta^2(N_1^2(t) + N_2^2(t)))\tilde{p}_2^u(t) + \theta N_2(t)\tilde{q}_2^u(t)] dt, \\ \quad - [\tilde{q}_2^u(t) + \theta N_2(t)\tilde{p}_2^u(t)] \overleftarrow{dB}_t - \theta N_1(t)\tilde{p}_2^u(t) dW_t, \\ \tilde{p}_2^u(T) = 0. \end{cases} \quad (6.10)$$

Peng and Shi [13] introduced a type of time-symmetric forward-backward stochastic differential equations (SFBSDE in short), i.e., so-called fully coupled forward-backward doubly stochastic differential equations (FBDSDE in short). Therefore, an optimal solution $(\tilde{p}_2^u, y^u, u)$ can be obtained by solving the following type of SFBSDE:

$$\begin{cases} dy_t^u = -[y_t^u + \tilde{p}_2^u(t) + \sigma \tilde{q}_2^u(t)] dt - [\sigma(\tilde{p}_2^u(t) + \sigma \tilde{q}_2^u(t))] \overleftarrow{dB}_t + z_t^u dW_t, \\ d\tilde{p}_2^u(t) = [-y_t^u + (1 + \theta^2(N_1^2(t) + N_2^2(t)))\tilde{p}_2^u(t) + \theta N_2(t)\tilde{q}_2^u(t)] dt \\ \quad - [\tilde{q}_2^u(t) + \theta N_2(t)\tilde{p}_2^u(t)] \overleftarrow{dB}_t - \theta N_1(t)\tilde{p}_2^u(t) dW_t, \\ y_T^u = \xi, \quad \tilde{p}_2^u(T) = 0. \end{cases} \quad (6.11)$$

Unfortunately, in such a system it is difficult to find the explicit solution. To solve this type of SFBSDE (6.11), we use a technique similar to the one used by Yong and Zhou [17]. We conjecture that the solution to (6.11)

is related by

$$\tilde{p}_2^u(t) = \varphi(t)y_t^u + \chi(t) \quad (6.12)$$

for some deterministic differentiable functions $\varphi(t)$ and $\chi(t)$. Applying Itô's formula to (6.12) gives

$$\begin{cases} d\tilde{p}_2^u(t) = [\dot{\varphi}(t)y_t^u - \varphi(t)y_t^u - \varphi(t)(\tilde{p}_2^u(t) + \sigma\tilde{q}_2^u(t)) + \dot{\chi}(t)] dt \\ \quad - \varphi(t)\sigma(\tilde{p}_2^u(t) + \sigma\tilde{q}_2^u(t))\overleftarrow{dB}_t + \varphi(t)z_t^u dW_t, \\ \tilde{p}_2^u(T) = 0. \end{cases} \quad (6.13)$$

Putting (6.12) into (6.13), we get

$$\begin{cases} d\tilde{p}_2^u(t) = [\dot{\varphi}(t) - \varphi^2(t) - \varphi(t)y_t^u - \varphi(t)\chi(t) - \varphi(t)\sigma\tilde{q}_2^u(t) + \dot{\chi}(t)] dt \\ \quad - [\varphi^2(t)\sigma y_t^u + \varphi(t)\chi(t)\sigma + \varphi(t)\sigma^2\tilde{q}_2^u(t)]\overleftarrow{dB}_t + \varphi(t)z_t^u dW_t, \\ \tilde{p}_2^u(T) = 0. \end{cases} \quad (6.14)$$

On the other hand, after substituting (6.12) into (6.10), we arrive at

$$\begin{cases} d\tilde{p}_2^u(t) = [((1 + \theta^2(N_1^2(t) + N_2^2(t)))\varphi(t) - 1)y_t^u \\ \quad + (1 + \theta^2(N_1^2(t) + N_2^2(t)))\chi(t) + \theta N_2(t)\tilde{q}_2^u(t)] dt \\ \quad - [\tilde{q}_2^u(t) + \theta N_2(t)\varphi(t)y_t^u + \theta N_2(t)\chi(t)]\overleftarrow{dB}_t - [\theta N_1(t)\varphi(t)y_t^u + \theta N_1(t)\chi(t)] dW_t, \\ \tilde{p}_2^u(T) = 0. \end{cases} \quad (6.15)$$

Equating the coefficients of (6.14) and (6.15), we have

$$(\tilde{p}_2^u(t), \tilde{q}_2^u(t)) = \left(\varphi(t)y_t^u + \chi(t), \frac{\varphi(t)(\sigma\varphi(t) - \theta N_2(t))y_t^u + (\varphi(t)\sigma - \theta N_2(t))\chi(t)}{1 - \varphi(t)\sigma^2} \right), \quad (6.16)$$

where $\varphi(t)$ is the solution to the Riccati-type equation

$$\begin{cases} \dot{\varphi}(t) - \varphi^2(t) - 2\varphi(t)\left(1 + \frac{1}{2}\theta^2(N_1^2(t) + N_2^2(t))\right) + 1 = 0, \\ \varphi(T) = 0, \end{cases} \quad (6.17)$$

and $\chi(t)$ is a solution to the following ordinary differential equation:

$$\begin{cases} \dot{\chi}(t) - (\varphi(t) + 1 + \theta^2(N_1^2(t) + N_2^2(t)))\chi(t) - (\theta N_2(t) + \varphi(t)\sigma)\tilde{q}_2(t) = 0, \\ \chi(T) = 0. \end{cases} \quad (6.18)$$

By using the same identification, we get

$$\chi(t) = -\frac{1}{\theta N_1(t)}\varphi(t)z_t^u - \varphi(t)y_t^u. \quad (6.19)$$

Finally, by (6.9) and (6.16), we can get the optimal control in the following state in feedback form:

$$u_t = \frac{(1 - \sigma\theta N_2(t))}{1 - \varphi(t)\sigma^2}\varphi(t)y_t^u + \frac{(1 - \sigma\theta N_2(t))}{1 - \varphi(t)\sigma^2}\chi(t). \quad (6.20)$$

Putting (6.19) in (6.20), we get

$$u_t = -\frac{(1 - \sigma\theta N_2(t))}{(1 - \varphi(t)\sigma^2)\theta N_1(t)}\varphi(t)z_t^u, \quad (6.21)$$

where $\varphi(t)$ is determined by (6.17).

Theorem 6.2. *We assume that the pair $(\varphi(t), \chi(t))$ has the solution of system (6.17) and (6.18). Then the optimal control of our linear-quadratic risk-sensitive stochastic optimal control problem (6.5), (6.6) has the state feedback form (6.21).*

6.2.1 Solution of the deterministic functions via Riccati equation

To the best of our knowledge, it is very hard to find the explicit solution to a Riccati equation in general. But in our case, we can find the explicit solution of

$$\begin{cases} \dot{\varphi}(t) - \varphi^2(t) + 2\varphi(t)K + 1 = 0, \\ \varphi(T) = 0, \end{cases} \quad (6.22)$$

where we set

$$K = -\left(1 + \frac{1}{2}\theta^2(N_1^2(t) + N_2^2(t))\right). \quad (6.23)$$

It is very easy to see that for the discriminate $\Delta = 4(K^2 + 1) > 0$ holds. We obtain

$$dt = \frac{d\varphi(t)}{\varphi^2(t) - 2\varphi(t)K - 1}.$$

Then

$$T - t = \frac{1}{2\sqrt{K^2 + 1}} \int_t^T \left\{ \frac{1}{(\varphi(s) - (K + \sqrt{K^2 + 1}))} - \frac{1}{(\varphi(s) - (K - \sqrt{K^2 + 1}))} \right\} d\varphi(s).$$

This implies

$$\begin{aligned} (T - t)2\sqrt{K^2 + 1} &= \ln|K + \sqrt{K^2 + 1}| - \ln|K - \sqrt{K^2 + 1}| \\ &\quad + \ln|\varphi(t) - (K - \sqrt{K^2 + 1})| - \ln|\varphi(t) - (K + \sqrt{K^2 + 1})|. \end{aligned}$$

Hence,

$$\frac{\varphi(t) - (K + \sqrt{K^2 + 1})}{\varphi(t) - (K - \sqrt{K^2 + 1})} = \pm \frac{K + \sqrt{K^2 + 1}}{K - \sqrt{K^2 + 1}} \exp(-(T - t)2\sqrt{K^2 + 1}).$$

This concludes

$$\varphi(t) = \frac{(K + \sqrt{K^2 + 1})(1 + \exp(-2(T - t)\sqrt{K^2 + 1}))}{(1 + (\frac{K + \sqrt{K^2 + 1}}{K - \sqrt{K^2 + 1}})\exp(-2(T - t)\sqrt{K^2 + 1}))}.$$

Substituting the equation $4(K^2 + 1) = \Delta$, we then have

$$\varphi(t) = \frac{(K + \frac{\sqrt{\Delta}}{2})(1 + \exp(-(T - t)\sqrt{\Delta}))}{(1 + (\frac{K + \frac{\sqrt{\Delta}}{2}}{K - \frac{\sqrt{\Delta}}{2}})\exp(-(T - t)\sqrt{\Delta}))}.$$

In fact, the Riccati equation has another solution:

$$\varphi(t) = \frac{(K + \frac{\sqrt{\Delta}}{2})(1 - \exp(-(T - t)\sqrt{\Delta}))}{(1 - (\frac{K + \frac{\sqrt{\Delta}}{2}}{K - \frac{\sqrt{\Delta}}{2}})\exp(-(T - t)\sqrt{\Delta}))}.$$

We must reject this solution because of the choose problem. If we denote

$$L = \frac{K + \frac{\sqrt{\Delta}}{2}}{K - \frac{\sqrt{\Delta}}{2}}, \quad \delta_1 = K + \frac{\sqrt{\Delta}}{2}, \quad \delta_2 = K - \frac{\sqrt{\Delta}}{2},$$

then we get

$$\varphi(t) = \frac{\delta_1 + \delta_2 L \exp(-(T - t)\sqrt{\Delta})}{1 + L \exp(-(T - t)\sqrt{\Delta})}. \quad (6.24)$$

We put

$$\alpha(t) = -(\varphi(t) + 1 + \theta^2(N_1^2(t) + N_2^2(t))), \quad \beta(t) = -(\theta N_2(t) + \varphi(t)\sigma)\tilde{q}_2(t). \quad (6.25)$$

We rewrite equation (6.18) as follows:

$$\begin{cases} \dot{\chi}(t) + \alpha(t)\chi(t) + \beta(t) = 0, \\ \chi(T) = 0, \end{cases} \quad (6.26)$$

The explicit solution to equation (6.26) is

$$\chi(t) = \left[\exp \left(\int_t^T \alpha(s) ds \right) \right] \left[\int_t^T -\beta(s) \exp \left(\int_t^s \alpha(r) dr \right) ds \right], \quad (6.27)$$

where $\alpha(t), \beta(t)$ are determined by (6.25).

Corollary 6.3. *The explicit solution of the Riccati equation (6.22) is given by (6.24) and equation (6.26) has an explicit solution given by (6.27), where the constant coefficients K and $\alpha(t), \beta(t)$ are given by (6.23) and (6.25) respectively.*

Corollary 6.4. *We assume that the pair $(\varphi(t), \chi(t))$ has the unique solution given by (6.24), (6.27). Then the optimal control of the problem (6.5), (6.6) has the state feedback from (6.21), where the constant coefficient K is given by (6.23), and $\alpha(t), \beta(t)$ are determined by (6.25).*

7 Conclusion and further works

This paper contains two main results. The first result is Theorem 4.3, which establishes the necessary optimality condition for the system of BDSDE with risk-sensitive performance, using a scheme very similar to the one by Chala [2]. The second main result, Theorem 5.1, suggests sufficient optimality conditions of BDSDE given in form of risk-sensitive performance. We acknowledge that this result is a good extension of the result established by Chala in [3]. The proof is based on the convexity conditions of the Hamiltonian function and the initial and terminal terms of the performance function. Note that the risk-sensitive control problem studied by Lim and Zhou in [10] is different to ours. By adding the jump diffusion term to our system (this result will be discussed in our next paper), in this case we can compare this result with Shi and Wu in [15]. On the other hand, in the case where the system is governed by mean-field fully coupled FBSDE (this result will be discussed in detail in our next paper), this result will hopefully generalize the result of Djehiche et al. in [5]. The maximum principle of risk-neutral obtained by [1, 8] and Yong [16] are similar to our Theorem 3.2, but the adjoint equations and maximum conditions heavily depend on the risk-sensitive parameter.

Acknowledgment: The authors wish to thank the referees and editors for their valuable comments and suggestions which led to improvements in the document.

Funding: This work is partially supported by PRFU project N: C00L03UN070120180002.

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