Research Article

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Comparison theorems for anticipated backward doubly stochastic differential equations with non-Lipschitz coefficients

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Abstract: In this work, we prove some comparison theorems of anticipated backward doubly stochastic differential equations with non-Lipschitz coefficients.

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1 Introduction

Backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng [4]. They proved an existence and uniqueness result under the Lipschitz condition. Since then many efforts have been made in relaxing the Lipschitz assumption of the generator of the BSDEs (see, among others, Mao [3] and Wang and Huang [7]). A few years later, Pardoux and Peng considered in [5] a new type of BSDEs, that is, a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals. These equations are extensively used in the study of stochastic partial differential equations (SPDEs). Their link with SPDEs in the case of Lipschitzian drift was established in [5]. The key point of solvency of such equations is the martingale representation theorem. In this spirit, Bally and Matoussi [2] gave the probabilistic representation of the solution in a Sobolev space of semilinear SPDEs in terms of BDSDEs.

On the other hand, Peng and Yang [6] introduced the following type of anticipated backward stochastic differential equations (ABSDEs in short):

$$\begin{cases} -dY_t = f(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)}) dt - Z_t dW_t, & 0 \leq t \leq T, \\ Y_t = \xi_t, \quad Z_t = \eta_t, & T \leq t \leq T + K, \end{cases}$$

where δ and ζ are given nonnegative deterministic functions. In these equations, the generator includes not only the values of solutions of the present, but also the future. In [6], Peng and Yang obtained the existence and uniqueness of the solution of ABSDEs under the Lipschitz assumption, gave the comparison theorem for one-dimensional ABSDEs and finally they solved a stochastic control problem by showing the duality between linear stochastic differential delay equations and ABSDEs.

Later, Zhang [8] improved Peng and Yang's comparison theorem result and omitted these assumptions. In [8], Zhang studied some comparison theorems for one-dimensional ABSDEs with non-Lipschitz coeffi-

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cients in the case when the generator function is allowed to contain the anticipated term of z , neither generator function needs to be necessarily monotone in the anticipated term of y . Recently, Aidara [1] extended Peng and Yang's results to anticipated backward doubly stochastic differential equations (ABDSDEs in short) with non-Lipschitz coefficients.

In this work, inspired by the method developed by Zhang [8], we show that the results obtained in [8] can be extended to ABDSDEs with non-Lipschitz coefficients.

The paper is organized as follows: In Section 2, we introduce some preliminaries. In Section 3, we prove some comparison theorems for one-dimensional ABDSDEs under non-Lipschitz coefficients.

2 Preliminaries

Let Ω be a non-empty set, $\mathcal{F}$ a σ -algebra of sets of Ω and $\mathbb{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ defines a probability space, which is assumed to be complete. For a fixed real number $0 < T \leq \infty$, we assume two mutually independent processes are given:

- An ℓ -dimensional Brownian motion $(B_t)_{0 \leq t \leq T}$.
- A d -dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$.

We consider the family $(\mathcal{F}_t)_{0 \leq t \leq T}$ given by

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \quad 0 \leq t \leq T, \quad \mathcal{G}_s = \mathcal{F}_{0,s}^W \vee \mathcal{F}_{s,T+K}^B, \quad 0 \leq s \leq T+K,$$

where for any process $\{\varphi_t\}_{t \geq 0}$ one has $\mathcal{F}_{s,t}^\varphi = \sigma\{\varphi_r - \varphi_s, s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\varphi = \mathcal{F}_{0,t}^\varphi$. Here $\mathcal{N}$ denotes the class of $\mathbb{P}$ -null sets of $\mathcal{F}$. Note that $(\mathcal{F}_t)_{0 \leq t \leq T}$ does not constitute a classical filtration.

For $k \in \mathbb{N}^*$, we consider the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$):

- $L^2(\mathcal{G}_T, \mathbb{R}^k)$, the space of $\mathcal{G}_T$ -measurable random variables such that $\mathbf{E}[|\xi_T|^2] < +\infty$.
- $\mathcal{S}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^k)$, the space of $\mathcal{G}_t$ -adapted càdlàg processes

$$\Psi : [0, T] \times \Omega \rightarrow \mathbb{R}^k, \quad \|\Psi\|_{\mathcal{S}^2(\mathbb{R}^k)}^2 = \mathbf{E}\left(\sup_{0 \leq t \leq T} |\Psi_t|^2\right) < +\infty.$$

- $\mathcal{M}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^{k \times d})$, the space of $\mathcal{G}_t$ -progressively measurable processes

$$\Psi : [0, T] \times \Omega \rightarrow \mathbb{R}^{k \times d}, \quad \|\Psi\|_{\mathcal{M}^2(\mathbb{R}^{k \times d})}^2 = \mathbf{E} \int_0^T |\Psi_t|^2 dt < +\infty.$$

- $\mathcal{C}_{\mathcal{G}}^2(0, T) = \mathcal{M}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^k) \times \mathcal{M}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^{k \times d})$ endowed with the norm

$$\|(Y, Z)\|_{\mathcal{C}_{\mathcal{G}}^2(0, T)}^2 = \|Y\|_{\mathcal{M}^2(\mathbb{R}^k)}^2 + \|Z\|_{\mathcal{M}^2(\mathbb{R}^{k \times d})}^2.$$

- $\mathcal{B}_{\mathcal{G}}^2(0, T) = \mathcal{S}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^k) \times \mathcal{M}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^{k \times d})$ endowed with the norm

$$\|(Y, Z)\|_{\mathcal{B}_{\mathcal{G}}^2(0, T)}^2 = \|Y\|_{\mathcal{S}^2(\mathbb{R}^k)}^2 + \|Z\|_{\mathcal{M}^2(\mathbb{R}^{k \times d})}^2.$$

- Let $\mathbf{S}$ be the set of all nondecreasing, continuous and concave functions $\rho(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\rho(0) = 0$, $\rho(s) > 0$ for $s > 0$ and $\int_{0+} \frac{du}{\rho(u)} = +\infty$.

Remark 2.1. For any $\rho \in \mathbf{S}$, we can find a pair of positive constants a and b such that $\rho(v) \leq a + bv$ for all $v \geq 0$.

Define $\mathcal{A} = \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$. For each $t \in [0, T]$, we assume two given functions:

$$\begin{aligned} f &: \mathcal{A} \times \mathcal{C}_{\mathcal{G}}^2(t, T+K) \rightarrow L^2(\mathcal{G}_t, \mathbb{R}^k), \\ g &: \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow L^2(\mathcal{G}_t, \mathbb{R}^{k \times \ell}). \end{aligned}$$

For notational simplicity we set $f(r, 0) = h(r, 0, 0, 0, 0)$ and for all $x, y \in \mathbb{R}^k$ we denote by $|x|$ the Euclidean norm of x and by $\langle x, y \rangle$ the Euclidean inner product.

Consider the ABDSDE

$$\begin{cases} Y_t = \xi_T + \int_t^T f(r, Y_r, Z_r, Y_{r+\delta(r)}, Z_{r+\zeta(r)}) dr + \int_t^T g(r, Y_r, Z_r) dB_r - \int_t^T Z_r dW_r, & 0 \leq t \leq T, \\ Y_t = \xi_t, \quad Z_t = \eta_t, & T \leq t \leq T + K, \end{cases} \quad (2.1)$$

where $\delta, \zeta : [0, T] \rightarrow \mathbb{R}_+$ are continuous functions satisfying the following conditions:

(A1) $t + \delta(t) \leq T + K$, $t + \zeta(t) \leq T + K$.

(A2) There exists $M \geq 0$ such that for $0 \leq t \leq T$ and a nonnegative integrable function h ,

$$\int_t^T h(r + \phi(r)) dr \leq M \int_t^{T+K} h(r) dr, \quad \phi \in \{\delta, \zeta\}.$$

Definition 2.2. A pair of processes (Y, Z) is called a solution to ABDSDE (2.1) if $(Y, Z) \in \mathcal{B}_{\mathcal{G}}^2(0, T + K)$ and if it satisfies (2.1).

In the following, we assume that f and g satisfy assumptions (H):

(H1) There exists a constant $c > 0$ such that

$$\begin{aligned} & |f(t, y, z, \theta(r), \varphi(r)) - f(t, y', z', \theta'(r), \varphi'(r))|^2 \\ & \leq c(\rho(|y - y'|^2) + |z - z'|^2) + \mathbf{E}^{\mathcal{F}_t}[\rho(|\theta(r) - \theta'(r)|^2) + |\varphi(r) - \varphi'(r)|^2] \end{aligned}$$

for all $(r, r') \in [t, T + K]$ and $(t, y, z, \theta(r), \varphi(r)), (t, y', z', \theta'(r), \varphi'(r)) \in \mathcal{A} \times \mathcal{C}_{\mathcal{G}}^2(t, T + K)$.

(H2) There exists a constant $0 < \alpha_1 < 1$ such that for any $(t, y, z), (t, y', z') \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$|g(t, y, z) - g(t, y', z')|^2 \leq \rho(|y - y'|^2) + \alpha_1 |z - z'|^2.$$

(H3) For any $(t, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$\mathbf{E} \left[\int_0^T |f(s, 0)|^2 ds \right] + \mathbf{E} \left[\int_0^T |g(s, y, z)|^2 ds \right] < +\infty.$$

Let us recall the following result given in [1, Theorem 4.2].

Theorem 2.3. Assume that assumptions (A1), (A2) and (H) are true and let $\xi_T \in L^2(\mathcal{G}_T, \mathbb{R}^k)$. Then for any

$$(\xi, \eta) \in \mathcal{S}_{[T, T+K]}^2(\mathcal{G}, \mathbb{R}^k) \times \mathcal{M}_{[T, T+K]}^2(\mathcal{G}, \mathbb{R}^{k \times d})$$

the ABDSDE (2.1) has a unique solution $(Y, Z) \in \mathcal{B}_{\mathcal{G}}^2(0, T + K)$.

We are now in a position to move on to study our main subject.

3 Comparison theorems for anticipated BDSDEs

In this section, our objective is to establish some comparison theorems for the following one-dimensional anticipated BDSDEs under non-Lipschitz coefficients:

$$\begin{cases} Y_t^{(i)} = \xi_T^{(i)} + \int_t^T f^i(r, Y_r^{(i)}, Z_r^{(i)}, Y_{r+\delta(r)}^{(i)}, Z_{r+\zeta(r)}^{(i)}) dr + \int_t^T g(r, Y_r^{(i)}, Z_r^{(i)}) dB_r - \int_t^T Z_r^{(i)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(i)} = \xi_t^{(i)}, \quad Z_t^{(i)} = \eta_t^{(i)}, & T \leq t \leq T + K, \end{cases} \quad (3.1)$$

where $i = 1, 2$ and (A1), (A2) and (H) hold. Then (3.1) has a unique solution by Theorem 2.3.

Let us first give a comparison theorem of BDSDEs under non-Lipschitz assumptions, which will play a key role in what follows.

Let $(Y^{(1)}, Z^{(1)})$, $(Y^{(2)}, Z^{(2)})$ be respectively the solutions of the following two one-dimensional BDSDEs:

$$Y_t^{(i)} = \xi_T^{(i)} + \int_t^T \tilde{f}^i(r, Y_r^{(i)}, Z_r^{(i)}) dr + \int_t^T g(r, Y_r^{(i)}, Z_r^{(i)}) dB_r - \int_t^T Z_r^{(i)} dW_r, \quad 0 \leq t \leq T,$$

where $\tilde{f}^i(t, y, z) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that the following conditions $(\tilde{H})$ are satisfied:

$(\tilde{H}1)$ There exists a constant $c > 0$ and $\rho \in \mathbf{S}$ such that for each $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$,

$$|\tilde{f}^i(t, y, z) - \tilde{f}^i(t, y', z')|^2 \leq c(\rho(|y - y'|^2) + |z - z'|^2).$$

$(\tilde{H}3)$ $(H3)$ holds.

Let us recall the following result, which will be useful in the sequel (the proof is omitted since it is an adaptation of [9, Theorem 5]).

Proposition 3.1. Assume that $\tilde{f}^i$ and $\tilde{g}$ satisfy $(\tilde{H})$ and assume $\xi_T^{(1)}, \xi_T^{(2)} \in L^2(\mathcal{G}_T, \mathbb{R})$. If $\xi_T^1 \geq \xi_T^2$ a.s. and for all $t \in [0, T]$ one has $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ such that $\tilde{f}^1(t, y, z) \geq \tilde{f}^2(t, y, z)$, then $Y_t^{(1)} \geq Y_t^{(2)}$ a.s. for all $t \in [0, T]$.

Now let us turn to the study of the comparison theorem for anticipated BDSDEs. We have our first comparison theorem as follows.

Theorem 3.2. Under the assumption $(\tilde{H})$, assume the following:

- (i) $\xi_t^{(1)} \geq \xi_t^{(2)}$ a.s., $T \leq t \leq T + K$.
- (ii) One has

$$f^1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq f^2(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(2)}(t)}^{(2)}) \text{ a.s., } 0 \leq t \leq T.$$

(iii) For all $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $\varphi(r) \in \mathcal{M}_{[0, T]}^2(\mathcal{G}, \mathbb{R}^d)$, $r \in [t, T + K]$, the function $f^2(t, y, z, \cdot, \varphi(r))$ is nondecreasing.

Then $Y_t^{(1)} \geq Y_t^{(2)}$ a.s., $0 \leq t \leq T + K$.

Proof. Set

$$\tilde{f}^1(r, y, z) = f^1(r, y, z, Y_{r+\delta(r)}^{(1)}, Z_{r+\zeta^{(1)}(r)}^{(1)}) \quad \text{for all } t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^d.$$

Then $(Y^{(1)}, Z^{(1)})$ is also the unique solution of

$$Y_t^{(1)} = \xi_T^{(1)} + \int_t^T \tilde{f}^1(r, Y_r^{(1)}, Z_r^{(1)}) dr + \int_t^T g(r, Y_r^{(1)}, Z_r^{(1)}) dB_r - \int_t^T Z_r^{(1)} dW_r, \quad 0 \leq t \leq T.$$

Let us introduce the following equation:

$$\begin{cases} Y_t^{(3)} = \xi_T^{(2)} + \int_t^T f^2(r, Y_r^{(3)}, Z_r^{(3)}, Y_{r+\delta(r)}^{(1)}, Z_{r+\zeta^{(2)}(r)}^{(2)}) dr + \int_t^T g(r, Y_r^{(3)}, Z_r^{(3)}) dB_r - \int_t^T Z_r^{(3)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(3)} = \xi_t^{(2)}, \quad Z_t^{(3)} = \eta_t^{(2)}, & T \leq t \leq T + K. \end{cases}$$

If we set

$$\tilde{f}^3(r, y, z) = f^2(r, y, z, Y_{r+\delta(r)}^{(1)}, Z_{r+\zeta^{(2)}(r)}^{(2)}) \quad \text{for all } t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^d,$$

then this equation is equivalent to

$$\begin{cases} Y_t^{(3)} = \xi_T^{(2)} + \int_t^T \tilde{f}^3(r, Y_r^{(3)}, Z_r^{(3)}) dr + \int_t^T g(r, Y_r^{(3)}, Z_r^{(3)}) dB_r - \int_t^T Z_r^{(3)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(3)} = \xi_t^{(2)}, \quad Z_t^{(3)} = \eta_t^{(2)}, & T \leq t \leq T + K, \end{cases}$$

which admits a unique solution

$$(Y^{(3)}, Z^{(3)}) \in \mathcal{S}_{[0,T]}^2(\mathcal{G}, \mathbb{R}) \times \mathcal{M}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^d)$$

by Theorem [9, Theorem 4].

Since $\xi_T^{(1)} \geq \xi_T^{(2)}$ a.s. and $\bar{f}^1(t, Y_t^{(1)}, Z_t^{(1)}) \geq \bar{f}^3(t, Y_t^{(1)}, Z_t^{(1)})$ a.s. for $t \in [0, T]$, by Proposition 3.1 we have $Y_t^{(1)} \geq Y_t^{(3)}$ a.s. for $t \in [0, T]$, and so

$$Y_t^{(1)} \geq Y_t^{(3)} \text{ a.s., } 0 \leq t \leq T + K.$$

Similarly, we obtain the following equation:

$$\begin{cases} Y_t^{(4)} = \xi_T^{(2)} + \int_t^T f^2(r, Y_r^{(3)}, Z_r^{(4)}, Y_{r+\delta(r)}^{(3)}, Z_{r+\zeta^{(2)}(r)}^{(2)}) dr + \int_t^T g(r, Y_r^{(3)}, Z_r^{(4)}) dB_r - \int_t^T Z_r^{(4)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(4)} = \xi_t^{(2)}, \quad Z_t^{(3)} = \eta_t^{(2)}, & T \leq t \leq T + K. \end{cases}$$

Since for all $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $\varphi(r) \in \mathcal{M}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^d)$, $r \in [t, T + K]$, the function $f^2(t, y, z, \cdot, \varphi(r))$ is nondecreasing and $Y_t^{(1)} \geq Y_t^{(3)}$ a.s., by Proposition 3.1 we know that for almost all t ,

$$Y_t^{(3)} \geq Y_t^{(4)} \text{ a.s., } 0 \leq t \leq T + K.$$

Now, for $n \geq 5$, we consider the following equation:

$$\begin{cases} Y_t^{(n)} = \xi_T^{(2)} + \int_t^T f^2(r, Y_r^{(n-1)}, Z_r^{(n)}, Y_{r+\delta(r)}^{(n-1)}, Z_{r+\zeta^{(2)}(r)}^{(2)}) dr + \int_t^T g(r, Y_r^{(n-1)}, Z_r^{(n)}) dB_r - \int_t^T Z_r^{(n)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(n)} = \xi_t^{(2)}, \quad Z_t^{(n)} = \eta_t^{(2)}, & T \leq t \leq T + K. \end{cases} \quad (3.2)$$

Thanks to Proposition 3.1, by induction we have

$$Y_t^{(1)} \geq Y_t^{(3)} \geq Y_t^{(4)} \geq Y_t^{(5)} \geq \dots \geq Y_t^{(n-1)} \geq Y_t^{(n)} \text{ a.s., } 0 \leq t \leq T + K.$$

From the proof of Theorem 2.3 we know that $(Y^{(n)}, Z^{(n)})$ is a Cauchy sequence in $\mathcal{B}_{\mathcal{F}}^2(\beta, a, [0, T + K], \mathbb{R})$. Denoting its limit by (Y, Z) and taking the limit in the iterative equation (3.2), we obtain that (Y, Z) satisfies the following anticipated BDSDE:

$$\begin{cases} Y_t = \xi_T^{(2)} + \int_t^T f^2(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta^{(2)}(s)}^{(2)}) ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T Z_s dW_s, & 0 \leq t \leq T, \\ Y_t = \xi_t^{(2)}, \quad Z_t = \eta_t^{(2)}, & T \leq t \leq T + K. \end{cases}$$

Then the uniqueness part of Theorem 2.3 leads to $Y_t = Y_t^{(2)}$ a.s. Since for almost all t ,

$$Y_t^{(1)} \geq Y_t^{(3)} \geq Y_t^{(4)} \geq Y_t^{(5)} \geq \dots \geq Y_t \text{ a.s.,}$$

it holds immediately that

$$Y_t^{(1)} \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T + K. \quad \square$$

Similarly to Theorem 3.2, we have the following theorem.

Theorem 3.3. Under $(\tilde{H})$, assume the following:

- (i) $\xi_t^{(1)} \geq \xi_t^{(2)}$ a.s., $T \leq t \leq T + K$.
- (ii) One has

$$f^1(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq f^2(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(2)}(t)}^{(2)}) \text{ a.s., } 0 \leq t \leq T.$$

(iii) For all $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $\varphi(r) \in \mathcal{M}_{[0, T]}^2(\mathcal{G}, \mathbb{R}^d)$, $r \in [t, T + K]$, the function $f^1(t, y, z, \cdot, \varphi(r))$ is nondecreasing.

Then $Y_t^{(1)} \geq Y_t^{(2)}$ a.s., $0 \leq t \leq T + K$.

Proof. Set

$$\tilde{f}^2(r, y, z) = f^2(r, y, z, Y_{r+\delta(r)}^{(2)}, Z_{r+\zeta^{(2)}(r)}^{(2)}) \quad \text{for all } t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^d.$$

Then $(Y^{(2)}, Z^{(2)})$ is also the unique solution of

$$Y_t^{(2)} = \xi_T^{(2)} + \int_t^T \tilde{f}^2(r, Y_r^{(2)}, Z_r^{(2)}) dr + \int_t^T g(r, Y_r^{(2)}, Z_r^{(2)}) dB_r - \int_t^T Z_r^{(2)} dW_r, \quad 0 \leq t \leq T.$$

Let us introduce the following equation:

$$\begin{cases} Y_t^{(3)} = \xi_T^{(1)} + \int_t^T f^1(r, Y_r^{(3)}, Z_r^{(3)}, Y_{r+\delta(r)}^{(1)}, Z_{r+\zeta^{(1)}(r)}^{(1)}) dr + \int_t^T g(r, Y_r^{(3)}, Z_r^{(3)}) dB_r - \int_t^T Z_r^{(3)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(3)} = \xi_t^{(1)}, \quad Z_t^{(3)} = \eta_t^{(1)}, & T \leq t \leq T + K. \end{cases}$$

If we set

$$\tilde{f}^3(r, y, z) = f^1(r, y, z, Y_{r+\delta(r)}^{(1)}, Z_{r+\zeta^{(1)}(r)}^{(1)}) \quad \text{for all } t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^d,$$

then this equation is equivalent to

$$\begin{cases} Y_t^{(3)} = \xi_T^{(1)} + \int_t^T \tilde{f}^3(r, Y_r^{(3)}, Z_r^{(3)}) dr + \int_t^T g(r, Y_r^{(3)}, Z_r^{(3)}) dB_r - \int_t^T Z_r^{(3)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(3)} = \xi_t^{(1)}, \quad Z_t^{(3)} = \eta_t^{(1)}, & T \leq t \leq T + K. \end{cases}$$

Since $\xi_T^{(1)} \geq \xi_T^{(2)}$ a.s. and $\tilde{f}^3(t, Y_t^{(2)}, Z_t^{(2)}) \geq \tilde{f}^2(t, Y_t^{(2)}, Z_t^{(2)})$ a.s. for $t \in [0, T]$, Proposition 3.1 leads to $Y_t^{(3)} \geq Y_t^{(2)}$ a.s. for $t \in [0, T]$, and thus

$$Y_t^{(3)} \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T + K.$$

Similarly, we obtain the following equation:

$$\begin{cases} Y_t^{(4)} = \xi_T^{(1)} + \int_t^T f^1(r, Y_r^{(3)}, Z_r^{(4)}, Y_{r+\delta(r)}^{(3)}, Z_{r+\zeta^{(1)}(r)}^{(1)}) dr + \int_t^T g(r, Y_r^{(3)}, Z_r^{(4)}) dB_r - \int_t^T Z_r^{(4)} dW_r, & 0 \leq t \leq T, \\ Y_t^{(4)} = \xi_t^{(1)}, \quad Z_t^{(4)} = \eta_t^{(1)}, & T \leq t \leq T + K. \end{cases}$$

With a similar argument and the assumptions on the generator f^1 , we have

$$Y_t^{(4)} \geq Y_t^{(3)} \text{ a.s., } 0 \leq t \leq T + K.$$

Now, for $m = 5, 6, \dots$, we introduce the following equation:

$$\begin{cases} Y_t^{(m)} = \xi_T^{(1)} + \int_t^T f^1(r, Y_r^{(m-1)}, Z_r^{(m)}, Y_{r+\delta(r)}^{(m-1)}, Z_{r+\zeta^{(1)}(r)}^{(1)}) dr + \int_t^T g(r, Y_r^{(m-1)}, Z_r^{(m)}) dB_r - \int_t^T Z_r^{(m)} dW_r, \\ \hspace{15em} 0 \leq t \leq T, \\ Y_t^{(m)} = \xi_t^{(1)}, \quad Z_t^{(m)} = \eta_t^{(1)}, \quad T \leq t \leq T + K. \end{cases}$$

Thanks to Proposition 3.1, by induction we have

$$Y_t^{(m)} \geq Y_t^{(m-1)} \geq \dots \geq Y_t^{(5)} \geq Y_t^{(4)} \text{ a.s., } 0 \leq t \leq T + K.$$

Similarly to the proof of Theorem 2.3, we can prove that $(Y^{(m)}, Z^{(m)})$ converges in $\mathcal{B}_{\mathcal{F}}^2(\beta, a, [0, T + K], \mathbb{R})$

to $(Y^{(1)}, Z^{(1)})$. Since for almost all t ,

$$Y_t^{(1)} \geq \dots \geq Y_t^{(5)} \geq Y_t^{(4)} \geq Y_t^{(3)} \geq Y_t^{(2)} \text{ a.s.},$$

it holds immediately that

$$Y_t^{(1)} \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T + K. \quad \square$$

We see from the above Theorems 3.2 and 3.3 that one of the generator functions is nondecreasing in the anticipated term of y . In what follows, our aim now is to eliminate this assumption and establish two other comparison theorems.

Theorem 3.4. *Let f^1 and f^2 satisfy $(\tilde{H})$. Assume that there exists $h = h(t, y, z, \theta(s), \varphi(r))$ which satisfies $(\tilde{H})$ and such that the following conditions hold:*

- (i) $\xi_t^{(1)} \geq \xi_t^{(2)} \text{ a.s., } T \leq t \leq T + K.$
- (ii) *One has*

$$f^1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq h(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(2)}(t)}^{(1)}) \text{ a.s., } 0 \leq t \leq T.$$

- (iii) *One has*

$$h(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(2)}(t)}^{(2)}) \geq f^2(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(2)}(t)}^{(2)}) \text{ a.s., } 0 \leq t \leq T.$$

- (iv) *For all $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $\varphi(r) \in \mathcal{M}_{[0, T]}^2(\mathcal{G}, \mathbb{R}^d)$, $r \in [t, T + K]$, the function $h(t, y, z, \cdot, \varphi(r))$ is nondecreasing.*

Then $Y_t^{(1)} \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T + K.$

Proof. By Theorem 2.3, the ABDSDE

$$\begin{cases} \bar{Y}_t = \xi_T^{(1)} + \int_t^T h(r, \bar{Y}_r, \bar{Z}_r, \bar{Y}_{r+\delta(r)}, Z_{r+\zeta^{(2)}(r)}^{(2)}) dr + \int_t^T g(r, \bar{Y}_r, \bar{Z}_r) dB_r - \int_t^T \bar{Z}_r dW_r, & 0 \leq t \leq T, \\ \bar{Y}_t = \xi_t^{(1)}, & T \leq t \leq T + K, \end{cases}$$

admits a unique solution $(\bar{Y}, \bar{Z}) \in \mathcal{S}_{[0, T]}^2(\mathcal{G}, \mathbb{R}) \times \mathcal{M}_{[0, T]}^2(\mathcal{G}, \mathbb{R}^d)$. By assumptions (i), (ii) and (iv), Theorem 3.2 shows that

$$Y_t^{(1)} \geq \bar{Y}_t \text{ a.s., } 0 \leq t \leq T + K. \quad (3.3)$$

On the other hand, in view of assumptions (i), (iii) and (iv), Theorem 3.3 yields

$$\bar{Y}_t \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T + K. \quad (3.4)$$

Hence gathering (3.3) and (3.4), we obtain

$$Y_t^{(1)} \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T + K,$$

as desired. $\square$

Similarly to Theorem 3.4, we can also assert the last comparison theorem.

Theorem 3.5. *Let f^1 and f^2 satisfy $(\tilde{H})$. Assume that there exists $h = h(t, y, z, \theta(s), \varphi(r))$ which satisfies $(\tilde{H})$ and such that the following conditions hold:*

- (i) $\xi_t^{(1)} \geq \xi_t^{(2)} \text{ a.s., } T \leq t \leq T + K.$
- (ii)

$$f^1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq h(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \text{ a.s., } 0 \leq t \leq T.$$

- (iii)

$$h(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq f^2(t, Y_t^{(2)}, Z_t^{(2)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(2)}(t)}^{(2)}) \text{ a.s., } 0 \leq t \leq T.$$

- (iv) *For all $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $\varphi(r) \in \mathcal{M}_{[0, T]}^2(\mathcal{G}, \mathbb{R}^d)$, $r \in [t, T + K]$, the function $h(t, y, z, \cdot, \varphi(r))$ is nondecreasing.*

Then $Y_t^{(1)} \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T + K.$

Proof. Similarly, by Theorem 2.3, the equation

$$\begin{cases} \bar{Y}_t = \xi_T^{(1)} + \int_t^T h(r, \bar{Y}_r, \bar{Z}_r, \bar{Y}_{r+\delta(r)}, Z_{r+\zeta^{(1)}(r)}^{(1)}) dr + \int_t^T g(r, \bar{Y}_r, \bar{Z}_r) dB_r - \int_t^T \bar{Z}_r dW_r, & 0 \leq t \leq T, \\ \bar{Y}_t = \xi_t^{(1)}, & T \leq t \leq T+K, \end{cases}$$

admits a unique solution $(\bar{Y}, \bar{Z}) \in \mathcal{S}_{[0,T]}^2(\mathcal{G}, \mathbb{R}) \times \mathcal{M}_{[0,T]}^2(\mathcal{G}, \mathbb{R}^d)$. By assumptions (i), (ii) and (iv), it follows from Theorem 3.2 that

$$Y_t^{(1)} \geq \bar{Y}_t \text{ a.s., } 0 \leq t \leq T+K. \quad (3.5)$$

In view of assumptions (i), (iii) and (iv), Theorem 3.3 yields

$$\bar{Y}_t \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T+K. \quad (3.6)$$

As a consequence, gathering inequalities (3.5) and (3.6), we obtain

$$Y_t^{(1)} \geq Y_t^{(2)} \text{ a.s., } 0 \leq t \leq T+K. \quad \square$$

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