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On the estimation of regime-switching Lévy models

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Abstract: The regime-switching Lévy model combines jump-diffusion under the form of a Lévy process, and Markov regime-switching where all parameters depend on the value of a continuous time Markov chain. We start by giving general stochastic results. Estimation is performed following a two-step procedure. The EM-algorithm is extended to this new class of jump-diffusion regime-switching models. Simulations are proposed, alongside an empirical application dedicated to the study of financial and commodity time series. When comparing the results with (i) non regime-switching models, and (ii) continuous regime-switching models (where the Lévy process is replaced by a classic Brownian motion), the Lévy regime-switching model outperforms other competitors.

Keywords: jump-diffusion; Lévy process; Markov-switching model; stochastic processes.

JEL Classification: C15; C53; Q40; G15.

1 Introduction

In recent years, many pure jump or jump-diffusion models have been suggested in the economic and statistical literatures to deal with (possibly large) discontinuities in price processes; see the reference textbook by Cont and Tankov (2004). Adding a jump component to a continuous component (leading to a mixture model) or considering a jump component only allows to fit the data better than with a continuous component only. As emphasized in Aït-Sahalia and Jacod (2009a,b), the presence of jumps and/or the absence of a continuous martingale also has important implications for portfolio choice or risk management activities. In addition, given the presence of jumps in the data, pure jump models are preferred by users as they are easier to handle for practical applications such as derivatives pricing or real-life problems such as valuation of insurance contracts [see Ballotta (2005) or Kassberg, Kiesel, and Liebmann (2008)] or real-option valuation (Martzoukos and Trigeorgis 2002).

Markov-switching models have been widely used in economics and finance since Hamilton (1989a,b) introduced them to estimate regime- or state-dependent variables. They have also been utilized to capture volatility in financial markets. Cai (1994) and Hamilton and Susmel (1994) introduce Markov-switching models to estimate high- and low-volatility regimes in financial data. Regimes constructed in this way are an important instrument for interpreting business cycles using Markov-switching models. They constitute an optimal inference on the latent state of the economy, whereby probabilities are assigned to the unobserved regimes ‘expansion’ and ‘contraction’ conditional on the available information set. Clearly, such an approach is useful when a series is thought to undergo shifts from one type of behavior to another and back again, but where the ‘forcing variable’ that causes the regime shifts is unobservable.

The paper provides an empirical study of the dynamics of a range of financial variables such as equity and commodity indices by introducing the regime-switching Lévy model. The key is that the parameters of a Lévy process may vary depending on the state of a Markov chain. Empirical results demonstrate that the

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newly introduced model outperforms a set of benchmark models. The paper adds to previous works by providing a new model that combines two strands of literature: (i) Lévy-type extensions of standard diffusion models, and (ii) switching processes which can incorporate different market regimes.

The central contribution is to develop a practicable methodology to estimate a Markov-switching model augmented by jumps, under the form of a Lévy process. This particular class of stochastic processes is entirely determined by a drift, a scaled Brownian motion and an independent pure-jump process. The estimation strategy relies on a two-step procedure: by estimating first the diffusion parameters in presence of switching, and second the Lévy jump component by means of separate normal inverse Gaussian (NIG) distributions fitted to each regime. Computationally, the expectation-maximization (EM) Algorithm is extended to this new class of jump-diffusion regime-switching model. By means of simulations and empirical applications to real indexes, we demonstrate the goodness-of-fit of the regime-switching Lévy model (versus Brownian regime-switching or non regime-switching models), and thereby illustrate the interest to resort to that kind of model in financial economics.

This paper extends jump-diffusion models mixed with Markov-switching to Lévy processes, whereas previous literature focused on the class of Poisson models. Das (2002) develops a Poisson-Gaussian model mixed with Markov-switching that offers a good statistical description of the fed funds short rate behavior. Huisman and Mahieu (2003), Weron, Bierbrauer, and Truck (2004) and Mari (2006) attempt to recover the main characteristics of electricity spot price dynamics. Besides the regime-switching mechanism, they specify in the jump-diffusion an independent homogenous Poisson process to capture the spike formation mechanism. Hansen and Poulsen (2000), Song, Yin, and Zhang (2006), Zhang, Elliott, and Siu (2012) also consider regime-switching jump-diffusion processes governed by random arrivals from a Poisson process. For recent applications on Markov-modulated jump-diffusion processes, see Elliott and Siu (2010) for quantitative risk management, Siu (2010) for bond pricing, Boyarchenko and Boyarchenko (2011) for American options pricing, or Lin, Lian, and Lia (2014) for the pricing of gold options. Yin, Song, and Zhang (2005), Xi (2008), Yin and Xi (2010) study further theoretical properties and numerical solutions of jump-diffusion processes with a random switching device.

The remainder of the paper is structured as follows. Section 2 introduces the rationale behind Lévy and Markov-switching modeling. Section 3 develops the stochastic model. Section 4 details the estimation method. Section 5 contains simulation results. Section 6 provides an empirical application. Section 7 concludes.

2 Background

In this preliminary section, we review the basic intuitions behind our modeling strategy. Lévy processes have many appealing properties in financial economics, and constitute the first building block of our model. Second, we recall the very intuitive interpretation of the aperiodic, irreducible and ergodic Markov chain. Third, we set the objective of the newly proposed regime-switching Lévy model.

2.1 Lévy jumps

Lévy processes can be thought of as a combination of a diffusion process and a jump process. Both Brownian motion (i.e. a pure diffusion process) and Poisson processes (i.e. pure jump processes) are Lévy processes. As such, Lévy processes represent a tractable extension of Brownian motion to infinitely divisible distributions. In addition, Lévy processes allow the modeling of discontinuous sample paths, whose properties match those of empirical phenomena such as financial time series. There have been many efforts to apply Lévy processes, such as the variance gamma (VG) model [Carr and Madan (1999)], and the NIG model [Rydberg (1997)]. Kijima (2002), Cont and Tankov (2004) and Schoutens (2003) are general books which discuss the use of Lévy processes in finance.

Jumps are discontinuous variations in assets' prices. By nature, jumps consist of rare and dramatic events that dominate the trading days during which they occur. In financial economics, jumps are expected

to appear due to dividend payments, microcrashes due to short-term liquidity challenges or news, such as macroeconomic announcements. Such events have been made partly accountable for the non-Gaussian feature of financial returns, as they can only be captured by fat-tailed distributions. By definition, jumps generate returns that lie outside their usual scale of value. Hence, the higher the jump activity, the higher the uncertainty for market participants. This is why measuring jumps matters.

Jumps are an essential building block of the underlying data-generating process in financial and commodity markets, both in the returns and volatility dynamics. The frequency of occurrence and the size of the jumps are found to be very different from one market to another. Kaeck (2013) studies several jump-diffusion processes (including Lévy processes) in the S&P 500 index and evaluates their performance in terms of option pricing and jump risk premia. Deaton and Laroque (1992) find empirical evidence that agricultural prices are agitated by jumps, which led to numerous theoretical contributions [see Casassus and Collin-Dufresne (2005), Liu and Tang (2011)]. Recently, Brooks and Prokopczuk (2013) have documented that a large negative return in the crude oil price should trigger a jump in its volatility (alongside other examples for soybean, etc.).

2.2 Markov-switching

The normal behavior of economies is occasionally disrupted by dramatic events that seem to produce quite different dynamics for the variables that economists study. Chief among these is the business cycle, in which economies depart from their normal growth behavior and a variety of indicators go into decline [Hamilton and Raj (2002)].

Following Hamilton (1989a,b), time series may be modeled by following different processes at different points in time, with the shifts between processes determined by the outcome of an unobserved Markov chain. In this framework, the parameters and the variance of an autoregressive process depend upon an unobservable regime variable, which represents the probability of being in a particular state of the world. As explained by Engle and Hamilton (1990), the basic idea is to decompose time series into a sequence of stochastic, segmented time trends. A complete description of the Markov-switching model requires the formulation of a mechanism that governs the evolution of the stochastic and unobservable regimes on which the parameters of the autoregression depend. Once a law has been specified for the states, the evolution of regimes can be inferred from the data. Typically, the regime-generating process is an ergodic Markov chain with a finite number of states defined by the transition probabilities, which determine the probability that volatility will switch to another regime, and thus the expected duration of each regime.¹

The regime at any given date is presumed to be the outcome of a Markov chain whose realizations are unobserved to the econometrician. The task facing the econometrician is to characterize the regimes and the law that governs the transitions between them. These parameters estimates can then be used to infer which regime the process was in at any historical date. Although the state of the business cycle is not observed directly by the econometrician, the statistical model implies an optimal way to form an inference about the unobserved variable and to evaluate the likelihood function of the observed data. The techniques developed in Hamilton (1996) rely on the EM algorithm. The standard errors are calculated considering the covariance matrix of the estimators [Bollerslev and Wooldridge (1992)]. The residuals are calculated as a weighted sum of the residuals in the four states (for a two-regime model), with weights given by the filtered probabilities.

In terms of recent applications of this technique, we may cite Li, Li, and Yu (2013), who consider a continuous-time regime-switching term structure model applied to monetary policy, and underline the relevance of switching regimes for term structure modeling. Besides, Tu (2010) shows that stock market displays regime switching between upturns and downturns. Therefore, the author recommends the use of regime switching models in portfolio decisions.

¹ The estimation routine generates two by-products. First, the regime probability at time t is the probability that state t will operate at t , conditional on information available up to $t-1$. The second by-product is the smooth probability, which is the probability of a particular state in operation at time t conditional on all information in the sample.

2.3 Regime-switching Lévy

In this paper, we choose to combine Markov-switching models with Lévy jump-diffusion to match the empirical characteristics of financial and commodity markets. From a statistical point of view, it makes sense to introduce a Markov chain with the existence of a Lévy jump in order to disentangle potentially normal economic regimes (e.g. with a Gaussian distribution) versus agitated economic regimes (e.g. crises periods with stochastic jumps). By combining these two features, we offer a new model that captures well the various crashes and rallies over the business cycle, that are captured by jumps, whereas the trend is simply modeled under a Gaussian framework. On the one hand, the benefits of resorting to regime-switching dynamics lie in disentangling different market regimes that do not have the same parameters of modeling. These separate dynamics are endogenous to the asset price, that need to be uncovered by the Markov chain. On the other hand, Lévy processes represent a very flexible class of stochastic processes, since they allow for the presence of a diffusion (e.g. a scaled Brownian motion) and/or the presence of an independent pure-jump process. Consequently, the regime-switching Lévy model allows identifying the presence of discontinuities for each market regime. This feature constitutes the objective of the proposed model.

Lévy processes are key to this study, since they enable us to measure the intensity of jumps. The higher the jump intensity during one market regime, the higher the need to include a pure-jump process. Conversely, in the absence of jumps during another market regime, then a continuous diffusion with a Brownian motion is indicated. The use of Lévy processes in finance was pioneered by Madan and Seneta (1990) and Madan, Carr, and Chang (1998) who paid particular attention to their use in option pricing. Recent extensions include Carr et al. (2003) who called these models CGMY after their own initials. Barndorff-Nielsen, Mikosch and Resnick (2001) further discuss the empirical fit of Lévy processes. Computationally, an elegant discussion of the EM-algorithm is given in Tanner (1996). In the next section, we formally introduce the notations for Lévy processes modulated by a Markov chain.

3 The stochastic model

Let $(\omega, \mathcal{F}, P)$ be a filtered probability space and T be a fixed terminal time horizon. We propose in this paper to model the dynamic of a sequence of historical values of price using a regime-switching stochastic jump-diffusion. This model is defined using the class of Lévy processes.

3.1 Lévy process

Definition 1 A Lévy process L_t is a stochastic process such that

1. $L_0 = 0$.
2. For all $s > 0$ and $t > 0$, we have that the property of stationary increments is satisfied. i.e. $L_{t+s} - L_t$ as the same distribution as L_s .
3. The property of independent increments is satisfied. i.e. for all $0 < t_0 < t_1 < \dots < t_n$, we have that $L_{t_i} - L_{t_{i-1}}$ are independent for all $i = 1, \dots, n$.
4. L has a Cadlag paths. This means that the sample paths of a Lévy process are right continuous and admit a left limits.

Remark 1 In a Lévy process, the discontinuities occur at random times.

3.2 Markov-switching

Definition 2 Let $(Z_t)_{t \in [0, T]}$ be a continuous time Markov chain on finite space $S := \{1, 2, \dots, K\}$. Denote $\mathcal{F}_t^Z := \{\sigma(Z_s); 0 \leq s \leq t\}$, the natural filtration generated by the continuous time Markov chain Z . The generator matrix of Z , denoted by Π^Z , is given by

$$\Pi_{ij}^Z \geq 0 \text{ if } i \neq j \text{ for all } i, j \in S \text{ and } \Pi_{ii}^Z = -\sum_{j \neq i} \Pi_{ij}^Z \text{ otherwise.} \quad (1)$$

Remark 2 The quantity Π_{ij}^Z represents the switch from state i to state j .

3.3 Regime-switching Lévy

Let us define the regime-switching Lévy Model:

Definition 3 For all $t \in [0, T]$, let Z_t be a continuous time Markov chain on finite space $S := \{1, \dots, K\}$ defined as in Definition 2. A regime-switching model is a stochastic process (X_t) which is solution of the stochastic differential equation given by

$$dX_t = \kappa(Z_t)(\theta(Z_t) - X_t)dt + \sigma(Z_t)dY_t \quad (2)$$

where $\kappa(Z_t)$, $\theta(Z_t)$ and $\sigma(Z_t)$ are functions of the Markov chain Z . Hence, they are constants which take values in $\kappa(S)$, $\theta(S)$ and $\sigma(S)$

$$\kappa(S) := \{\kappa(1), \dots, \kappa(K)\} \in \mathbb{R}^{K^+}, \quad \theta(S) := \{\theta(1), \dots, \theta(K)\}, \quad \sigma(S) := \{\sigma(1), \dots, \sigma(K)\} \in \mathbb{R}^{K^+}.$$

where Y is a stochastic process which could be a Brownian motion or a Lévy process.

Remark 3 The following classic notations apply:

- κ denotes the mean-reverting rate;
- θ denotes the long-run mean;
- σ denotes the volatility of X .

Remark 4

- In this model, there are two sources of randomness: the stochastic process Y appearing in the dynamics of X , and the Markov chain Z . There exists one randomness due to the market information which is the initial continuous filtration $\mathcal{F}$ generated by the stochastic process Y ; and another randomness due to the Markov chain Z , $\mathcal{F}^Z$.
- In our model, the Markov chain Z infers the unobservable state of the economy, i.e. expansion or recession. The processes Y^i estimated in each state, where $i \in S$, capture: a different level of volatility in the case of Brownian motion (i.e. $Y^i \equiv W^i$), or a different jump intensity level of the distribution (and a possible skewness) in the case of Lévy process (i.e. $Y^i \equiv L^i$).

Remark 5 One could propose to use a regime-switching stochastic volatility model, à la Heston, to better capture the flexibility of the volatility changes and levels. Nevertheless, this kind of model increases dramatically the computational burden during the simulations without improving greatly the empirical fit.

3.4 NIG distribution

We recall the main properties of the NIG distribution. Indeed, we assume that a Lévy process L follows a NIG distribution. Note the variance-Gamma could have been an alternative at this stage (Kaishev and Dimitrova 2009). The NIG family of distribution was introduced by Barndorff-Nielsen and Halgreen (1977). The NIG density belongs to the family of normal variance-mean mixtures, i.e. one of the most commonly used parametric densities in financial economics.

Taking $\delta > 0$, $\alpha \geq 0$, then the density function of a NIG variable $\text{NIG}(\alpha, \beta, \delta, \mu)$ is given by

$$f_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha}{\pi} \exp(\delta \sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)) \frac{K_1(\alpha \delta \sqrt{1 + (x - \mu)^2 / \delta^2})}{\sqrt{1 + (x - \mu)^2 / \delta^2}}. \quad (3)$$

where K_ν is the third Bessel kind fonction with index ν . It can be represented with the following integral

$$K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu-1} \exp\left(-\frac{1}{2}z(y+y^{-1})\right) dy.$$

For a given real ν , the function K_ν satisfies the differential equation given by

$$x^2 y'' + xy' - (x^2 + \nu^2)y = 0.$$

This class of distribution is stable by convolution as the classic normal distribution. i.e.

$$\text{NIG}(\alpha, \beta, \delta_1, \mu_1) * \text{NIG}(\alpha, \beta, \delta_2, \mu_2) = \text{NIG}(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2).$$

Lemma 1 If $X \sim \text{NIG}(\alpha, \beta, \delta, \mu)$ then for any $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$, we have that

$$Y = aX + b \sim \left(\frac{\alpha}{a}, \frac{\beta}{a}, a\delta, a\mu + b\right).$$

The log cumulative function of a NIG variable is given by

$$\phi^{\text{NIG}}(z) = \mu z + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}), \quad \text{for all } |\beta + z| < \alpha. \quad (4)$$

The first moments are given by

$$\mathbb{E}[X] = \mu + \frac{\delta\beta}{\gamma}, \quad \text{Var}[X] = \frac{\delta\alpha^2}{\gamma^3}. \quad (5)$$

with $\gamma = \sqrt{\alpha^2 - \beta^2}$. And finally the Lévy measure of a $\text{NIG}(\alpha, \beta, \delta, \mu)$ law is

$$F_{\text{NIG}}(dx) = e^{\beta x} \frac{\delta\alpha}{\pi|x|} K_1(\alpha|x|) dx. \quad (6)$$

Remark 6 Each parameter in $\text{NIG}(\alpha, \beta, \delta, \mu)$ distributions can be interpreted as having a different effect on the shape of the distribution:

- α – tail heaviness of steepness.
- β – skewness.
- δ – scale.
- μ – location.

4 Estimation

This section covers the methodology pertaining to the estimation task. How can we estimate the parameters given that the underlying process is a regime-switching Lévy? To perform this task, we rely on a two-step approach by estimating (i) model parameters in a regime-switching Brownian process, and (ii) the distribution parameters. Whilst detecting the jump dynamics simultaneously with the regime switches would constitute a more robust methodology, our paper is the first of its kind in the literature to tackle this complex estimation issue. As demonstrated below, our two-step methodology that works well on empirical data. In what follows, we extend the EM algorithm to the class of Lévy regime-switching and explain how the likelihood can be evaluated. In the following sub-sections, the two-step estimation strategy as well as the initialization choice for the parameters are detailed.

4.1 EM algorithm

The EM algorithm used to estimate the regime-switching Lévy model in this paper is a generalization and extension of the EM-algorithm developed in Hamilton (1989a,b).

Our aim is to fit a regime-switching Lévy model such as (2) where the stochastic process Y is a Lévy process that follows a NIG distribution. Thus the optimal set of parameters to estimate is $\hat{\Theta} := (\hat{\kappa}_i, \hat{\theta}_i, \hat{\sigma}_i, \hat{\alpha}_i, \hat{\beta}_i, \hat{\delta}_i, \hat{\mu}_i, \hat{\Pi})$, for $i \in \mathcal{S}$.

We have the three parameters of the dynamics of X , the four parameters of the density of the Lévy process L , and the transition matrix of the Markov chain Z . Because the number of parameters grows rapidly in this class of jump-diffusion regime-switching models, direct maximization of the total log-likelihood is not practicable. To bypass this problem, we propose a method in two successive steps to estimate the global set of parameters. Let us first discretize the model, and then detail the two-step procedure.

Discretization

We first take for the stochastic process Y a Brownian motion W . Moreover, suppose that the size of historical data is $M+1$. Let Γ denote the corresponding increasing sequence of time from which the data values are taken:

$$\Gamma = \{t_j; 0 = t_0 \leq t_1 \leq \dots \leq t_M = T\}, \text{ with } \Delta_t = t_j - t_{j-1} = 1.$$

The discretized version of model (2) writes

$$X_{t+1} = \kappa(Z_t)\theta(Z_t) + (1 - \kappa(Z_t))X_t + \sigma(Z_t)\epsilon_{t+1}, \quad (7)$$

where $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$ (since the process Y is a Brownian motion). We denote by $\mathcal{F}_{t_k}^X$ the vector of historical values of the process X until time $t_k \in \Gamma$. Thus, $\mathcal{F}_{t_k}^X$ is the vector of the $k+1$ last values of the discretized model and therefore, $\mathcal{F}_{t_k}^X = (X_{t_0}, X_{t_1}, \dots, X_{t_k})$.

Remark 7 The filtration generated by the Markov chain Z (i.e. $\mathcal{F}^Z$) is the one generated by the history values of Z in the time sequence Γ .

For simplicity of notation, we will write in the sequel the model (7) as

$$X_{t+1} = \kappa_i \theta_i + (1 - \kappa_i)X_t + \sigma_i \epsilon_{t+1}.$$

This means that at time $t \in [0, T]$, the Markov chain Z is in state $i \in \mathcal{S}$ (i.e. $Z_t = i$) and Z jumps at time $t_j \in \Gamma$, $j \in \{0, 1, \dots, M-1\}$.

Step 1: Estimation of the regime-switching model (2) in the Brownian case

In the first step based on the EM-algorithm, the complete parameter space estimate $\hat{\Theta}$ is split into: $\hat{\Theta}_1 := (\hat{\kappa}_i, \hat{\theta}_i, \hat{\sigma}_i, \hat{\Pi})$, for $i \in \mathcal{S}$, which corresponds to the first subset of diffusion parameters. Recall that, we estimate the parameters of the discretized model (7).

Note the regimes are determined with respect to the κ, θ, σ parameters. As an artefact linked to our two-step estimation method, any bias in the transition probabilities can be carried over in the 2nd-step.

Step 2: Estimation of the parameters of the Lévy process fitted to each regime

Using the regime classification obtained in the previous step, we estimate the second subset of parameters $\hat{\Theta}_2 := (\hat{\alpha}_i, \hat{\beta}_i, \hat{\delta}_i, \hat{\mu}_i)$, for $i \in \mathcal{S}$, which corresponds to the NIG distribution parameters of the Lévy jump process fitted for each regime.

4.2 Step 1: The regime-switching model

We use the EM-algorithm where the set of parameters $\Theta = \hat{\Theta}_1 := (\hat{\kappa}_i, \hat{\theta}_i, \hat{\sigma}_i, \hat{\Pi})$ is estimated by an iterative two-step procedure.

1. Starting with an initial vector set $\hat{\Theta}_1^{(0)} := (\kappa_i^{(0)}, \theta_i^{(0)}, \sigma_i^{(0)}, \Pi^{(0)})$, for all $i \in \mathcal{S}$. Fix $N \in \mathbb{N}$, the maximum number of iterations we authorize for this method (for the steps 2 and 3 of EM-algorithm). And fix a positive constant ε as a convergence constant for the estimated log-likelihood function.
2. Assume that we are at the $n+1 \leq N$ steps, calculation in the previous iteration of the algorithm yields the vector set $\hat{\Theta}_1^{(n)} := (\kappa_i^{(n)}, \theta_i^{(n)}, \sigma_i^{(n)}, \Pi^{(n)})$.

The expectation procedure or E-step

We evaluate the smoothed and filtered probabilities. The filtered probability is given by the probability such that the Markov chain Z is in regime $i \in \mathcal{S}$ at time t with respect to $\mathcal{F}_t^X$. The smoothed probability is given by the probability such that the Markov chain Z is in regime $i \in \mathcal{S}$ at time t with respect to all the historical data $\mathcal{F}_T^X$. Because the Markov chain is unobserved, inference on the underlying regimes is given by the following equations.

Filtered probability:

For all $i \in \mathcal{S}$ and $k = \{1, 2, \dots, M\}$, evaluate the quantity

$$\begin{aligned} P(Z_{t_k} = i | \mathcal{F}_{t_k}^X; \hat{\Theta}_1^{(n)}) &= \frac{P(Z_{t_k} = i, X_{t_k} | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})}{f(X_{t_k} | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})} \\ &= \frac{P(Z_{t_k} = i | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}) f(X_{t_k} | Z_{t_k} = i; \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})}{\sum_{j \in \mathcal{S}} P(Z_{t_k} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}) f(X_{t_k} | Z_{t_k} = j; \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})} \end{aligned} \quad (8)$$

with

$$\begin{aligned} P(Z_{t_k} = i | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}) &= \sum_{j \in \mathcal{S}} P(Z_{t_k} = i, Z_{t_{k-1}} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}) \\ &= \sum_{j \in \mathcal{S}} P(Z_{t_k} = i, Z_{t_{k-1}} = j | \hat{\Theta}_1^{(n)}) P(Z_{t_{k-1}} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}) \\ &= \sum_{j \in \mathcal{S}} \Pi_{ji}^{(n)} P(Z_{t_{k-1}} = j | \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}) \end{aligned} \quad (9)$$

where $f(X_{t_k} | Z_{t_k} = i; \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})$ is the density of the process X at time t_k conditional that the process is in regime $i \in \mathcal{S}$. Observed by (7), that given $\mathcal{F}_{t_{k-1}}^X$, the process X_{t_k} has a conditional Gaussian distribution with mean

$$\kappa_i^{(n)} \theta_i^{(n)} + (1 - \kappa_i^{(n)}) X_{t_{k-1}}$$

and standard deviation $\sigma_i^{(n)}$, whose density function is given by

$$f(X_{t_k} | Z_{t_k} = i; \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}) = \frac{1}{\sqrt{2\pi}\sigma_i^{(n)}} \exp \left\{ -\frac{(X_{t_k} - (1 - \kappa_i^{(n)})X_{t_{k-1}} - \kappa_i^{(n)}\theta_i^{(n)})^2}{2(\sigma_i^{(n)})^2} \right\}. \quad (10)$$

Smoothed probability:

For all $i \in \mathcal{S}$ and $k = \{M-1, M-2, \dots, 1\}$,

$$P(Z_{t_k} = i | \mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)}) = \sum_{j \in \mathcal{S}} \left(\frac{P(Z_{t_k} = i | \mathcal{F}_{t_k}^X; \hat{\Theta}_1^{(n)}) P(Z_{t_{k+1}} = j | \mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)}) \Pi_{ij}^{(n)}}{P(Z_{t_{k+1}} = j | \mathcal{F}_{t_k}^X; \hat{\Theta}_1^{(n)})} \right). \quad (11)$$

Note that smoothing inferences on the state of the Markov chain can also be computed using an algorithm developed by Kim (1994).

The maximization step, or M-step

The likelihood is obtained as a by-product of the EM-algorithm. We need to know the explicit density function of our model, which can be derived from the characteristic function of X . We obtain explicit

formula of the maximum likelihood estimator of the first subset of parameters $\hat{\Theta}_1$. The maximum likelihood estimates $\hat{\Theta}_1^{(n+1)}$ for all model parameters are given, for all $i \in \mathcal{S}$, by

$$\begin{aligned}\theta_i^{(n+1)} &= \frac{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})(X_{t_k} - (1-\kappa_i^{(n+1)})X_{t_{k-1}})]}{\kappa_i^{(n+1)} \sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})]}, \\ \kappa_i^{(n+1)} &= \frac{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})X_{t_{k-1}}B_1]}{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})X_{t_{k-1}}B_2]}, \\ \sigma_i^{2(n+1)} &= \frac{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})(X_{t_k} - \kappa_i^{(n+1)}\theta_i^{(n+1)} - (1-\kappa_i^{(n+1)})X_{t_{k-1}})^2]}{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})]},\end{aligned}$$

where

$$\begin{aligned}B_1 &= X_{t_k} - X_{t_{k-1}} - \frac{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})(X_{t_k} - X_{t_{k-1}})]}{\sum_{k=2}^M P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})}, \\ B_2 &= \frac{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})X_{t_{k-1}}]}{\sum_{k=2}^M [P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})]} - X_{t_{k-1}}.\end{aligned}$$

Finally, the transition probabilities are estimated according to the following formula

$$\Pi_{ij}^{(n+1)} = \frac{\sum_{k=2}^M \left[P(Z_{t_k}=j|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)}) \frac{\Pi_{ij}^{(n)} P(Z_{t_{k-1}}=i|\mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})}{P(Z_{t_k}=j|\mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})} \right]}{\sum_{k=2}^M [P(Z_{t_{k-1}}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})]}. \quad (12)$$

3. Denote by $\hat{\Theta}_1^{(n+1)} := (\kappa_i^{(n+1)}, \theta_i^{(n+1)}, \sigma_i^{(n+1)}, \Pi^{(n+1)})$, the new parameters of the algorithm and use them in step 2 until the convergence of the EM-algorithm. In fact, we stop the procedure if one of the following conditions are verified:
 - (a) We have performed N times the procedure.
 - (b) The difference between the log-likelihood at step $n+1 \leq N$ denoted by $\log L(n+1)$ and at step n , satisfies the relation

$$\log L(n+1) - \log L(n) < \varepsilon. \quad (13)$$

Remark 8

1. Since the log-likelihood function is increasing after each iteration of the procedure, we do not need to take the absolute value of the left-hand side of inequality (13).
2. In our case (i.e. regime-switching), the standard log-likelihood function without regime-switching $\sum_{k=1}^M \log(f(X_{t_k}|\mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)}))$ has to be weighted with the corresponding smoothed inference. Each observation X_{t_k} belongs to the i th state with probability $P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})$. Hence, the regime-switching log-likelihood function is:

$$L(\Theta) = \sum_{k=1}^M \sum_{i \in \mathcal{S}} \log(f(X_{t_k}|Z_{t_k}=i; \mathcal{F}_{t_{k-1}}^X; \hat{\Theta}_1^{(n)})) P(Z_{t_k}=i|\mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)}). \quad (14)$$

Note the log-likelihood of the RS-Lévy model is the combination of two log-likelihoods. The first one is given by Eq. (14) for the switching dynamics. The second one is given by Eq. (15) for the NIG distribution of the Lévy process.

3. The (quasi) maximum likelihood estimators are consistent (Kim 1994). In our setting, the convergence of the estimated set of parameters $\hat{\Theta}_1^{(n)}$ to the true optimal values is guaranteed, since our density distribution of X given by (10) belongs to the class of exponential families.

4.3 Step 2: Lévy distribution fitted to each regime

Following the EM algorithm-based estimation of the regime-switching model (2), we now proceed with the remaining set of parameters $\hat{\Theta}_2$ by fitting a NIG distribution for each regime:

$$X(\text{Regime 1}) \sim L_1(\alpha^1, \beta^1, \delta^1, \mu^1)$$

and

$$X(\text{Regime 2}) \sim L_2(\alpha^2, \beta^2, \delta^2, \mu^2)$$

with L_1 and L_2 two separate NIG Lévy distributions.

The estimation of the distribution parameters is achieved by constrained maximum likelihood:

$$\Phi^1 := \{\alpha^1, \beta^1, \delta^1, \mu^1\}$$

and

$$\Phi^2 := \{\alpha^2, \beta^2, \delta^2, \mu^2\}$$

We denote by $r_t := \log(X_t) - \log(X_{t-1})$.

Proposition 1 The log-likelihood function of the sequence of log-returns with a NIG distribution $(\alpha, \beta, \delta, \mu)$ is given by

$$L(\alpha, \beta, \delta, \mu) = n \log\left(\frac{\alpha}{\pi}\right) + n\delta\gamma + \sum_{t=0}^{n-1} [\beta\delta\tau_t - \log c_t + \log K_1(\alpha\delta c_t)], \quad (15)$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. For any $t=0, 1, \dots, n-1$, $\tau_t = \frac{r_t - \mu}{\delta}$ and $c_t = \sqrt{1 + \tau_t^2}$.

Proof. We propose to use the maximum likelihood method to estimate the set of parameters $(\alpha, \beta, \delta, \mu)$. Denote for all $t=0, 1, \dots, n-1$, $\tau_t = \frac{r_t - \mu}{\delta}$ and $c_t = \sqrt{1 + \tau_t^2}$. Then, the log-likelihood function following (3) is given by

$$\begin{aligned} L(\alpha, \beta, \delta, \mu) &= \sum_{t=0}^{n-1} \log f_{\text{NIG}}(r_t; \alpha, \beta, \delta, \mu) \\ &= \sum_{t=0}^{n-1} \left[\log\left(\frac{\alpha}{\pi}\right) \alpha\gamma + \beta\delta\tau_t + \log\left(\frac{K_1(\alpha\delta c_t)}{c_t}\right) \right] \\ &= n \log\left(\frac{\alpha}{\pi}\right) + n\delta\gamma + \sum_{t=0}^{n-1} [\beta\delta\tau_t - \log c_t + \log K_1(\alpha\delta c_t)]. \end{aligned}$$

To compute the standard errors of our estimators, we give the first partial derivatives of each parameter: ■

Proposition 2 We have that

$$\begin{aligned} \frac{\partial L}{\partial \gamma} &= \frac{2n\gamma}{\alpha^2} + n\delta - \sum_{t=0}^{n-1} \left(\frac{\gamma\delta}{\alpha} c_t R_1(\alpha\delta c_t) \right) \\ \frac{\partial L}{\partial \delta} &= n\gamma + \frac{n}{\delta} - \sum_{t=0}^{n-1} \left(\frac{\alpha}{c_t} R_1(\alpha\delta c_t) \right) \\ \frac{\partial L}{\partial \beta} &= 2n \frac{\beta}{\alpha^2} + \sum_{t=0}^{n-1} \left(\delta\tau_t - \frac{\beta\delta}{\alpha} c_t R_1(\alpha\delta c_t) \right) \\ \frac{\partial L}{\partial \mu} &= -n\beta + \sum_{t=0}^{n-1} \left(\alpha \frac{\tau_t}{c_t} R_1(\alpha\delta c_t) \right) \end{aligned}$$

where for all $x > 0$,

$$R_1(x) = \frac{K_2(x)}{K_1(x)}$$

To compute the standard errors of the maximum likelihood estimators, we have to evaluate the Fisher matrix $\mathcal{J}$ which is given by:

$$\mathcal{J} = \mathbb{E} \left[\frac{\partial L(X; \theta)}{\partial \theta} \frac{\partial L(X; \theta)}{\partial \theta'} \right]$$

where θ and θ' are both parameters of the NIG distribution [i.e. $(\alpha, \beta, \delta, \mu)$]. We have the following equality:

$$\mathbb{E} \left[\frac{\partial L(X; \theta)}{\partial \theta} \frac{\partial L(X; \theta)}{\partial \theta'} \right] = -\mathbb{E} \left[\frac{\partial^2 L(X; \theta)}{\partial \theta \partial \theta'} \right]$$

Thus, to estimate the Fisher matrix with regard to the time series, we can evaluate the empirical average of the Hessian matrix:

$$\hat{\mathcal{J}} = \frac{1}{n} \sum_{i=0}^{n-1} \left[\frac{\partial L(r_i; \hat{\theta})}{\partial \theta} \frac{\partial L(r_i; \hat{\theta})}{\partial \theta'} \right]$$

Finally, the diagonal terms of the inverse matrix of $\hat{\mathcal{J}}$ give the estimations of the variance of our maximum likelihood estimators.

Moreover, to take into account the constraints on the NIG parameters (i.e. the positivity of γ and δ), the optimization is performed according to the following variables $(\gamma', \delta', \beta, \mu)$ with $\gamma' = \log(\sqrt{\alpha^2 - \beta^2})$ and $\delta' = \log(\delta)$.

4.4 Initialization choice

What remains now is deciding how to start up the algorithm. The initialization of the estimation is performed by the method of moments. Indeed, if $r_t \sim \text{NIG}(\alpha, \beta, \delta, \mu)$, then we can derive the first four moments:

$$\begin{aligned} m_1 &= \mu + \delta\beta\gamma^{-1}, & m_2 &= \delta\alpha^2\gamma^{-3} \\ m_3 &= 3\delta\beta\alpha^2\gamma^{-5} & \text{and } m_4 &= 3\delta\alpha^2(\alpha^2 + 4\beta^2)\gamma^{-7}. \end{aligned}$$

The initial set of parameters $(\alpha, \beta, \delta, \mu)$ is estimated by injecting, in these equations, the Monte Carlo approximation of m_1, m_2, m_3 and m_4 computed with observations $(\hat{m} + \hat{s}\varepsilon_0, \hat{m} + \hat{s}\varepsilon_1, \dots, \hat{m} + \hat{s}\varepsilon_{n-1})$. This gives

$$\begin{aligned} \hat{\gamma} &= \frac{3}{\bar{S}\sqrt{3\bar{\gamma}_2 - 5\bar{\gamma}_1^2}}, & \hat{\beta} &= \frac{\bar{\gamma}_1\bar{S}\hat{\gamma}^2}{3} \\ \hat{\delta} &= \frac{\bar{S}^2\hat{\gamma}^3}{\hat{\beta}^2 + \hat{\gamma}^2} & \text{and } \hat{\mu} &= \bar{X} - \hat{\beta}\frac{\hat{\delta}}{\hat{\gamma}}. \end{aligned}$$

where $\bar{X}$ and $\bar{S}$ are the sample mean and variance, respectively, and $\bar{\gamma}_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}$, $\bar{\gamma}_2 = \frac{\mu_4}{\mu_2^2} - 2$ with $\mu_t = \frac{1}{n} \sum_{j=0}^{n-1} (\varepsilon_j - \bar{X})^t$.

5 Simulated processes

In this section, we report simulation results based on Monte Carlo (MC) experiments to harness the performance of the regime-switching Lévy model. The entire simulation study is mainly composed of three parts formulated in three different scenarios as follows: (i) the Gaussian case with regime-switching dynamics, (ii) the jump case with no regime-switching evidence, and (iii) the regime-switching Lévy case.

5.1 Gaussian case (no jump) with regime-switching dynamics

We now simulate the MC trajectories of a regime-switching model without jump of length $M=1000$ values. The underlying economic model is a regime-switching mean-reverting model with a Gaussian noise. Equation (7) becomes:

$$X_{t+1} = \kappa_i \theta_i + (1 - \kappa_i) X_t + \sigma_i \epsilon_{t+1},$$

with $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$.

The true parameters values are given in Table 1.

For each estimation of the parameters during the MC experiments, we evaluate the quadratic errors which are given by:

$$\frac{1}{MC} \sum_{k=1}^{MC} (\hat{p} - p)^2$$

where p represents the true value of the parameter given in Table 1, and $\hat{p}$ the estimated values. We are setting up a comparison between the Normal and the NIG distributions on fair grounds, by taking jumps of high intensity (i.e. low value of the α parameter). The results are detailed in Table 2. We have computed percentage errors (i.e. MSE/parameter) in Table 3.

With only 200 simulations, our estimation process delivers a satisfactory performance, as the estimated parameters are very close to their true values. Moreover, as expected, when the number of MC experiments increases, we obtain a much higher convergence speed.

Table 1: Parameters in the Gaussian case with regime-switching.

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	p_{11}^z	p_{22}^z
0.80	0.60	12.50	33.33	1.00	1.30	0.90	0.95

Table 2: Quadratic errors obtained for the Gaussian case with regime-switching.

Parameter	200	500	1000
κ_1	0.000081	0.000082	0.000077
κ_2	0.000102	0.000096	0.000091
θ_1	0.005669	0.005813	0.005567
θ_2	0.006271	0.006120	0.005975
σ_1	0.006369	0.006006	0.005748
σ_2	0.005840	0.005521	0.005361
p_{11}^z	0.000269	0.000293	0.000296
p_{22}^z	0.000065	0.000073	0.000077

Table 3: Percentage errors obtained for the Gaussian case with regime-switching.

Parameter	200	500	1000
κ_1	0.000101	0.000103	0.000096
κ_2	0.000170	0.000160	0.000152
θ_1	0.000454	0.000465	0.000445
θ_2	0.000188	0.000184	0.000179
σ_1	0.006369	0.006006	0.005748
σ_2	0.004492	0.004247	0.004124
p_{11}^z	0.000299	0.000326	0.000329
p_{22}^z	0.000068	0.000077	0.000081

5.2 Jump case with no regime-switching evidence

We now simulate the *MC* trajectories of without regime-switching model but with jumps of length $M=1000$ values. From 1000 replications onwards, the size of the quadratic error becomes very small (at the 10^{-4} decimal level), hence we judge that the computational burden to augment the number of replications is not necessary. The underlying economic model is a mean-reverting model driven by a Lévy process. Equation (7) becomes:

$$X_{t+1} = \kappa\theta + (1-\kappa)X_t + \sigma\epsilon_{t+1}.$$

with $\epsilon_{t+1} \sim L_1$ where L_t is a Lévy process with L_1 follows a NIG distribution of parameters $(\alpha, \beta, \delta, \mu)$ as mentioned in Section 3.4. The true parameters values of the diffusion (i.e. κ, θ and σ) are given in state 2 of the parameters reproduced in Table 1. The values of the NIG parameters are given in Table 4.

The quadratic errors obtained are detailed in Table 5. The computed percentage errors (i.e. MSE/parameter) are displayed in Table 6.

With 200 simulations, our estimation process delivers a satisfactory performance, as the estimated parameters are very close to their true values.

It could be interesting to look the obtained results if we increase the intensity of jumps. To do so, we recall that – in the special case of a NIG distribution – the parameter α represents this intensity. Whenever α goes to zero, the intensity of jumps increases. Whenever α increases, the corresponding distribution converges in law to a Gaussian one.

The quadratic errors obtained are detailed in Table 7. We proceed with $MC=500$ simulations.

The higher the jump intensity, the better the quality of our estimation process. These simulation results demonstrate the stability of our estimation process with respect to the jump intensity parameter of the distribution. Conversely, if we decrease this intensity and take a value corresponding to a Gaussian distribution, then our estimation method delivers inferior results. Indeed, in this case we would like to fit a non continuous distribution with a continuous one.

Table 4: Parameters in the NIG case without regime-switching.

α	β	δ	μ
0.6969	0.1143	0.6690	-0.1112

Table 5: Quadratic errors obtained for the NIG case without regime-switching.

Parameter	200	500	1000
α	0.009520 (0.0045)	0.008330 (0.0032)	0.008832 (0.0030)
β	0.002468 (0.0079)	0.002471 (0.0072)	0.002571 (0.0070)
δ	0.002697 (0.0023)	0.002401 (0.0020)	0.002468 (0.0019)
μ	0.010070 (0.0043)	0.010103 (0.0032)	0.010024 (0.0030)

Table 6: Percentage errors obtained for the NIG case without regime-switching.

Parameter	200	500	1000
α	0.013660	0.011953	0.012673
β	0.021592	0.021619	0.022493
δ	0.004031	0.003589	0.003689
μ	-0.090558	-0.090854	-0.090144

Table 7: Quadratic errors obtained for the NIG case without regime-switching for different intensity of jumps.

Parameter	Base	2 Times less	+40%	+100%
Intensity α	0.6969	1.3938	0.5575	0.3484
α	0.008330	0.051475	0.005974	0.003722
β	0.002471	0.017163	0.001876	0.001318
δ	0.002401	0.020510	0.001202	0.000326
μ	0.010103	0.120527	0.004324	0.000706

Table 8: Parameters in the NIG case with regime-switching.

	α	β	δ	μ
State 1	0.6969	0.1143	0.6690	-0.1112
State 2	0.3484	0.0292	0.3448	-0.0290

5.3 Regime-switching Lévy case

We now simulate the MC trajectories of a regime-switching model with jump of again a length $M=1000$ values. The underlying economic model is a mean-reverting model driven by a Lévy process. Equation (7) becomes:

$$X_{t+1} = \kappa_i \theta_i + (1 - \kappa_i) X_t + \sigma_i \epsilon_{i,t+1}.$$

with $\epsilon_{i,t+1} \sim L_1^i$, $i \in \{1, 2\}$, where L_t^i is a regime switching Lévy process with L_1^i follows a NIG distribution of parameters $(\alpha_i, \beta_i, \delta_i, \mu_i)$. The parameters values of the diffusion are such given in Table 1. The values of the NIG parameters are given in Table 8.

The quadratic errors obtained are detailed in Table 9. The computed percentage errors (i.e. MSE/parameter) are displayed in Table 10.

Table 9: Quadratic errors obtained for the NIG case with regime-switching. (Standard errors in parenthesis).

Parameter	200	500	1000
α_1	0.020919 (0.1390)	0.019237 (0.1357)	0.020012 (0.1363)
α_2	0.058022 (0.2780)	0.055931 (0.2625)	0.056467 (0.2727)
β_1	0.005684 (0.0749)	0.005250 (0.0724)	0.005558 (0.0739)
β_2	0.006749 (0.0890)	0.006717 (0.0884)	0.006395 (0.0874)
δ_1	0.005541 (0.0720)	0.005554 (0.0734)	0.005861 (0.0744)
δ_2	0.035433 (0.01720)	0.036566 (0.01795)	0.037043 (0.01764)
μ_1	0.009978 (0.0065)	0.010089 (0.0065)	0.009961 (0.0066)
μ_2	0.011478 (0.0085)	0.011435 (0.0080)	0.011405 (0.0083)

Table 10: Percentage errors obtained for the NIG case with regime-switching.

Parameter	200	500	1000
α_1	0.030017	0.027604	0.028716
α_2	0.166538	0.160537	0.162075
β_1	0.049729	0.045932	0.048626
β_2	0.231130	0.230034	0.219007
δ_1	0.008283	0.008302	0.008761
δ_2	0.102764	0.106050	0.107433
μ_1	-0.089730	-0.090728	-0.089577
μ_2	-0.395793	-0.394310	-0.393276

Table 11: Quadratic errors obtained for the NIG case with regime-switching for different intensity of jumps.

Parameter	Base	2 Times less	+40%	+100%
Intensity α	0.6969	1.3938	0.5575	0.3484
α_1	0.019237	0.183248	0.012910	0.010163
α_2	0.055931	0.319214	0.035005	0.013191
β_1	0.005250	0.082296	0.003934	0.002661
β_2	0.006717	0.061504	0.004227	0.002120
δ_1	0.005554	0.048475	0.002749	0.000932
δ_2	0.036566	0.065636	0.026593	0.013529
μ_1	0.010089	0.119121	0.004383	0.000754
μ_2	0.011435	0.137012	0.004931	0.00085

With 200 simulations, our estimation process delivers a satisfactory performance, as the estimated parameters are very close to their true values. It is important to notice that our estimated values behave well in terms of NIG parameters values in both regimes.

It could be interesting again to look the obtained results if we increase the intensity of jumps. The quadratic errors obtained are detailed in Table 11. We proceed with $MC=500$ simulations.

The higher the intensity of jumps, the better the empirical fit for this last specification with jumps and regime-switching. These results are very satisfactory in light of the innovative estimation technique proposed in this paper for the Lévy regime switching model.

We obtain similar conclusions when we decrease the intensity of jumps.

6 Empirical fit

In this section, we propose an empirical application to financial and commodity time series that are well-known to exhibit a jump-type behavior [Jorion (1988), Chen and Insley (2012)]. Besides, we test the relative performance of the regime-switching Lévy model vs. other regime-switching and non-switching models.

The data is retrieved from Thomson financial datastream over the period going from January 25, 1983 to January 25, 2013 with a weekly frequency, totaling 1118 observations. The characteristics for each time series are given in Table 12. Note that GSCI stands for the Goldman Sachs Commodity Index (with specific sub-indices for agricultural products, energy prices and industrial metals).

Figure 1 displays the raw data. For each time series, a table reports the results of: (i) the set of diffusion parameters, and (ii) the NIG density parameters of the Lévy jump process fitted to each regime. The remaining problem in this work is to specify the number of regimes in the Markov chain. For simplicity, we proceed with two regimes that relate to the ‘boom’ and ‘bust’ phases of the business cycle.

We also report a plot where each regime is reported with a different color [e.g. blue (red) corresponds to regime 1 (regime 2)]. To provide the reader with a clearer picture, we have chosen to plug the regimes

Table 12: Description of the time series.

Variable	Description
Financial markets	
S&P 500	Standard & Poor's 500 Equity index
Commodity markets	
GSCI agricultural	GSCI sub-index for agricultural products
GSCI energy	GSCI sub-index for energy prices
GSCI industrial metals	GSCI sub-index for industrial metals

Figure 1: Raw data.

From left to right: the top panel represents the S&P 500 and S&P GSCI agricultural index. The bottom panel represents the S&P GSCI energy and S&P GSCI industrial metals indices. The source of the data is Thomson Financial Datastream.

identified back into the raw (non-stationary) data. Of course, all the estimates were performed on log-returns $X_t := \log(P_t) - \log(P_{t-1})$, e.g. stationary data. Below this first plot, the filtered and smoothed probabilities are displayed. They reflect the regime switches at stake.

6.1 Equities

In Table 13, the parameter α represents the jump intensity. The lower the α , the higher the jump intensity in a given regime. We observe that $\alpha=0.01$ during regime 2, which indicates a high jump intensity. On the contrary, $\alpha=1.20$ during regime 1, which points out a low jump intensity.

Inspecting the top panel of Figure 2, we observe that the regime 1 is characterized by a rather stable price path for the S&P 500 index from January 1983 to May 1996. Visually, the model captures one main regime switch in June 1996. Beyond that point, the data enters the regime 2 which is characterized by ups and downs, and thus a higher jump activity.

β is the skewness parameter: $\beta < 0$ ($\beta > 0$) implies a density skewed to the left (right). The skewness of the density increases as β increases. In the case where β is equal to 0, the density is symmetric around μ . During regime 1, $\beta = -0.30$ indicates that there are more positive than negative jumps. During regime 2, $\beta = 0.00$ reflects a balance between positive and negative jumps, as illustrated in Figure 2. δ is the scale parameter representing a measure of the spread of the returns, ranging from 1 to 13 across the two regimes.

In Table 13, of particular interest is the volatility parameter σ : it is more than 20 times higher during regime 2 than during regime 1. The relatively higher jump intensity during regime 2 therefore translates into higher volatility levels. The mean-reverting parameters κ are close to zero in both regimes for the S&P 500.

When inspecting the filtered and smoothed probabilities in the middle and bottom panels of Figure 2, we notice that the probability to stay in the current regime is very high (e.g. close to unity). This information is also visible in the last column of Table 13: the probability to stay in regime 1 (in regime 2) is equal to 99.5% (99.8%). If there is some general form of persistence in the chain (e.g. high probability of staying in a given regime), then this could have important implications for the computation of the value-at-risk and dynamic portfolio allocation, because the benefits of portfolio diversification would be less volatile.

What we learn mainly from these probability graphs is that the stochastic process fitted to each regime does not have the same jumps characteristics during the sample period. Indeed, there are periods of time with an obvious presence of jumps (recorded during regime 2) in the asset price, and others without. Hence, this first set of results illustrates the interest of resorting to the regime-switching Lévy model.

6.2 Agricultural products

Moving to commodities, we start our investigation with the GSCI sub-index for agricultural products. In Table 14, the α parameter is lower during regime 2 ($\alpha=0.08$) than during regime 1 ($\alpha=1.36$). Hence, we find

Table 13: Estimated parameters for S&P 500.

Process	κ	θ	σ	P_{ii}^z
State 1	0.0019 (0.0025)	290.9094 (0.8519)	41.2816 (2.6814)	0.9956
State 2	0.0090 (0.0051)	1282.5548 (6.0167)	996.7976 (55.5512)	0.9983
NIG	α	β	δ	μ
State 1	1.2041 (56.4564)	-0.3071 (0.0048)	1.0001 (2.8219)	0.2638 (2.6200)
State 2	0.0172 (0.1711)	0.0033 (0.1451)	13.2360 (0.7253)	-2.5410 (0.0034)

Standard errors are given in parentheses below parameter estimates.

Figure 2: S&P 500.

Table 14: Estimated parameters for agricultural products.

Process	κ	θ	σ	p_{ii}^z
State 1	0.0095 (0.0048)	214.0224 (0.9996)	19.6198 (0.9579)	0.9897
State 2	0.0129 (0.0100)	408.9781 (3.6866)	265.3126 (22.4640)	0.9722
NIG	α	β	δ	μ
State 1	1.3630 (52.7875)	0.1580 (0.0032)	0.8883 (2.8222)	-0.1037 (2.6186)
State 2	0.0810 (0.6113)	0.0144 (0.3096)	18.1844 (0.6249)	-3.2855 (0.0059)

Standard errors are given in parentheses below parameter estimates.

again a higher jump intensity during the second regime. This finding is confirmed by the higher volatility prevalent during regime 2. The σ parameter is equal to 265 in regime 2 (versus 19 in regime 1). The mean reversion parameter κ is also higher during regime 2, as the agricultural sub-index recovers towards its long-run mean following a jump. During regime 1, $\beta=0.15$ implying that there are more negative than positive jumps. During regime 2, $\beta=0.01$ suggesting another balance between positive and negative jumps.

In Figure 3, we observe that one main regime-switch has occurred in March 2007, concomitantly with the run-up in most commodity prices at that time period. According to the filtered probabilities, there are more frequent regime-switches (from regime 2 to regime 1) towards the end of the sample. Note that the probability of staying in the current regime is very high (above 97%), as indicated in the last column of Table 14. Overall,

Figure 3: S&P GSCI agricultural spot.

we have reached for agricultural commodities the same kind of qualitative comments as in the case of the S&P 500 index, with one tranquil and one turbulent regime. Commodities are agitated by jumps as well, which can be captured adequately – alongside the business cycle fluctuations – by the regime-switching Lévy model.

6.3 Energy

Partly repeating some of the discussion above, there are a number of common features across these results. The GSCI energy sub-index exhibits a higher jump intensity during regime 2, as indicated by the α parameter close to 0.09 in Table 15. This jump intensity is associated with an important level of volatility: the σ parameter is equal to 153 during regime 2. During both regimes, the β coefficient is found to be negative, which suggests a slight asymmetry in favor of positive (rather than negative) jumps.

Visually, we observe in the top panel of Figure 4 the presence of one large jump in this time series, mirroring the effects of the July 2008 oil price swing. During 2000–2001, more frequent switches are visible from regime 1 to regime 2 according to the filtered and smoothed probabilities. Although the Lévy jump process behaves differently during both regimes (e.g. with a more stable regime 1 compared to the agitated regime 2), the probability to stay in the current regime is very high (above 98%) according to the last column of Table 15.

Past November 2001, the GSCI energy sub-index enters regime 2, which is characterized by a higher volatility level and jumps. This result can be explained by the de-regulation of commodity markets following the Commodity Futures Modernization Act (CFMA) signed by President Clinton in December 2000. Since then, more and more non-commercial players have entered commodity markets (such as hedge funds or investment banks proprietary trading), with rising levels of risks and prices.

6.4 Industrial metals

Industrial metals are characterized by a high jump intensity during regime 2 ($\alpha=0.08$ in Table 16), associated with a high level of volatility ($\sigma=223$ in Table 16) and a higher level of mean-reversion ($\kappa=0.0104$). The probability to stay in the current regime is above 99%. During regime 1, the positivity of the β coefficient ($=0.16$) highlights that there are more negative than positive jumps. During regime 2, β is approximately equal to zero ($=-0.0033$), which suggests that the density is centered around the long-term mean.

In Figure 5, we observe one main jump occurring during the summer 2008 for the GSCI industrial metals sub-index, with a similar interpretation as in the energy case (in fact most commodity prices dropped during that time period). There are mainly two regime switches (June 1988–February 1991 and 2006–2013), the main one occurring towards the end of the sample period. The jump activity is quite remarkable for this time series, similarly to the jumps detected in the GSCI energy sub-index previously.

Across these empirical applications, we notice that the transition probabilities are generally smoother for the S&P 500 and commodities (especially the latest category relating to industrial metals). One interesting

Table 15: Estimated parameters for energy.

Process	κ	θ	σ	P_{ii}^z
State 1	0.0160 (0.0061)	73.3400 (0.4569)	7.5799 (0.4115)	0.9882
State 2	0.0094 (0.0065)	280.5037 (1.5607)	153.8423 (10.3807)	0.9840
NIG	α	β	δ	μ
State 1	1.0696 (14.6059)	-0.0396 (0.0018)	0.6546 (1.6527)	0.0242 (2.7937)
State 2	0.0923 (0.2798)	-0.0197 (0.1594)	12.9941 (0.4451)	2.8423 (0.0104)

Standard errors are given in parentheses below parameter estimates.

Figure 4: S&P GSCI energy.

Table 16: Estimated parameters for industrial metals.

Process	κ	θ	σ	p_{ii}^z
State 1	0.0032 (0.0038)	173.3371 (0.5903)	13.9216 (0.7368)	0.9954
State 2	0.0104 (0.0064)	373.4790 (2.1929)	223.8479 (15.7486)	0.9943
NIG	α	β	δ	μ
State 1	1.4724 (24.7672)	0.1640 (0.0017)	1.2667 (2.8714)	-0.1420 (2.5911)
State 2	0.0862 (0.2009)	-0.0033 (0.2473)	18.0277 (0.4609)	-0.6983 (0.0063)

Standard errors are given in parentheses below parameter estimates.

Figure 5: S&P GSCI industrial metals.

implication can be drawn from the regime-switching Lévy model in these specific case studies: if the transition probabilities are smoother, then the gain from portfolio diversification will also be smoother (which might imply a smoother pattern for the VaR and portfolio weights).

6.5 Benchmark models

In this subsection, we compare various regime-switching models against their one-regime counterparts. To this end, we conduct a ‘horse race’ between the following models:

– Regime-switching Lévy

It is the regime-switching model given by (2) where the process $Y \equiv L$ is a Lévy process with $L_1 \sim \text{NIG}(\alpha, \beta, \delta, \mu)$.

$$dX_t = \kappa(Z_t)(\theta(Z_t) - X_t)dt + \sigma(Z_t)dL_t$$

– Regime-switching Gaussian

It is the regime-switching model given by (2) where the process $Y \equiv W$ is a Brownian motion.

$$dX_t = \kappa(Z_t)(\theta(Z_t) - X_t)dt + \sigma(Z_t)dW_t$$

– Lévy

It is the model given by (2) where the process $Y \equiv L$ is a Lévy process with $L_1 \sim \text{NIG}(\alpha, \beta, \delta, \mu)$ and where there are no regime switches. Consequently, the corresponding stochastic differential equation does not depend on the Markov chain Z .

$$dX_t = \kappa(\theta - X_t)dt + \sigma dL_t$$

– Gaussian

This model is typically a Vasicek model, given by (2), where the process $Y \equiv W$ is a Brownian motion without regime switches. The corresponding stochastic differential equation does not depend on the Markov chain Z either.

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

To evaluate the goodness-of-fit of these competing models, we report the log-likelihood values obtained by each model with the EM-algorithm. Because the regime-switching Lévy model and the competing models are not directly nested, note we cannot perform a likelihood-ratio test to verify whether the increase in the likelihood is significant. Furthermore, we calculate the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) given by

$$\text{AIC} = -2\ln(L(\hat{\Theta})) + 2k \quad \text{and} \quad \text{BIC} = -2\ln(L(\hat{\Theta})) + k\ln(n), \quad (16)$$

where $L(\hat{\Theta})$ is the log-likelihood value obtained with the estimated parameters $\hat{\Theta}$ found with our two-step procedure, k is the degree of freedom of each model, and n the number of observations. The preferred model minimizes the AIC or BIC values.

What is interesting is to compare the log-likelihood of the different models (all the results are in Table 17). For the S&P 500, the likelihood of the RS Lévy model is 1709 points higher (in absolute value) than that of the RS Gaussian model, which is a very large improvement. Hence, we assess that the highest performance is reached for the jump-diffusion regime-switching model. This comment applies for all time series. We can also compare our model with the class of non regime-switching models. In Table 17, we notice that the regime-switching Lévy model delivers consistently higher log-likelihood values. Regarding the magnitude of the changes in the log-likelihood for the GSCI energy, they are reflecting the occurrence of many jumps with a high volatility level. Those jumps have indeed a dramatic impact on the log-likelihood for this series.

Another interesting comparison is possible in Table 18 across the information criteria. In the case of the S&P 500, if we compare our model with other competitors, we notice that the lowest values of the AIC and BIC are systematically obtained by the regime-switching Lévy. This result confirms the first diagnostic based on the log-likelihood in the previous table. The same comment holds for other time series.

Nevertheless, we need to weigh these diagnostic tests against those delivered by regime-switching classification indicators. Indeed, a given model could fit the data well and obtain a high log-likelihood value. However, it could perform poorly at discriminating the data into the corresponding regimes. That is why we introduce below some regime-switching classification indicators.

Table 17: Log likelihood values.

	RS-Lévy	RS-Gaussian	Gaussian	Lévy
Financial markets				
S&P 500	−2977.70	−4687.36	−5160.77	−7808.78
Commodity markets				
GSCI agricultural	−1550.80	−3596.69	−4048.74	−3794.30
GSCI industrial metals	−204.96	−3364.07	−3926.52	−3650.79
GSCI energy	−233.66	−3611.60	−4106.69	−5559.39

Bold characters indicate the best model.

Table 18: Akaike information criterion (AIC) and Bayesian information criterion (BIC).

	RS-Lévy		RS-Gaussian		Lévy		Gaussian	
	BIC	AIC	BIC	AIC	BIC	AIC	BIC	AIC
Financial markets								
S&P 500	6011.6	5971.4	9430.9	9390.7	15645.6	15625.5	10342.6	10327.5
Commodity markets								
GSCI agricultural	3157.8	3117.6	7249.5	7209.4	7616.7	7596.6	8118.5	8103.5
GSCI industrial metals	466.1	425.9	6784.3	6744.1	7329.6	7309.6	7874.1	7859.0
GSCI energy	523.5	483.3	7279.4	7239.2	1114.7	11126.8	8234.4	8219.4

Bold characters indicate the best model.

6.6 Regime-switching classification

An ideal model is one that classifies the regimes sharply, with smoothed probabilities close to either zero or one. In order to measure the quality of regime classification, we propose the following two measures:

- (1) The regime classification measure (RCM) introduced by Ang and Bekaert (2002) and generalized for multiple states by Baele (2005).

$$\text{RCM}(K) = 100 \cdot \left(1 - \frac{K}{K-1} \frac{1}{T} \sum_{k=1}^N \sum_{Z_{t_k}} \left(P(Z_{t_k} = i | \mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)}) - \frac{1}{K} \right)^2 \right), \quad (17)$$

where the quantity $P(Z_{t_k} = i | \mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)})$ is the smoothed probability and $\hat{\Theta}_1^{(n)}$ is the vector of parameters estimated. The constant serves to normalize the statistic to be between 0 and 100. Good regime classification is associated with low RCM statistic value: a value of 0 means perfect regime classification, and a value of 100 implies that no information about the regimes is revealed. Consequently, even if a model has the highest log-likelihood value, its RCM needs to be close to zero.

- (2) The smoothed probability indicator: a good classification for the data can be achieved when the smoothed probability is less than 0.1 or greater than 0.9. This means that the data at time $t \in [0, T]$ is in one of the regimes at the 10% error level.

Table 19: Regime classification measure (RCM) and smoothed probability indicator.

	RCM	$p^{10\%}$
Financial markets		
S&P 500	0.70	99.37
Commodity markets		
GSCI agricultural	7.60	92.22
GSCI industrial metals	4.69	95.08
GSCI energy	2.61	97.50

Table 19 contains the corresponding indicators for each time series.

For the S&P 500, we observe that the RCM statistic is close to zero (0.70). In that case, the regime-switching Lévy model is able to discriminate perfectly between the two regimes. Similar conclusions can be reached for other time series, with RCM statistics varying from 3 to 8 for commodities.

In the second column of 19, we notice that the smoothed probability indicator is equal to 99.37% for the S&P 500. Again, we conclude that the discrimination between regime is accurately performed by the model, as it is very close to the upper bound of 100%. The same comment applies for commodities (with probabilities above 92%). With this battery of diagnostic tests, we have established the robustness of the results obtained with the Lévy regime-switching model.

7 Conclusion

Present work is devoted to the estimation of parameters of a regime-switching model driven by a latent Markov chain which influences the parameters of an endogenously driven Lévy process. The resulting model merges two strands of literatures: jump-diffusion under the form of a Lévy process, and Markov regime-switching. After explaining the motivation and the model, we discretize the dynamics to estimate the parameters using the EM-algorithm. The technique is illustrated on two-state models using real-world index prices, for which contrasted jump dynamics are identified.

Besides its appealing features, the regime-switching Lévy model allows analytic computation and follows a two-step estimation procedure. In a first step, we estimate the stochastic diffusion parameters in presence of regime-switching. The evaluation of the log-likelihood is achieved by extending Hamilton's EM-algorithm (because of the unobserved Markov chain). In a second step, we estimate the density parameters. We assume that the Lévy process follows a NIG distribution, and its dependence is given. Conditionally on the first step estimates, two separate Lévy processes are fitted (one for each regime) with separate NIG distributions. We further study the properties of the our regime-switching Lévy estimation procedure based on simulated processes.

An application of this model to financial and commodity indices illustrates its good behavior. A comparison of our regime-switching Lévy model with (i) non regime-switching models, and (ii) continuous regime-switching models (where the Lévy process is replaced by a classic Brownian motion) shows that our model overall achieves a better performance. We have a strong idea about which regime we are in at every point in time as the smoothed probabilities are close to either zero or one most of the time. Examining more closely the transition matrix, we document a high persistence to stay in the current regime (above 90%), which can be useful for portfolio diversification. Extensions to the pricing of derivatives and hedging strategies can be pursued.

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