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Bayesian multivariate Beveridge–Nelson decomposition of I(1) and I(2) series with cointegration

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Abstract: The dynamic IS equation implies that if the real interest rate is I(1), then so is the output growth rate with possible cointegration, and log output is I(2). This paper extends the Beveridge–Nelson decomposition to such a case, and develops a Bayesian method to obtain error bands. The method is valid whether log output is I(1) or I(2). The paper applies the method to US data to estimate the natural rates (or their permanent components) and gaps of output, inflation, interest, and unemployment jointly, and finds that allowing for cointegration gives much bigger estimates of the gaps for all variables.

Keywords: natural rate; output gap; trend-cycle decomposition; trend inflation; unit root; vector error correction model (VECM).

1 Introduction

Distinguishing between growth and cycles is fundamental in macroeconomics. One can define growth as the time-varying steady state, or the permanent component, and cycles as deviations from the steady state, or the transitory component. One may interpret the permanent and transitory components as the natural rate and gap, respectively, though some economists may disagree with such interpretation, in which case one can consider the permanent component of the natural rate.¹ If shocks affecting the two components differ, then policy prescriptions for promoting growth and stabilizing cycles differ. Thus it is useful to decompose economic fluctuations into the two components.

Among such decomposition methods, this paper focuses on the multivariate Beveridge–Nelson (B–N) decomposition, which decomposes a multivariate I(1) or CI(1,1) series into a random walk permanent component and an I(0) transitory component, assuming a linear time series model such as a VAR model or a vector error-correction model (VECM) for the differenced series. In practice, however, some series may be I(2), e.g., log output in some countries, in which case one must decompose I(1) and I(2) series jointly. Murasawa (2015) develops the multivariate B–N decomposition of I(1) and I(2) series.

As Murasawa (2015) shows, for non-US data, the B–N decomposition assuming I(1) log output often gives an unreasonable estimate of the output gap, perhaps because of possible structural breaks in the mean output growth rate; see Figure 1. Kamber, Morley, and Wong (2018, p. 563) explain,

¹ Phelps (1995) defines the natural rate as the current stationary rate (p. 16) or the equilibrium steady state path (pp. 29–30). Woodford (2003, pp. 8–9) defines the natural rate as the equilibrium rate under flexible prices, which may not be in the steady state. Kiley (2013) compares alternative definitions empirically using a new Keynesian DSGE model for the US, and obtains similar estimates of the output gap. Laubach and Williams (2016) and Holston et al. (2017) note that these long-run and short-run views are complementary. Del Negro et al. (2017) also distinguish the natural rate and its low-frequency component. See also Glick (2020, sec. 2.2) for alternative definitions of the natural rate of interest r^* .

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Figure 1: Output gap estimates in Japan given by the multivariate B–N decomposition assuming I(1) or I(2) log output. The plots replicate those in Murasawa (2015, Figures 2 and 3). The shaded areas indicate the recessions determined by the Cabinet Office.

... if there is a large reduction in the long-run growth rate, a forecasting model that fails to account for it will keep anticipating faster growth than actually occurs after the break, leading to a persistently negative estimate of the output gap based on the BN decomposition.²

Assuming I(2) log output and hence I(1) output growth rate, one introduces a stochastic trend in the output growth rate, which captures possible structural breaks in the mean growth rate automatically in real time without specifying break dates *a priori*. Thus the B–N decomposition assuming I(2) log output gives a more ‘reasonable’ estimate of the output gap that fluctuates around 0; see Figure 1.

This paper extends Murasawa (2015) in two ways. First, we allow for cointegration in the multivariate B–N decomposition of I(1) and I(2) series. Recall that the consumption Euler equation in a simple macroeconomic model implies a dynamic IS equation such that for all t ,

$$E_t(\Delta \ln Y_{t+1}) = \frac{1}{\sigma}(r_t - \rho) \quad (1)$$

where Y_t is output, r_t is the real interest rate, ρ is the discount rate, and σ is the curvature of the utility of consumption; see Galí (2015, pp. 21–23). This equation implies that if $0 < \sigma < \infty$, then the output growth rate and the real interest rate are of the same order of integration; thus if the real interest rate is I(1), then so is the output growth rate with possible cointegration, and log output is I(2). This observation motivates our development of the multivariate B–N decomposition of I(1) and I(2) series with cointegration.^{3,4}

Second, we apply Bayesian analysis to obtain error bands for the components, building on recent developments in Bayesian analysis of a VECM. Since cointegrating vectors require normalization, our parameter of interest is in fact the cointegrating space rather than cointegrating vectors. Strachan and Inder (2004) use a matrix angular central Gaussian (MACG) distribution proposed by Chikuse (1990) as a prior on the cointegrating space. Koop, León-González, and Strachan (2010) propose two Gibbs samplers, a collapsed Gibbs sampler and a parameter-augmented Gibbs sampler, for posterior simulation of such a model. Since one often has prior information on the steady state of a system, Villani (2009) specifies a prior on the steady state form of a VECM. Since some hyperparameters such as the tightness (shrinkage) hyperparameter on the

² In this quote, Kamber et al. (2018) seem to consider the *output growth rate gap*. If one fails to account for a large reduction in the true mean growth rate μ^* , then the output growth rate $\Delta \ln Y_t$ tends to be below the assumed mean growth rate μ ; i.e., the output growth rate gap $\Delta \ln Y_t - \mu$ indeed tends to be negative. If log output is I(1), so that the output growth rate is I(0), then with positive serial correlation, the future $\Delta \ln Y_t - \mu$ tends to be negative, implying that the current output gap is *positive*, as shown in Figure 1.

³ Since the B–N decomposition requires only a reduced-form forecasting model, we do not assume Eq. (1), which is just a motivating example for considering some cointegration. Though we assume that $\{\Delta \ln Y_t\}$ is I(1), our method is valid even if it is I(0), since we can treat an I(0) series as an I(1) series cointegrated with itself, i.e., ‘pseudo’ cointegration; see Fisher et al. (2016, section 2.2).

⁴ One can think of this paper as a reduced-form VECM version of Laubach and Williams (2003), though we include more variables. To estimate the natural rate of interest, they start from the consumption Euler equation, and assume I(2) log output. In one specification, they also assume cointegration between the output growth rate and the real interest rate.

VAR coefficients are difficult to choose, Giannone, Lenza, and Primiceri (2015) use a hierarchical prior. Using these ideas, we develop a Gibbs sampler to simulate the joint posterior distribution of the components.⁵

As an application, we simulate the joint posterior distribution of the natural rates (or their permanent components) and gaps of output, inflation, interest, and unemployment in the US during 1950Q1–2018Q4. To apply the Bayesian multivariate B–N decomposition of I(1) and I(2) series with cointegration, we assume a four-variate VAR model for the output growth rate, the CPI inflation rate, the short-term interest rate, and the unemployment rate, and estimate it in the VECM form. The Bayes factors give decisive evidences that the cointegrating rank is 2. The posterior medians of the gaps seem reasonable compared to previous works that focus on a particular natural rate or gap. The posterior probability of positive gap is useful when the sign of the gap is uncertain. The Phillips curve and Okun's law hold between the gaps despite that we do not impose such relations. The dynamic IS equation holds between the natural rates of output and interest, though the evidence is weak. Comparisons of alternative model specifications show not only that assuming I(2) log output gives a more 'reasonable' estimate of the output gap, but also that allowing for cointegration gives much bigger estimates of the gaps for all variables.

The paper proceeds as follows. Section 2 reviews the literature on the B–N decomposition. Section 3 derives the multivariate B–N decomposition of I(1) and I(2) series with cointegration. Section 4 specifies our model and prior, and explains our posterior simulation and model evaluation. Section 5 applies the method to US data. Section 6 discusses remaining issues. The Appendix gives the details of the derivation of our algorithm. We use the notation proposed by Abadir and Magnus (2002).

2 Literature

Beveridge and Nelson (1981) give operational definitions of the permanent and transitory components, show that one can express any I(1) series as the sum of a random walk permanent component and an I(0) transitory component, and propose the B–N decomposition of a univariate I(1) series, assuming an ARIMA model.⁶

Multivariate extension of the B–N decomposition is straightforward. Evans (1989a, 1989b) and Evans and Reichlin (1994) apply the B–N decomposition to a multivariate series consisting of I(0) and I(1) series, assuming a VAR model for the stationarized series. Evans and Reichlin (1994) show that the transitory components are no smaller with the multivariate B–N decomposition than with the univariate one. This is because the transitory components are 'forecastable movements' (Rotemberg and Woodford 1996), and multivariate models forecast no worse than univariate models, using more information. King et al. (1991) and Cochrane (1994) apply the B–N decomposition to a CI(1,1) series, assuming a VECM.

Morley (2002) gives a general framework for the B–N decomposition, using a state space representation of the assumed linear time series model. Garratt, Robertson, and Wright (2006) note that if the state vector is observable as in a VAR model or a VECM, then the transitory component is an explicit weighted sum of the observables given the model parameters; thus the multivariate B–N decomposition based on a VAR model or a VECM is transparent. They also note that the result of the multivariate B–N decomposition depends strongly on the assumed cointegrating rank.

The B–N decomposition also applies to an I(2) series. Newbold and Vougas (1996), Oh and Zivot (2006), and Oh, Zivot, and Creal (2008) extend the B–N decomposition to a univariate I(2) series. Murasawa (2015) extends the method to a multivariate series consisting of I(1) and I(2) series.

One can apply Bayesian analysis to obtain error bands for the components. This approach is useful especially when the state vector is observable as in a VAR model or a VECM, in which case the components are

⁵ Since we assume a VECM as the true reduced-form model, the B–N decomposition *defines* the permanent and transitory components; hence it suffices for our error bands to reflect only parameter uncertainty. If we instead assume an unobserved components (UC) model as the true model, then the B–N decomposition *estimates* the components, and we must consider estimation errors as well. See Morley (2011) on this distinction.

⁶ Beveridge and Nelson (1981) reverse the sign of the transitory component. Nelson (2008) explains why they had to do so.

explicit functions of the model parameters and observables; thus the joint posterior distribution of the model parameters directly translates into that of the components. Murasawa (2014) uses a Bayesian VAR model to obtain error bands for the components.

Kiley (2013) uses a Bayesian DSGE model, but gives no error band for the components. Del Negro et al. (2017) use a Bayesian DSGE model and give error bands for the components. They also use a multivariate unobserved components (UC) model, where the permanent components have a factor structure and the transitory components follow a VAR model. Bayesian analysis of a UC model requires state smoothing, since the state vector is unobservable given the model parameters; thus it is less straightforward than that of a VAR model. Morley and Wong (2020) use a large Bayesian VAR model and give error bands for the components, but they do not take parameter uncertainty into account.⁷ There seems no previous work that uses a Bayesian VECM to obtain error bands for the components.⁸

3 Model specification

3.1 VAR model

Let for $d = 1, 2$, $\{x_{t,d}\}$ be an N_d -variate $I(d)$ sequence. Let $N := N_1 + N_2$. Let for all t , $x_t := (x'_{t,1}, x'_{t,2})'$, $y_{t,1} := x_{t,1}$, $y_{t,2} := \Delta x_{t,2}$, and $y_t := (y'_{t,1}, y'_{t,2})'$, where the prime ($'$) denotes the transpose of a vector or a matrix, so that $\{y_t\}$ is $I(1)$. Assume also that $\{y_t\}$ is $CI(1,1)$ with cointegrating rank r .⁹ Let for $d = 1, 2$, $\mu_d := E(\Delta y_{t,d})$. Let $\mu := (\mu'_1, \mu'_2)'$. Let $\{y_t^*\}$ be such that for all t ,

$$y_t = \alpha + \mu t + y_t^* \quad (2)$$

where α is an intercept vector. Assume a $\text{VAR}(p+1)$ model for $\{y_t^*\}$ such that for all t ,¹⁰

$$\Pi(L)y_t^* = u_t \quad (3)$$

$$\{u_t\} \sim \text{WN}(\Sigma) \quad (4)$$

where $\Pi(L)$ is a $(p+1)$ th-order polynomial matrix in the lag operator L and $\text{WN}(\Sigma)$ means ‘(multivariate) white noise sequence with variance–covariance matrix Σ ’.

3.2 VECM representation

Write

$$\Pi(L) = \Pi(1)L + \Phi(L)(1-L)$$

⁷ Morley and Wong (2020) assume a UC model as the true model, and interpret the B–N decomposition as a way to estimate the components; hence their error bands reflect estimation errors of the components, but not parameter uncertainty.

⁸ Cogley et al. (2010) use a Bayesian time-varying parameter VAR model, which is a nonlinear time series model. They still apply the B–N decomposition and give error bands for the components, pretending at each date that the VAR coefficients no longer vary. They justify their approach as an approximation based on an ‘anticipated-utility’ model. See also Cogley and Sargent (2002, 2005) and Cogley et al. (2005).

⁹ Our assumption includes the case where some series in $\{y_t\}$ are $I(0)$, since we can treat an $I(0)$ series as an $I(1)$ series cointegrated with itself, i.e., ‘pseudo’ cointegration; see Fisher, Huh, and Pagan (2016, section 2.2).

¹⁰ By starting from a VAR model for $I(1)$ series $\{y_t\}$, we exclude cointegration among $I(2)$ series $\{x_{t,2}\}$ nor multicointegration (polynomial cointegration) studied by Johansen (1995), Granger and Lee (1989, 1990), and Gregoir and Laroque (1993, 1994). We leave extensions of our Bayesian multivariate B–N decomposition to such cases for future work.

where $\Phi(L) := (\Pi(L) - \Pi(1)L)/(1 - L)$. Then we have a VECM of order p for $\{\Delta y_t^*\}$ such that for all t ,

$$\begin{aligned}\Phi(L)\Delta y_t^* &= -\Pi(1)y_{t-1}^* + u_t \\ &= -\Lambda\Gamma'y_{t-1}^* + u_t\end{aligned}\quad (5)$$

where $\Lambda, \Gamma \in \mathbb{R}^{N \times r}$. Since $\{y_t^*\}$ is CI(1,1), the roots of $\det(\Pi(z)) = 0$ must lie on or outside the unit circle. This requirement gives an implicit restriction on the VECM parameters $(\Phi(\cdot), \Lambda, \Gamma)$.

We can write for all t ,

$$\begin{aligned}\Phi(L)(\Delta y_t - \mu) &= -\Lambda\Gamma'[y_{t-1} - \alpha - \mu(t-1)] + u_t \\ &= -\Lambda[\Gamma'y_{t-1} - \beta - \delta(t-1)] + u_t\end{aligned}\quad (6)$$

where $\beta := \Gamma'\alpha$ and $\delta := \Gamma'\mu$. Though slightly different, the last expression is essentially a steady state VECM suggested by Villani (2009, p. 633). We have for all t ,

$$E(\Delta y_t) = \mu \quad (7)$$

$$E(\Gamma'y_t) = \beta + \delta t \quad (8)$$

which help us to specify an informative prior on (μ, β, δ) . Thus a steady state VECM is useful for Bayesian analysis.

3.3 State space representation

Assume that $p \geq 1$. We have for all t ,

$$\begin{aligned}\Gamma'y_t^* &= \Gamma'(y_{t-1}^* + \Phi_1\Delta y_{t-1}^* + \cdots + \Phi_p\Delta y_{t-p}^* - \Lambda\Gamma'y_{t-1}^* + u_t) \\ &= \Gamma'\Phi_1\Delta y_{t-1}^* + \cdots + \Gamma'\Phi_p\Delta y_{t-p}^* + (I_r - \Gamma'\Lambda)\Gamma'y_{t-1}^* + \Gamma'u_t\end{aligned}$$

or

$$\begin{aligned}\Gamma'y_t - \beta - \delta t &= \Gamma'\Phi_1(\Delta y_{t-1} - \mu) + \cdots + \Gamma'\Phi_p(\Delta y_{t-p} - \mu) \\ &\quad + (I_r - \Gamma'\Lambda)[\Gamma'y_{t-1} - \beta - \delta(t-1)] + \Gamma'u_t\end{aligned}\quad (9)$$

Let s_t be a state vector such that for all t ,

$$s_t := \begin{pmatrix} \Delta y_t - \mu \\ \vdots \\ \Delta y_{t-p+1} - \mu \\ \Gamma'y_t - \beta - \delta t \end{pmatrix}$$

which is I(0) and observable given the model parameters. A state space representation of the steady state VECM is for all t ,

$$s_t = As_{t-1} + Bs_t \quad (10)$$

$$\Delta y_t = \mu + Cs_t \quad (11)$$

$$\{z_t\} \sim \text{WN}(I_N) \quad (12)$$

where

$$A := \begin{bmatrix} \Phi_1 & \cdots & \Phi_p & -\Lambda \\ & I_{(p-1)N} & \mathbf{0}_{(p-1)N \times N} & \mathbf{0}_{(p-1)N \times r} \\ \Gamma'\Phi_1 & \cdots & \Gamma'\Phi_p & I_r - \Gamma'\Lambda \end{bmatrix}$$

$$\mathbf{B} := \begin{bmatrix} \Sigma^{1/2} \\ \mathbf{0}_{(p-1)N \times N} \\ \Gamma' \Sigma^{1/2} \end{bmatrix}$$

$$\mathbf{C} := \begin{bmatrix} \mathbf{I}_N & \mathbf{0}_{N \times (p-1)N} & \mathbf{0}_{N \times r} \end{bmatrix}$$

with $\mathbf{0}_{m \times n}$ denoting an $m \times n$ null matrix. Note that $\{\mathbf{s}_t\}$ is I(0) if and only if the eigenvalues of $\mathbf{A}$ lie inside the unit circle.

We have for all t , for $h \geq 0$,

$$\mathbb{E}_t(\Delta \mathbf{y}_{t+h}) = \boldsymbol{\mu} + \mathbf{C}\mathbf{A}^h \mathbf{s}_t$$

or

$$\mathbb{E}_t(\Delta \mathbf{x}_{t+h,1}) = \boldsymbol{\mu}_1 + \mathbf{C}_1 \mathbf{A}^h \mathbf{s}_t \quad (13)$$

$$\mathbb{E}_t(\Delta^2 \mathbf{x}_{t+h,2}) = \boldsymbol{\mu}_2 + \mathbf{C}_2 \mathbf{A}^h \mathbf{s}_t \quad (14)$$

where

$$\mathbf{C}_1 := \begin{bmatrix} \mathbf{I}_{N_1} & \mathbf{0}_{N_1 \times N_2} & \mathbf{0}_{N_1 \times (p-1)N} & \mathbf{0}_{N_1 \times r} \end{bmatrix}$$

$$\mathbf{C}_2 := \begin{bmatrix} \mathbf{0}_{N_2 \times N_1} & \mathbf{I}_{N_2} & \mathbf{0}_{N_2 \times (p-1)N} & \mathbf{0}_{N_2 \times r} \end{bmatrix}$$

3.4 Multivariate B–N decomposition

Introducing cointegration changes the state space model, but the formulae for the multivariate B–N decomposition of I(1) and I(2) series given by Murasawa (2015, Theorem 1) remain almost unchanged. Let $\mathbf{x}_t^*$ and $\mathbf{c}_t$ be the B–N permanent and transitory components in $\mathbf{x}_t$, respectively.

Theorem 1. Suppose that the eigenvalues of $\mathbf{A}$ lie inside the unit circle. Then for all t ,

$$\mathbf{x}_{t,1}^* = \lim_{T \rightarrow \infty} (\mathbb{E}_t(\mathbf{x}_{t+T,1}) - T\boldsymbol{\mu}_1) \quad (15)$$

$$\mathbf{x}_{t,2}^* = \lim_{T \rightarrow \infty} \left\{ \mathbb{E}_t(\mathbf{x}_{t+T,2}) - T^2 \frac{\boldsymbol{\mu}_2}{2} - T \left[\frac{\boldsymbol{\mu}_2}{2} + \Delta \mathbf{x}_{t,2} + \mathbf{C}_2(\mathbf{I}_{pN+r} - \mathbf{A})^{-1} \mathbf{A} \mathbf{s}_t \right] \right\} \quad (16)$$

$$\mathbf{c}_{t,1} = -\mathbf{C}_1(\mathbf{I}_{pN+r} - \mathbf{A})^{-1} \mathbf{A} \mathbf{s}_t \quad (17)$$

$$\mathbf{c}_{t,2} = \mathbf{C}_2(\mathbf{I}_{pN+r} - \mathbf{A})^{-2} \mathbf{A}^2 \mathbf{s}_t \quad (18)$$

Proof. See Murasawa (2015, pp. 158–159). □

Let

$$\mathbf{W} := \begin{bmatrix} -\mathbf{C}_1(\mathbf{I}_{pN+r} - \mathbf{A})^{-1} \mathbf{A} \\ \mathbf{C}_2(\mathbf{I}_{pN+r} - \mathbf{A})^{-2} \mathbf{A}^2 \end{bmatrix}$$

Then for all t ,

$$\mathbf{c}_t = \mathbf{W} \mathbf{s}_t \quad (19)$$

where $\mathbf{W}$ depends only on the VECM coefficients and $\{\mathbf{s}_t\}$ is observable given the model parameters. This observation is useful for Bayesian analysis of $\{\mathbf{c}_t\}$.

4 Bayesian analysis

4.1 Conditional likelihood function

Assume Gaussian innovations for Bayesian analysis, and write the VECM as for all t ,

$$\Delta \mathbf{y}_t - \boldsymbol{\mu} = \Phi_1(\Delta \mathbf{y}_{t-1} - \boldsymbol{\mu}) + \cdots + \Phi_p(\Delta \mathbf{y}_{t-p} - \boldsymbol{\mu}) - \Lambda[\Gamma' \mathbf{y}_{t-1} - \boldsymbol{\beta} - \Gamma' \boldsymbol{\mu}(t-1)] + \mathbf{u}_t \quad (20)$$

$$\{\mathbf{u}_t\} \sim \text{IN}_N(\mathbf{0}_N, \mathbf{P}^{-1}) \quad (21)$$

where $\text{IN}_N(\mathbf{0}_N, \mathbf{P}^{-1})$ means ‘independent N -variate normal distributions with mean vector $\mathbf{0}_N$ (null vector) and variance–covariance matrix $\mathbf{P}^{-1}$ ’. Let $\boldsymbol{\psi} := (\boldsymbol{\beta}', \boldsymbol{\mu}')'$, $\boldsymbol{\Phi} := [\Phi_1, \dots, \Phi_p]$, and $\mathbf{Y} := [\mathbf{y}_0, \dots, \mathbf{y}_T]$.¹¹ By the prediction error decomposition, the joint pdf of $\mathbf{Y}$ is¹²

$$\begin{aligned} p(\mathbf{Y}|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma) &= p(\Delta \mathbf{y}_T, \dots, \Delta \mathbf{y}_{p+1}, \mathbf{s}_p|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma) \\ &= \prod_{t=p+1}^T p(\Delta \mathbf{y}_t|\mathbf{s}_{t-1}, \boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma) p(\mathbf{s}_p|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma) \end{aligned} \quad (22)$$

Our Bayesian analysis relies on $\prod_{t=p+1}^T p(\Delta \mathbf{y}_t|\mathbf{s}_{t-1}, \boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma)$, which is equivalent to the conditional likelihood function of $(\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma)$ given $\mathbf{s}_p$.

4.2 Identification

Identification of (Λ, Γ) from $\text{II}(1)$ requires some restrictions, though such identification is unnecessary for the multivariate B–N decomposition. Let

$$\Lambda_* := \Lambda(\Gamma' \Gamma)^{1/2}, \quad \Gamma_* := \Gamma(\Gamma' \Gamma)^{-1/2}$$

Then $\Lambda_* \Gamma_*' = \Lambda \Gamma'$ and $\Gamma_*' \Gamma_* = \mathbf{I}_r$. These restrictions are common, but do not identify the signs of Λ_* and Γ_* .

Alternatively, we can apply linear normalization. Write $\Gamma := [\Gamma_1', \Gamma_2']'$, where Γ_1 is $r \times r$ and Γ_2 is $(N-r) \times r$. Let

$$\bar{\Lambda} := \Lambda \Gamma_1, \quad \bar{\Gamma} := \Gamma \Gamma_1^{-1}$$

Then we can identify $(\bar{\Lambda}, \bar{\Gamma})$. Let $\bar{\boldsymbol{\beta}} := \bar{\Gamma}' \boldsymbol{\alpha} = (\Gamma_1^{-1})' \boldsymbol{\beta}$ and $\bar{\boldsymbol{\psi}} := (\bar{\boldsymbol{\beta}}', \boldsymbol{\mu}')'$ correspondingly. Note that $\boldsymbol{\alpha}$ is not identifiable from the VECM.

4.3 Prior

4.3.1 Steady state parameters

Assume that

$$p(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma) = p(\boldsymbol{\alpha}, \boldsymbol{\mu}) p(\boldsymbol{\Phi}, \mathbf{P}, \Lambda, \Gamma) \quad (23)$$

¹¹ With a slight abuse of notation, $\boldsymbol{\Phi}$ denotes the VAR coefficient matrices in $\boldsymbol{\Phi}(\cdot)$.

¹² We use the same $p(\cdot)$ to denote different distributions, which is common in the Bayesian literature; see, e.g., Gelman et al. (2014, p. 6).

Assume a normal prior on (α, μ) such that

$$\begin{pmatrix} \alpha \\ \mu \end{pmatrix} \sim N_{2N} \left(\begin{pmatrix} \alpha_0 \\ \mu_0 \end{pmatrix}, \begin{bmatrix} Q_{0,\alpha}^{-1} & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & Q_{0,\mu}^{-1} \end{bmatrix} \right) \quad (24)$$

where $Q_{0,\alpha}$ and $Q_{0,\mu}$ are the prior precision matrices of α and μ , respectively. Since $\beta := \Gamma' \alpha$, this prior implies a prior on ψ such that

$$\psi | \Gamma \sim N_{r+N}(\psi_0, Q_0^{-1}) \quad (25)$$

where

$$\psi_0 := \begin{pmatrix} \Gamma' \alpha_0 \\ \mu_0 \end{pmatrix}, \quad Q_0 := \begin{bmatrix} (\Gamma' Q_{0,\alpha}^{-1} \Gamma)^{-1} & \mathbf{0}_{r \times r} \\ \mathbf{0}_{N \times N} & Q_{0,\mu} \end{bmatrix}$$

which depends on Γ in general.

4.3.2 VAR parameters

Let S be the set of (Φ, Λ, Γ) such that the eigenvalues of Λ lie inside the unit circle, so that $\{y_t^*\}$ is CI(1,1). Write

$$p(\Phi, P, \Lambda, \Gamma) \propto p^*(\Phi, P, \Lambda, \Gamma) [(\Phi, \Lambda, \Gamma) \in S] \quad (26)$$

where $p^*(.)$ is the pdf with no truncation of (Φ, Λ, Γ) and $[.]$ is the indicator function that truncates the support of $p^*(.)$ to S . Assume that

$$p^*(\Phi, P, \Lambda, \Gamma) = p^*(\Phi, P) p^*(\Lambda, \Gamma) \quad (27)$$

Assume a hierarchical normal-Wishart prior on (Φ, P) such that

$$\Phi | P, \nu \sim N_{N \times pN}(\mathbf{M}_0; P^{-1}, (\nu D_0)^{-1}) \quad (28)$$

$$P \sim W_N(k_0; S_0^{-1}) \quad (29)$$

$$\nu \sim \text{Gam}\left(\frac{A_0}{2}, \frac{B_0}{2}\right) \quad (30)$$

where ν is a hyperparameter that controls the tightness of the prior on the VAR coefficients, $W_N(k_0; S_0^{-1})$ means ‘an $(N \times N)$ -variate Wishart distribution with k_0 degrees of freedom and scale matrix S_0^{-1} ’, and $\text{Gam}(A_0/2, B_0/2)$ means ‘a gamma distribution with shape $A_0/2$ and rate $B_0/2$ ’. Since it is often difficult to choose ν *a priori*, we assume a gamma prior on ν .

4.3.3 Cointegrating space

Assume that

$$p^*(\Lambda, \Gamma) = p^*(\Lambda) p^*(\Gamma) \quad (31)$$

Assume a normal prior on Λ such that

$$\Lambda \sim N_{N \times r}(\mathbf{0}_{N \times r}; (\eta_0 G_0)^{-1}, I_r) \quad (32)$$

where $\eta_0 G_0$ is the prior precision matrix of each column of Λ .

Let $V_r(\mathbb{R}^N)$ be the r -dimensional Steifel manifold in $\mathbb{R}^N$, i.e.,

$$V_r(\mathbb{R}^N) := \{H \in \mathbb{R}^{N \times r} : H' H = I_r\}$$

Let $H_0 \in V_r(\mathbb{R}^N)$. Let $H(\cdot)$ be such that $\forall \tau \geq 0$,

$$H(\tau) := H_0 H_0' + \tau H_{0\perp} H_{0\perp}'$$

where $\mathbf{H}_{0\perp} \in V_{N-r}(\mathbb{R}^N)$ lies in the orthogonal complement of the column space of $\mathbf{H}_0$, so that $\mathbf{H}(0) = \mathbf{H}_0\mathbf{H}_0'$ is of rank r and $\mathbf{H}(1) = \mathbf{I}_N$. Assume a normal prior on $\mathbf{\Gamma}$ such that

$$\mathbf{\Gamma} \sim N_{N \times r}(\mathbf{0}_{N \times r}; \mathbf{H}(\tau_0)^{-1}, \mathbf{I}_r) \quad (33)$$

where $\tau_0 \neq 0$. This prior on $\mathbf{\Gamma}$ implies a prior on $\mathbf{\Gamma}_*$ such that

$$\mathbf{\Gamma}_* \sim \text{MACG}_{N \times r}(\mathbf{H}(\tau_0)^{-1}) \quad (34)$$

where $\tau_0 := 1$ implies the flat prior on $V_r(\mathbb{R}^N)$; see Chikuse (2003, sec. 2.4.2). Note that $p^*(\mathbf{\Lambda}_*|\mathbf{\Gamma})$ is normal, but $p^*(\mathbf{\Lambda}_*|\mathbf{\Gamma}_*)$ is not; see Koop, León-González, and Strachan (2010, p. 232), who use these priors on $(\mathbf{\Lambda}, \mathbf{\Gamma})$ for their parameter-augmented Gibbs sampler.

4.4 Posterior simulation

We simulate $p(\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Lambda}, \mathbf{\Gamma}, \nu | \mathbf{Y})$ by a Gibbs sampler consisting of five blocks:

1. Draw $\boldsymbol{\psi}$ from $p(\boldsymbol{\psi}|\boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Lambda}, \mathbf{\Gamma}, \nu, \mathbf{Y}) = p(\boldsymbol{\psi}|\boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Lambda}, \mathbf{\Gamma}, \mathbf{Y})$.
2. Draw $(\boldsymbol{\Phi}, \mathbf{P})$ from $p^*(\boldsymbol{\Phi}, \mathbf{P}|\boldsymbol{\psi}, \mathbf{\Lambda}, \mathbf{\Gamma}, \nu, \mathbf{Y})$.
3. Draw $\mathbf{\Lambda}$ from $p^*(\mathbf{\Lambda}|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Gamma}, \nu, \mathbf{Y}) = p^*(\mathbf{\Lambda}|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Gamma}, \mathbf{Y})$.
4. Draw $\mathbf{\Gamma}$ from $p^*(\mathbf{\Gamma}|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Lambda}, \nu, \mathbf{Y}) = p^*(\mathbf{\Gamma}|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Lambda}, \mathbf{Y})$. Accept the draw if $(\boldsymbol{\Phi}, \mathbf{\Lambda}, \mathbf{\Gamma}) \in S$; otherwise go back to step 2 and draw another $(\boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Lambda}, \mathbf{\Gamma})$.
5. Draw ν from $p(\nu|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Lambda}, \mathbf{\Gamma}, \mathbf{Y}) = p(\nu|\boldsymbol{\Phi}, \mathbf{P})$.

The first block builds on Villani (2009); the second block is standard; the third and fourth blocks follow the parameter-augmented Gibbs sampler proposed by Koop, León-González, and Strachan (2010); the fifth block is standard. See the Appendix for the details of each block.

4.5 Bayes factor

We use the Bayes factor for Bayesian model selection. When choosing between nested models with certain priors, the Savage–Dickey (S–D) density ratio gives the Bayes factor without estimating the marginal likelihoods; see Wagenmakers et al. (2010) for a tutorial on the S–D method.

We choose the cointegrating rank r . Consider comparing the following two models (hypotheses):

$$H_0: \text{rk}(\Pi(1)) = 0 \quad \text{vs} \quad H_r: \text{rk}(\Pi(1)) = r \quad (35)$$

Koop, León-González, and Strachan (2008, pp. 451–452) note that the problem is the same as comparing the following two nested models:

$$H_0: \mathbf{\Lambda} = \mathbf{0}_{N \times r} \quad \text{vs} \quad H_r: \mathbf{\Lambda} \neq \mathbf{0}_{N \times r} \quad (36)$$

The restriction $(\boldsymbol{\Phi}, \mathbf{\Lambda}, \mathbf{\Gamma}) \in S$ is necessary only for the B–N decomposition, and unnecessary for forecasting in general; hence we ignore this restriction for the moment, so that the priors on $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ become independent.¹³ Then the S–D density ratio for H_0 versus H_r is

$$B_{0,r} = \frac{p(\mathbf{\Lambda} = \mathbf{0}_{N \times r} | \mathbf{Y}; H_r)}{p(\mathbf{\Lambda} = \mathbf{0}_{N \times r} | H_r)} \quad (37)$$

The prior gives the denominator directly. For the numerator, we have

$$p(\mathbf{\Lambda} | \mathbf{Y}; H_r) = E(p(\mathbf{\Lambda} | \boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \mathbf{\Gamma}, \mathbf{Y}; H_r) | \mathbf{Y}; H_r) \quad (38)$$

¹³ If the priors on $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ are dependent, then we can use the generalized S–D density ratio proposed by Verdinelli and Wasserman (1995).

Let $\{\boldsymbol{\psi}_l, \boldsymbol{\Phi}_l, \mathbf{P}_l, \boldsymbol{\Gamma}_l\}_{l=1}^L$ be posterior draws. Let

$$\hat{p}(\Lambda = \mathbf{O}_{N \times r} | \mathbf{Y}; H_r) := \frac{1}{L} \sum_{l=1}^L p(\Lambda = \mathbf{O}_{N \times r} | \boldsymbol{\psi}_l, \boldsymbol{\Phi}_l, \mathbf{P}_l, \boldsymbol{\Gamma}_l, \mathbf{Y}; H_r)$$

See Appendix A.4 for the conditional posterior $p(\Lambda | \boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Gamma}, \mathbf{Y})$. An estimator of the S–D density ratio for H_0 versus H_r is

$$\hat{B}_{0,r} = \frac{\hat{p}(\Lambda = \mathbf{O}_{N \times r} | \mathbf{Y}; H_r)}{p(\Lambda = \mathbf{O}_{N \times r} | H_r)} \quad (39)$$

5 Application

5.1 Data

We consider joint estimation of the natural rates (or their permanent components) and gaps of the following four macroeconomic variables in the US:

1. **Output** Let Y_t be output. Assume that $\{\ln Y_t\}$ is I(2), so that $\{\Delta \ln Y_t\}$ is I(1).
2. **Inflation rate** Let P_t be the price level and $\pi_t := \ln(P_t/P_{t-1})$ be the inflation rate. Assume that $\{\pi_t\}$ is I(1).
3. **Interest rate** Let I_t be the 3-month nominal interest rate (annual rate in per cent), $i_t := \ln(1 + I_t/400)$, $r_t := i_t - E_t(\pi_{t+1})$ be the ex ante real interest rate, and $\hat{r}_t := i_t - \pi_{t+1}$ be the ex post real interest rate. Assume that $\{r_t\}$ is I(1).¹⁴
4. **Unemployment rate** Let L_t be the labor force, E_t be employment, and $U_t := -\ln(E_t/L_t)$ be the unemployment rate. Assume that $\{U_t\}$ is I(1).

Table 1 describes the data, which are available from FRED (Federal Reserve Economic Data). When monthly series are available, i.e., except for real GDP, we take the 3-month arithmetic means of monthly series each quarter to obtain quarterly series, from which we construct the quarterly inflation, interest, and unemployment rates as defined above.

Let for all t ,

$$\mathbf{x}_t := \begin{pmatrix} \pi_t \\ \hat{r}_t \\ U_t \\ \ln Y_t \end{pmatrix}, \quad \mathbf{y}_t := \begin{pmatrix} \pi_t \\ \hat{r}_t \\ U_t \\ \Delta \ln Y_t \end{pmatrix}$$

The sample period of $\{\mathbf{y}_t\}$ is 1948Q1–2018Q4 (284 observations). Table 2 shows summary statistics of $\{\mathbf{y}_t\}$ and $\{\Delta \mathbf{y}_t\}$ multiplied by 100. Note that $\{\pi_t\}$, $\{\hat{r}_t\}$, and $\{\Delta \ln Y_t\}$ are quarterly rates (not annualized).

5.2 Preliminary analyses

As preliminary analyses, we check if $\{\mathbf{y}_t\}$ is indeed I(1), though our method is valid even if some series in $\{\mathbf{y}_t\}$ are I(0). For each I(0) series, we treat it as an I(1) series cointegrated with itself, and increase the cointegrating rank by 1 as ‘pseudo’ cointegration; see Fisher, Huh, and Pagan (2016, sec. 2.2).

Table 3 shows the results of the ADF and ADF-GLS tests for unit root, with or without constant and/or trend terms in the ADF regression.¹⁵ The results depend on the number of lags included in the ADF regression, since the ADF test suffers from size distortion with short lags and low power with long lags. The ADF-GLS test mitigates the problem except when there is no constant nor trend term in the ADF regression, in which case

¹⁴ We can estimate the interest rate gap even if we observe $\{\hat{r}_t\}$ instead of $\{r_t\}$. See Murasawa (2014, pp. 499–500).

¹⁵ We use gretl 2021a for our preliminary analyses.

Table 1: Data.

Variable	Description
Y_t	Real GDP (billions of chained 2012 dollars, SA, AR)
P_t	CPI for all urban consumers: all items (1982–84 = 100, SA)
I_t	3-month treasury bill: secondary market rate (% , AR)
L_t	Civilian labor force (thousands of persons, SA)
E_t	Civilian employment level (thousands of persons, SA)

SA means ‘seasonally-adjusted’; AR means ‘annual rate’.

Table 2: Summary statistics.

Variable	Min.	1st qu.	Median	Mean	3rd qu.	Max.
$100\pi_t$	−2.32	0.39	0.75	0.84	1.10	3.98
$100\hat{r}_t$	−3.65	−0.24	0.23	0.19	0.55	2.69
$100U_t$	2.64	4.75	5.72	5.95	7.07	11.28
$100\Delta \ln Y_t$	−2.63	0.31	0.76	0.78	1.25	3.85
$100\Delta \pi_t$	−3.85	−0.26	0.00	−0.01	0.28	1.98
$100\Delta \hat{r}_t$	−1.95	−0.27	0.02	0.01	0.25	3.81
$100\Delta U_t$	−1.03	−0.20	−0.07	0.00	0.12	1.79
$100\Delta^2 \ln Y_t$	−2.79	−0.71	−0.02	0.00	0.62	4.70

the ADF test is asymptotically point optimal. The level 0.05 ADF-GLS test rejects $H_0 : \{y_{t,i}\} \sim I(1)$ in favor of $H_1 : \{y_{t,i}\} \sim I(0)$ for $\{\Delta \ln Y_t\}$ (and possibly for $\{\pi_t\}$, whose results are mixed). Hence these unit root tests do not support our assumption that $\{y_t\}$ is $I(1)$.

Table 4 shows the results of the KPSS stationarity tests, with or without a trend term. The results depend on the lag truncation parameter for the Newey–West estimator of the long-run error variance. With a trend term, the level 0.05 KPSS test rejects $H_0 : \{y_{t,i}\} \sim I(0)$ in favor of $H_1 : \{y_{t,i}\} \sim I(1)$ except for $\{\Delta \ln Y_t\}$. With no trend term, the test rejects $H_0 : \{y_{t,i}\} \sim I(0)$ in favor of $H_1 : \{y_{t,i}\} \sim I(1)$ except for $\{\hat{r}_t\}$. Though the results for $\{\hat{r}_t\}$ and $\{\Delta \ln Y_t\}$ are mixed, these stationarity tests support our assumption that $\{y_t\}$ is $I(1)$.

Overall, these unit root and stationarity tests confirm that $\{\Delta y_t\}$ is $I(0)$, but are inconclusive if each component of $\{y_t\}$ is $I(1)$. We can still proceed with our assumption that $\{y_t\}$ is $I(1)$, allowing for ‘pseudo’ cointegration.

As preliminary analyses, we also check if $\{y_t\}$ is $CI(1,1)$. Table 5 shows the results of the Engle–Granger cointegration tests, with or without a trend term in the cointegrating regression and for each choice of the left-hand side (LHS) variable. The results depend on the number of lags included in the ADF regression for the residual series, but the level 0.05 test rejects the null of no cointegration in favor of the alternative of cointegration only when we put $\Delta \ln Y_t$ on the LHS of the cointegrating regression.

Table 6 shows the results of Johansen’s cointegration tests, with unrestricted constant and restricted trend terms in the VECM. The results depend on the number of lags included in the VECM, but overall, the tests suggest that the cointegrating rank is 2, possibly including ‘pseudo’ cointegration.

Since the results of these cointegration tests are mixed, we estimate VECMs with different cointegrating ranks, and use the Bayes factor to choose the cointegrating rank.

5.3 Model specification

For our data, $N := 4$. To select p , we fit VAR models to $\{y_t\}$ up to $\text{VAR}(8)$, i.e., $p = 7$, and check model selection criteria. The common estimation period is 1950Q1–2018Q4. Table 7 summarizes the results of lag

Table 3: Unit root tests.

Variable	Const.	Trend	ADF			ADF-GLS	
			Lags	τ	p -value	Lags	τ
π_t	Yes	Yes	4	−3.73	0.02	4	−3.40**
$\hat{r}_t$	Yes	Yes	7	−3.89	0.01	5	−2.70*
U_t	Yes	Yes	13	−2.87	0.17	12	−2.19
$\Delta \ln Y_t$	Yes	Yes	11	−5.78	0.00	1	−8.30***
π_t	Yes	No	4	−3.69	0.00	12	−0.75
$\hat{r}_t$	Yes	No	7	−3.90	0.00	5	−1.58
U_t	Yes	No	13	−3.01	0.03	12	−1.49
$\Delta \ln Y_t$	Yes	No	2	−8.30	0.00	1	−6.36***
$\Delta \pi_t$	Yes	No	3	−11.88	0.00	0	−13.90***
$\Delta \hat{r}_t$	Yes	No	8	−8.72	0.00	0	−21.61***
ΔU_t	Yes	No	12	−5.21	0.00	0	−7.47***
$\Delta^2 \ln Y_t$	Yes	No	14	−7.52	0.00	0	−25.16***
$\Delta \pi_t$	No	No	3	−11.90	0.00		
$\Delta \hat{r}_t$	No	No	8	−8.75	0.00		
ΔU_t	No	No	12	−5.22	0.00		
$\Delta^2 \ln Y_t$	No	No	14	−7.53	0.00		

For the ADF-GLS test, *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively. For the number of lags included in the ADF regression, we use the default choice in gretl 2021a with maximum 15, where the lag order selection criteria are AIC for the ADF test, and a modified AIC using the Perron and Qu (2007) method for the ADF-GLS test. With no constant nor trend in the ADF regression, the ADF test is asymptotically point optimal; hence the ADF-GLS test is unnecessary.

Table 4: KPSS stationarity tests.

Variable	Trend	LM
π_t	Yes	0.53***
$\hat{r}_t$	Yes	0.32***
U_t	Yes	0.23***
$\Delta \ln Y_t$	Yes	0.03
π_t	No	0.58**
$\hat{r}_t$	No	0.34
U_t	No	0.63**
$\Delta \ln Y_t$	No	0.48**
$\Delta \pi_t$	No	0.03
$\Delta \hat{r}_t$	No	0.04
ΔU_t	No	0.07
$\Delta^2 \ln Y_t$	No	0.01

** and *** denote significance at the 5 and 1% levels, respectively. The lag truncation parameter for the Newey–West estimator of the long-run error variance is 5 (the default value for our sample length in gretl 2021a).

order selection.¹⁶ The level 0.05 LR test fails to reject $H_0: \{\mathbf{y}_t\} \sim \text{VAR}(7)$ against $H_1: \{\mathbf{y}_t\} \sim \text{VAR}(8)$ and AIC

¹⁶ The standard lag order selection criteria are valid even when $\{\mathbf{y}_t\}$ is I(1); see Kilian and Lütkepohl (2017, pp. 99–100).

selects VAR(7), but BIC and HQC select much smaller models. Since a high-order VAR model covers low-order VAR models as special cases, to be conservative, we choose a VAR(8) model for $\{y_t\}$, i.e., $p = 7$, and impose a shrinkage prior on the VAR coefficients in the VECM.

For the prior on α , we set $\alpha_0 := \mathbf{0}_N$ and $\mathbf{Q}_{0,\alpha} := \mathbf{I}_N$. For the prior on μ , we set $\mu_0 := \hat{\mu}$, where $\hat{\mu}$ is the sample mean of $\{\Delta y_t\}$, and $\mathbf{Q}_{0,\mu} := \mathbf{I}_N$.

Following Kadiyala and Karlsson (1997) and Bańbura, Giannone, and Reichlin (2010), we set

$$\begin{aligned} \mathbf{M}_0 &:= \mathbf{0}_{N \times pN} \\ \mathbf{D}_0 &:= \text{diag}(1, \dots, p)^2 \otimes \text{diag}(s_1, \dots, s_N)^2 \\ k_0 &:= N + 2 \\ \mathbf{S}_0 &:= (k_0 - N - 1) \text{diag}(s_1, \dots, s_N)^2 \end{aligned}$$

where for $i = 1, \dots, N$, s_i^2 is an estimate of $\text{var}(u_{t,i})$ based on the univariate AR($p + 1$) model with constant and trend terms for $\{y_{t,i}\}$.

For the prior on $\mathbf{A}$, we set $\eta_0 := 1$ and $\mathbf{G}_0 := \mathbf{I}_N$. For the prior on $\mathbf{\Gamma}$, we set $\tau_0 := 1$, which gives the flat prior on the cointegrating space.

For the prior on v , we set $A_0 := 1$ and $B_0 := 1$, i.e., $v \sim \chi^2(1)$; hence the tightness hyperparameter on the VAR coefficients tends to be small, implying potentially mild shrinkage toward $\mathbf{M}_0 := \mathbf{0}_{N \times pN}$.

Overall, our priors are weakly informative in the sense of Gelman et al. (2014, p. 55).

5.4 Bayesian computation

We run our Gibbs sampler on R 4.0.5 developed by R Core Team (2021). We use the ML estimate of $(\Phi, \mathbf{P}, \mathbf{A}, \mathbf{\Gamma})$ for their initial values.¹⁷ With poor initial values, the restriction that the eigenvalues of $\mathbf{A}$ lie inside the unit circle does not hold, and the iteration cannot start. Hence the choice of initial values seems important when one applies the B–N decomposition. Once the iteration starts, the restriction rarely binds for our sample.

To check convergence of the Markov chain generated by our Gibbs sampler to its stationary distribution, we perform convergence diagnoses discussed in Robert and Casella (2009, ch. 8) and available in the coda package for R. Given the diagnoses, we discard the initial 1000 draws, and use the next 4000 draws for our posterior inference. At significance level 0.05, Geweke's Z-score rejects equality of the means of the first 10% and the last 50% of the posterior sample (the default choice in the coda package) for 6 out of 141 identified parameters (applying linear normalization to $\mathbf{\Gamma}$). Since the rejection rate is close to the significance level, 1000 draws are enough as burn-in. The minimum effective sample size among the 141 identified parameters

Table 5: Engle–Granger cointegration tests.

LHS	Trend	τ	p -Value
π_t	Yes	−3.79	0.21
$\hat{r}_t$	Yes	−3.58	0.30
U_t	Yes	−3.28	0.45
$\Delta \ln Y_t$	Yes	−8.33	0.00
π_t	No	−3.60	0.16
$\hat{r}_t$	No	−3.57	0.17
U_t	No	−3.19	0.32
$\Delta \ln Y_t$	No	−7.85	0.00

The number of lags included in the ADF regression is 4 (the default value for our sample length in gretl 2021a).

Table 6: Johansen's cointegration tests (unrestricted constant and restricted trend in the VECM).

Rank	Trace	<i>p</i> -Value	λ -max	<i>p</i> -Value
0	142.77	0.00	83.00	0.00
1	59.78	0.00	41.81	0.00
2	17.97	0.35	12.00	0.43
3	5.97	0.47	5.97	0.48

The number of lags included in the VECM is 4 (the default value for our sample length in gretl 2021a).

Table 7: Lag order selection.

Lag	Log-lik	LR	<i>p</i> -Value	AIC	BIC	HQC
1	4400.07			−31.71	−31.40	−31.58
2	4752.06	703.98	0.00	−34.15	−33.62*	−33.93
3	4782.56	61.01	0.00	−34.25	−33.52	−33.96*
4	4800.35	35.58	0.00	−34.26	−33.32	−33.88
5	4814.33	27.96	0.03	−34.25	−33.09	−33.79
6	4834.26	39.87	0.00	−34.28	−32.91	−33.73
7	4855.60	42.67	0.00	−34.32*	−32.74	−33.68
8	4868.15	25.10	0.07	−34.29	−32.51	−33.58

For AIC, BIC, and HQC, * denotes the selected model. The LR test statistic for testing $H_0 : \{\mathbf{y}_t\} \sim \text{VAR}(p-1)$ versus $H_1 : \{\mathbf{y}_t\} \sim \text{VAR}(p)$ follows $\chi^2(16)$ under H_0 .

is 238. Gelman et al. (2014, p. 267) write, ‘... 100 independent draws are enough for reasonable posterior summaries.’ Hence 4000 draws are enough for posterior inference.

To select the cointegrating rank, we set $p := 7$, assume the above priors, and compute the S–D density ratios for $r = 0$ versus $r = 1, 2, 3$, respectively, from which we derive the posterior probabilities of $r = 0, 1, 2, 3$, assuming the flat prior on r . We find that the posterior probability of $r = 2$ is often numerically 1, which is consistent with the results of Johansen's cointegration tests.¹⁸ Thus we set $r := 2$ in the following analysis.

5.5 Empirical results

Figure 2 plots the actual rates and our point estimates (posterior medians) of the natural rates (or their permanent components) of the four variables. For ease of comparison of the actual and natural rates, we omit error bands for the natural rates, which are identical to those for the gaps. Figure 3 plots our point estimates of the gaps and their 95% error bands. Since a VECM has both the level and difference series, to avoid confusion in estimation, we do not annualize quarterly rates. To compare our results with previous works that use annualized quarterly rates, one must multiply our inflation and interest rates by 4 or 400.

Our estimate of the natural rate of inflation is smoother than typical univariate estimates of trend inflation in the CPI; e.g., Faust and Wright (2013, p. 22). Our estimate looks close to a recent estimate of trend inflation in the US CPI in Morley, Piger, and Rasche (2015, p. 894) based on a bivariate UC model for the inflation

¹⁷ The urca package for R is useful for ML estimation of a VECM.

¹⁸ The S–D density ratio may be inaccurate if the marginal posterior pdf of $\mathbf{A}$ at $\mathbf{O}_{N \times r}$ is extremely small. In such a case, one should check the result by running the Gibbs sampler multiple times. Wagenmakers et al. (2010, p. 182) write, ‘the qualitative conclusion is evident and reliable ... even if the quantitative result is not.’

Figure 2: Actual rates (solid red) and the posterior medians of the natural rates (dashed green). The shaded areas are the NBER recessions.

and unemployment rates, which assumes independent shocks to the trend and gap components and allows for structural breaks in the variances of these shocks. Our estimate is more volatile, however, than a recent estimate of trend inflation in the PCE price index by Chan, Clark, and Koop (2018, p. 21) that uses information in survey inflation expectations. Hwu and Kim (2019) estimate trend inflation in the PCE price index using a univariate UC model with correlated shocks to the trend and gap components, Markov-switching gap persistence, and Markov-switching association between inflation and inflation uncertainty. Without the last feature, their estimate looks close to ours; see Hwu and Kim (2019, p. 2314).

Our estimate of the natural rate of interest is more volatile than a recent estimate by Del Negro et al. (2017, p. 237) based on a VAR model with common trends (or a multivariate UC model with independent shocks to the trend and gap components and a factor structure for the trend components) for short- and long-term interest rates, inflation and its survey expectations, and some other variables. Their estimate is smooth partly because they impose tight priors on the variances of the shocks to the trend components.

Figure 3: Posterior medians of the gaps and their 95% error bands (posterior 0.025- and 0.975-quantiles). The shaded areas are the NBER recessions.

With the ‘loosest possible prior’, their estimate is as volatile as ours; see Del Negro et al. (2017, p. 272).¹⁹ Despite different definitions of the natural rate, their estimate based on a DSGE model looks close to ours; see Del Negro et al. (2017, p. 237).²⁰ Estimates by Laubach and Williams (2016, p. 60), Holston, Laubach, and Williams (2017, p. S61), and Lewis and Vazquez-Grande (2019) are close to trend output growth by construction; hence their estimates are quite different from those by Del Negro et al. (2017) and ours, especially before 1980.²¹

¹⁹ Their ‘loosest possible prior’ on the variance–covariance matrix of the shocks to the trend components is an inverse Wishart prior with the minimum degree of freedom that gives a finite mean; see Del Negro et al. (2017, p. 271).

²⁰ Del Negro et al. (2017) clearly distinguish the natural rate and its low-frequency component, estimating the former by a DSGE model and the latter by a VAR model. To construct the real interest rate, they use the PCE inflation rate instead of the CPI inflation rate; hence their real interest rate is on average slightly higher than ours.

²¹ Holston, Laubach, and Williams (2017, p. S63) writes,

Our estimate of the natural rate of unemployment looks more volatile than a recent estimate by Morley, Piger, and Rasche (2015, p. 898), obtained as a by-product of estimation of trend inflation. A possible reason for the difference is that they assume independent shocks to the trend and gap components, which may not hold in practice. The two estimates may coincide if one allows for dependence between the shocks, as Morley, Nelson, and Zivot (2003) show for the univariate trend-cycle decomposition. Despite the difference in volatility, our estimate of the unemployment rate gap looks close to that by Morley, Piger, and Rasche (2015, p. 901) in terms of the sign and magnitude. Recent estimates of the natural rate and gap by Crump et al. (2019, pp. 176, 180) are as smooth as those by Morley, Piger, and Rasche (2015), perhaps partly because they also assume independent shocks to the trend and gap components.²²

Our estimate of the output gap is at most about $\pm 5\%$ of the output level, and looks close to a recent estimate by Morley and Wong (2020, p. 9) based on a large Bayesian VAR(4) model with 23 variables. Our estimate also looks close to their estimate based on a Bayesian VAR(4) model with four variables using output growth, the unemployment rate, the CPI inflation rate, and the growth rate of industrial production, assuming that they are all $I(0)$; see Morley and Wong (2020, p. 8). Though output growth may or may not be $I(0)$ in the US, it may be clearly $I(1)$ in some countries or regions, in which case their method may give an unreasonable estimate of the output gap with a strong upward or downward trend, as Murasawa (2015) shows for the Japanese data.

Figure 4 plots the posterior probability of positive gap for the four variables. This probability index is useful if the sign of the gap is of interest. Even if the 95% error band for the gap covers 0, the posterior probability of positive gap may be close to 0.025 or 0.975. Indeed, the probability index is often below 0.25 or above 0.75; hence we are often quite sure about the sign of the gap. Moreover, Figure 4 shows the relation between the gaps more clearly than Figure 3.

Figure 5 shows the relation between the gaps more directly. The left panels are the scatter plots of the posterior medians of the gaps in each quarter. The right panels are the posterior pdfs of the correlation coefficients between the gaps. We see that the Phillips curves and Okun's law hold between the gaps, though we do not impose such relations. Thus our estimates of the gaps seem mutually consistent from a macroeconomic point of view.

Figure 6 shows the relation between the trend output growth rate $\Delta \ln Y_{t+1}^*$ and the natural rate of interest r_t^* , i.e., the dynamic IS curve; see Eq. (1). The left panel is the scatter plot of the posterior medians of $\Delta \ln Y_{t+1}^*$ and r_t^* in each quarter. The right panel is the posterior pdf of the slope coefficient, i.e., $1/\sigma$ in Eq. (1), obtained by regressing $\Delta \ln Y_{t+1}^*$ on r_t^* for each posterior draw of $\{\Delta \ln Y_t^*, r_t^*\}$. On average, the slope is positive as expected, but the evidence is weak. The result is similar to Hamilton et al. (2016) and Lunsford and West (2019), who study the determinants of r_t^* and find positive but low and insignificant long-run correlations between $\Delta \ln Y_t$ and r_t in the postwar US.

... we assume a one-for-one relationship between the trend growth rate of output and the natural rate of interest, which corresponds to assuming $\sigma = 1$ in Eq. (1).

where Eq. (1) in their paper is the consumption Euler equation in a steady state.

²² Crump et al. (2019) distinguish the trend unemployment rate $\bar{u}$ and the natural rate of unemployment u^* . They construct $\bar{u}$ from estimates of the trend job separation and finding rates based on a multivariate UC model with a factor structure for the rates across demographic groups, assuming independent shocks to the trend and gap components. Treating $\bar{u}$ as exogenous and relying on a New Keynesian (wage) Phillips curve, they estimate u^* using the CPI, survey inflation expectations, and various wage data, again assuming independent shocks to the natural rate and gap. Allowing for dependence between these shocks seems an interesting and important topic for future work.

Figure 4: Posterior probability of positive gap. The shaded areas are the NBER recessions.

5.6 Comparison of alternative model specifications

Figure 7 compares point estimates of the gaps under three alternative assumptions, i.e., $I(1)$ log output, $I(2)$ log output with no cointegration, and $I(2)$ log output with cointegration.²³ For ease of comparison, we omit error bands here.

The result assuming $I(1)$ log output is similar to that in Murasawa (2014, p. 513), who assumes a VAR(12) model with no constant term for the centered stationarized series, and chooses the tightness hyperparameter by the empirical Bayes method. In contrast to the result for Japan in Figure 1, for the US, the multivariate B–N decomposition assuming $I(1)$ log output gives a ‘reasonable’ estimate of the output gap that fluctuates around 0. However, the output gap is persistently positive since 2010, which may not be sensible.

Assuming $I(2)$ log output changes the estimate of the output gap. However, it hardly changes the estimates of other gaps, similar to the result for Japan in Murasawa (2015), which makes sense because the B–N transitory

²³ For the first two cases with no cointegration, we assume VAR(7) models for the stationarized series, and use the same prior and Gibbs sampler as those for our VECM (except for the loading and cointegrating matrices).

Figure 5: Phillips curves and Okun's law. The left panels are the scatter plots of the posterior medians of the gaps in each quarter. The right panels are the posterior pdfs of the correlation coefficients between the gaps.

components are ‘forecastable movements’, and whether log output is I(1) or I(2), the two VAR models use the information in log output for forecasting other variables in almost the same way.²⁴ The output gap now keeps fluctuating around 0 since 2010, perhaps because I(2) log output captures a possible structural break in the mean output growth rate after the 2008 Great Recession.

Allowing for cointegration gives much bigger estimates of the gaps for all variables. The result follows partly because a VECM uses more predictors, i.e., the error correction terms, than a VAR model; see

²⁴ A VAR(p) model with $\{\Delta^2 \ln Y_t\}$ includes $\Delta^2 \ln Y_{t-p} := \Delta \ln Y_{t-p} - \Delta \ln Y_{t-p-1}$ as a predictor and hence use the information in $\Delta \ln Y_{t-p-1}$, but a VAR(p) model with $\{\Delta \ln Y_t\}$ does not. If p is large enough, then the coefficient on $\Delta \ln Y_{t-p-1}$ tends to be small, especially with our shrinkage prior on the VAR coefficients.

Figure 6: Dynamic IS curve. The left panel is the scatter plot of the posterior medians of $\Delta \ln Y_{t+1}^*$ and r_t^* in each quarter. The right panel is the posterior pdf of the slope coefficient obtained by regressing $\Delta \ln Y_{t+1}^*$ on r_t^* for each posterior draw of $\{\Delta \ln Y_t^*, r_t^*\}$.

Evans and Reichlin (1994). However, the result also implies that for all variables, the coefficients on the error correction terms are large enough, if not significantly different from 0, to obtain such different estimates of the gaps. These bigger estimates of the gaps, or the associated estimates of the natural rates, are often close to other recent estimates that focus on a particular natural rate or gap, as already noted.

6 Discussion

The dynamic IS equation implies that if the real interest rate is $I(1)$, then so is the output growth rate with possible cointegration, and log output is $I(2)$. We extend the multivariate B–N decomposition to such a case, and apply Bayesian analysis to obtain error bands for the components. To implement this idea, we consider a steady state VECM with a hierarchical prior on the VAR coefficients, and develop a Gibbs sampler for posterior simulation of the parameters and the components. Applying the method to US data with weakly informative priors, we obtain a reasonable joint estimate of the natural rates (or their permanent components) and gaps of output, inflation, interest, and unemployment.

The B–N decomposition assuming $I(1)$ log output may give a trending estimate of the output gap if the mean output growth rate has structural breaks. Assuming $I(2)$ log output and hence $I(1)$ output growth rate, we introduce a stochastic trend in the output growth rate, which captures possible structural breaks in the mean output growth rate. Thus the B–N decomposition assuming $I(2)$ log output gives a reasonable estimate of the output gap that fluctuates around 0. Since the B–N transitory components are ‘forecastable movements’, and a VECM forecasts no worse than a VAR model, allowing for cointegration gives larger estimates of the gaps for all variables. Since we can treat an $I(0)$ series as an $I(1)$ series cointegrated with itself, our method is valid even if the output growth rate is $I(0)$, or log output is $I(1)$.

The B–N decomposition has some advantages over other approaches:

1. The B–N decomposition relies on a reduced-form forecasting model; hence it requires fewer assumptions than the decomposition based on a structural model.
2. The B–N decomposition requires no state smoothing if the state vector is observable as in our VECM; hence it can handle a large state vector more easily than the decomposition based on a UC model, which involves latent variables.

Thus the multivariate B–N decomposition is useful for estimating the natural rates and gaps of many variables jointly. Most, if not all, previous works focus on the natural rate or gap of one variable, using other variables only to improve estimation. We estimate the natural rates and gaps of four variables, which are equally of interest.

Since a reduced-form VECM is the most basic forecasting model for cointegrated series, our method gives a benchmark joint estimate of the natural rates and gaps of major macroeconomic variables. One can compare

Figure 7: Posterior medians of the gaps assuming I(1) log output (solid red), I(2) log output with no cointegration (dashed green), and I(2) log output with cointegration (longdash blue). The shaded areas are the NBER recessions.

this benchmark estimate with alternative estimates based on other forecasting models or DSGE models such as those in Del Negro et al. (2017). Our method is useful especially for non-US data, where log output is often clearly I(2). Application to non-US data is an interesting and important topic for future work.

One can possibly refine our estimate in two ways. First, one can use a larger VECM. Second, one can introduce nonlinearity into our VECM; e.g., Markov-switching, stochastic volatility, or more general time-varying parameters. Since the B–N decomposition may not apply to nonlinear models, and nonlinear models may become unnecessary with more variables, the first direction seems more promising. The following four variables are of particular interest:

1. PCE price index *in addition to* the CPI, which makes our results comparable to various previous works that use whichever price index.
2. long-run (LR) interest rate *in addition to* the short-run (SR) interest rate, which gives an estimate of the trend term premium, or the natural yield curve; cf. Brzoza-Brzezina and Kołowski (2014) and Imakubo, Kojima, and Nakajima (2018)
3. nominal yields on Aaa and Baa corporate bonds, which give estimates of the trend liquidity premium and the trend safety premium, or the trend liquidity and safety convenience yields; cf. Del Negro et al. (2017)

Our VECM allows for cointegration among I(1) series, but does not allow for cointegration among I(2) series nor multicointegration (polynomial cointegration). Such extensions are useful if we have multiple I(2) series; e.g., log output and the price level (instead of the inflation rate). We leave such extensions for future work.

Lastly, to obtain a monthly instead of quarterly joint estimate of the natural rates and gaps of output and other variables, it seems straightforward to extend the B–N decomposition of mixed-frequency series proposed by Murasawa (2016) to I(1) and I(2) series with cointegration.

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A. Appendix: Gibbs sampler

A.1 Useful lemmas

Our Gibbs sampler relies on the following two familiar results in Bayesian analysis of normal linear models, which we state as lemmas for ease of reference.

Lemma 1. *Suppose that*

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$\mathbf{u} \sim N_n(\mathbf{0}_n, \mathbf{P}^{-1})$$

and

$$\boldsymbol{\beta}|\mathbf{P} \sim N_k(\boldsymbol{\mu}_0, \mathbf{D}_0^{-1})$$

Then

$$\boldsymbol{\beta}|\mathbf{P}, \mathbf{y}, \mathbf{X} \sim N_k(\boldsymbol{\mu}_1, \mathbf{D}_1^{-1})$$

where

$$\mathbf{D}_1 := \mathbf{X}'\mathbf{P}\mathbf{X} + \mathbf{D}_0$$

$$\boldsymbol{\mu}_1 := \mathbf{D}_1^{-1}(\mathbf{X}'\mathbf{P}\mathbf{X}\mathbf{b}_{\text{GLS}} + \mathbf{D}_0\boldsymbol{\mu}_0)$$

with $\mathbf{b}_{\text{GLS}} := (\mathbf{X}'\mathbf{P}\mathbf{X})^{-1}\mathbf{X}'\mathbf{P}\mathbf{y}$.

Proof. See Koop (2003, pp. 118–121). □

Lemma 2. *Suppose that*

$$\mathbf{Y} = \mathbf{X}\mathbf{B}' + \mathbf{U}$$

$$\mathbf{U} \sim N_{n \times m}(\mathbf{0}_{n \times m}; \mathbf{I}_n, \mathbf{P}^{-1})$$

and

$$\mathbf{B}|\mathbf{P} \sim N_{m \times k}(\mathbf{M}_0; \mathbf{P}^{-1}, \mathbf{D}_0^{-1})$$

$$\mathbf{P} \sim W_m(k_0; \mathbf{S}_0^{-1})$$

Then

$$\begin{aligned} \mathbf{B}|\mathbf{P}, \mathbf{Y}, \mathbf{X} &\sim \mathbf{N}_{m \times k}(\mathbf{M}_1; \mathbf{P}^{-1}, \mathbf{D}_1^{-1}) \\ \mathbf{P}|\mathbf{Y}, \mathbf{X} &\sim \mathbf{W}_m(k_1; \mathbf{S}_1^{-1}) \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_1 &:= \mathbf{X}'\mathbf{X} + \mathbf{D}_0 \\ \mathbf{M}_1 &:= (\mathbf{B}_{\text{OLS}}\mathbf{X}'\mathbf{X} + \mathbf{M}_0\mathbf{D}_0)\mathbf{D}_1^{-1} \\ k_1 &:= n + k_0 \\ \mathbf{S}_1 &:= (\mathbf{B}_{\text{OLS}} - \mathbf{M}_0) [(\mathbf{X}'\mathbf{X})^{-1} + \mathbf{D}_0^{-1}]^{-1} (\mathbf{B}_{\text{OLS}} - \mathbf{M}_0)' + \mathbf{S} + \mathbf{S}_0 \end{aligned}$$

with $\mathbf{B}_{\text{OLS}} := \mathbf{Y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$ and $\mathbf{S} := (\mathbf{Y} - \mathbf{X}\mathbf{B}'_{\text{OLS}})'(\mathbf{Y} - \mathbf{X}\mathbf{B}'_{\text{OLS}})$.

Proof. See Drèze and Richard (1983, pp. 539–541). □

A.2 Steady state parameters

Write the VECM as for all t ,

$$\Phi(L)\Delta\mathbf{y}_t - \Phi(1)\boldsymbol{\mu} = -\Lambda[\Gamma'\mathbf{y}_{t-1} - \boldsymbol{\beta} - \Gamma'\boldsymbol{\mu}(t-1)] + \mathbf{u}_t$$

or

$$\Phi(L)\Delta\mathbf{y}_t + \Lambda\Gamma'\mathbf{y}_{t-1} = \Lambda\boldsymbol{\beta} + [\Phi(1) + (t-1)\Lambda\Gamma']\boldsymbol{\mu} + \mathbf{u}_t \quad (40)$$

Let for all t ,

$$\begin{aligned} \mathbf{w}_t &:= \Phi(L)\Delta\mathbf{y}_t + \Lambda\Gamma'\mathbf{y}_{t-1} \\ \mathbf{Z}_t &:= \begin{bmatrix} \Lambda & \Phi(1) + (t-1)\Lambda\Gamma' \end{bmatrix} \end{aligned}$$

Then for all t ,

$$\mathbf{w}_t = \mathbf{Z}_t\boldsymbol{\psi} + \mathbf{u}_t$$

Let

$$\mathbf{w} := \begin{pmatrix} \mathbf{w}_{p+1} \\ \vdots \\ \mathbf{w}_T \end{pmatrix}, \quad \mathbf{Z} := \begin{bmatrix} \mathbf{Z}_{p+1} \\ \vdots \\ \mathbf{Z}_T \end{bmatrix}, \quad \mathbf{u} := \begin{pmatrix} \mathbf{u}_{p+1} \\ \vdots \\ \mathbf{u}_T \end{pmatrix}$$

Then we have a normal linear model for $\mathbf{w}$ given $\mathbf{Z}$ such that

$$\mathbf{w} = \mathbf{Z}\boldsymbol{\psi} + \mathbf{u} \quad (41)$$

$$\mathbf{u} \sim \mathbf{N}_{N(T-p)}(\mathbf{0}_{N(T-p)}, \mathbf{I}_{T-p} \otimes \mathbf{P}^{-1}) \quad (42)$$

Theorem 2. Suppose that

$$\boldsymbol{\psi}|\Phi, \mathbf{P}, \Lambda, \Gamma \sim \mathbf{N}_{r+N}(\boldsymbol{\psi}_0, \mathbf{Q}_0^{-1})$$

Then

$$\boldsymbol{\psi}|\Phi, \mathbf{P}, \Lambda, \Gamma, \mathbf{w} \sim \mathbf{N}_{r+N}(\boldsymbol{\psi}_1, \mathbf{Q}_1^{-1}) \quad (43)$$

where

$$\mathbf{Q}_1 := \mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{Z} + \mathbf{Q}_0$$

$$\boldsymbol{\psi}_1 = \mathbf{Q}_1^{-1}[\mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{w} + \mathbf{Q}_0\boldsymbol{\psi}_0]$$

Proof. Let $\boldsymbol{\psi}_{\text{GLS}}$ be the GLS estimator of $\boldsymbol{\psi}$, i.e.,

$$\boldsymbol{\psi}_{\text{GLS}} = [\mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{Z}]^{-1}\mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{w}$$

By Lemma 1,

$$\begin{aligned}\mathbf{Q}_1 &:= \mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{Z} + \mathbf{Q}_0 \\ \boldsymbol{\psi}_1 &:= \mathbf{Q}_1^{-1}[\mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{Z}\boldsymbol{\psi}_{\text{GLS}} + \mathbf{Q}_0\boldsymbol{\psi}_0]\end{aligned}$$

where

$$\mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{Z}\boldsymbol{\psi}_{\text{GLS}} = \mathbf{Z}'(\mathbf{I}_{T-p} \otimes \mathbf{P})\mathbf{w}$$

□

A.3 VAR parameters

Let for all t ,

$$\mathbf{e}_t := \Gamma' \mathbf{y}_t - \boldsymbol{\beta} - \delta t, \quad \mathbf{s}_t^* := \begin{pmatrix} \Delta \mathbf{y}_t - \boldsymbol{\mu} \\ \vdots \\ \Delta \mathbf{y}_{t-p+1} - \boldsymbol{\mu} \end{pmatrix}$$

Write the VECM as for all t ,

$$\Delta \mathbf{y}_t - \boldsymbol{\mu} + \Lambda \mathbf{e}_{t-1} = \Phi \mathbf{s}_{t-1}^* + \mathbf{u}_t \quad (44)$$

Let

$$\mathbf{Y}_* := \begin{bmatrix} (\Delta \mathbf{y}_{p+1} - \boldsymbol{\mu} + \Lambda \mathbf{e}_p)' \\ \vdots \\ (\Delta \mathbf{y}_T - \boldsymbol{\mu} + \Lambda \mathbf{e}_{T-1})' \end{bmatrix}, \quad \mathbf{X} := \begin{bmatrix} \mathbf{s}_p^{*'} \\ \vdots \\ \mathbf{s}_{T-1}^{*'} \end{bmatrix}, \quad \mathbf{U} := \begin{bmatrix} \mathbf{u}_{p+1}' \\ \vdots \\ \mathbf{u}_T' \end{bmatrix}$$

Then we have a normal linear model for $\mathbf{Y}_*$ given $\mathbf{X}$ such that

$$\mathbf{Y}_* = \mathbf{X}\Phi' + \mathbf{U} \quad (45)$$

$$\mathbf{U} \sim N_{(T-p) \times N}(\mathbf{0}_{(T-p) \times N}, \mathbf{I}_{T-p}, \mathbf{P}^{-1}) \quad (46)$$

Let $\Phi_{\text{OLS}} := \mathbf{Y}_*' \mathbf{X}(\mathbf{X}' \mathbf{X})^{-1}$ and $\mathbf{S} := (\mathbf{Y}_* - \mathbf{X}\Phi_{\text{OLS}})'(\mathbf{Y}_* - \mathbf{X}\Phi_{\text{OLS}})$.

Theorem 3. Suppose that

$$\Phi | \mathbf{P}, \boldsymbol{\psi}, \Lambda, \Gamma, \nu \sim N_{N \times pN}(\mathbf{M}_0; \mathbf{P}^{-1}, (\nu \mathbf{D}_0)^{-1})$$

$$\mathbf{P} | \boldsymbol{\psi}, \Lambda, \Gamma, \nu \sim W_N(k_0; \mathbf{S}_0^{-1})$$

Then

$$\Phi | \mathbf{P}, \boldsymbol{\psi}, \Lambda, \Gamma, \nu, \mathbf{Y}_*, \mathbf{X} \sim N_{N \times pN}(\mathbf{M}_1; \mathbf{P}^{-1}, \mathbf{D}_1^{-1}) \quad (47)$$

$$\mathbf{P} | \boldsymbol{\psi}, \Lambda, \Gamma, \nu, \mathbf{Y}_*, \mathbf{X} \sim W_N(k_1; \mathbf{S}_1^{-1}) \quad (48)$$

where

$$\mathbf{D}_1 := \mathbf{X}' \mathbf{X} + \nu \mathbf{D}_0$$

$$\mathbf{M}_1 := (\Phi_{\text{OLS}} \mathbf{X}' \mathbf{X} + \mathbf{M}_0 \nu \mathbf{D}_0) \mathbf{D}_1^{-1}$$

$$k_1 := T - p + k_0$$

$$\mathbf{S}_1 := (\boldsymbol{\Phi}_{\text{OLS}} - \mathbf{M}_0) \left[(\mathbf{X}'\mathbf{X})^{-1} + (\nu\mathbf{D}_0)^{-1} \right]^{-1} (\boldsymbol{\Phi}_{\text{OLS}} - \mathbf{M}_0)' + \mathbf{S} + \mathbf{S}_0$$

Proof. Apply Lemma 2. □

A.4 Loading matrix

Write the VECM as for all t ,

$$\boldsymbol{\Phi}(\mathbf{L})(\Delta \mathbf{y}_t - \boldsymbol{\mu}) = -\Lambda \mathbf{e}_{t-1} + \mathbf{u}_t \quad (49)$$

Let

$$\mathbf{W} := \begin{bmatrix} [\boldsymbol{\Phi}(\mathbf{L})(\Delta \mathbf{y}_{p+1} - \boldsymbol{\mu})]' \\ \vdots \\ [\boldsymbol{\Phi}(\mathbf{L})(\Delta \mathbf{y}_T - \boldsymbol{\mu})]' \end{bmatrix}, \quad \mathbf{E} := \begin{bmatrix} -\mathbf{e}'_p \\ \vdots \\ -\mathbf{e}'_{T-1} \end{bmatrix}$$

Then we have a normal linear model for $\mathbf{W}$ given $\mathbf{E}$ such that

$$\begin{aligned} \mathbf{W} &= \mathbf{E}\boldsymbol{\Lambda}' + \mathbf{U} \\ \mathbf{U} &\sim N_{(T-p) \times N}(\mathbf{0}_{(T-p) \times N}, \mathbf{I}_{T-p}, \mathbf{P}^{-1}) \end{aligned}$$

or

$$\begin{aligned} \mathbf{W}' &= \Lambda \mathbf{E}' + \mathbf{U}' \\ \mathbf{U}' &\sim N_{N \times (T-p)}(\mathbf{0}_{N \times (T-p)}, \mathbf{P}^{-1}, \mathbf{I}_{T-p}) \end{aligned}$$

Let $\boldsymbol{\lambda} := \text{vec}(\boldsymbol{\Lambda})$. Then we have a normal linear model for $\text{vec}(\mathbf{W}')$ such that

$$\text{vec}(\mathbf{W}') = (\mathbf{E} \otimes \mathbf{I}_N) \boldsymbol{\lambda} + \text{vec}(\mathbf{U}') \quad (50)$$

$$\text{vec}(\mathbf{U}') \sim N_{N(T-p)}(\mathbf{0}_{N(T-p)}, \mathbf{I}_{T-p} \otimes \mathbf{P}^{-1}) \quad (51)$$

Theorem 4. Suppose that

$$\boldsymbol{\lambda} | \boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Gamma} \sim N_{Nr}(\mathbf{0}_{Nr}, \mathbf{U}_0^{-1})$$

where $\mathbf{U}_0 := \mathbf{I}_r \otimes \eta_0 \mathbf{G}_0$. Then

$$\boldsymbol{\lambda} | \boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Gamma}, \mathbf{W}, \mathbf{E} \sim N_{Nr}(\boldsymbol{\lambda}_1, \mathbf{U}_1^{-1}) \quad (52)$$

where

$$\mathbf{U}_1 = \mathbf{E}'\mathbf{E} \otimes \mathbf{P} + \mathbf{U}_0 \quad (53)$$

$$\boldsymbol{\lambda}_1 = \mathbf{U}_1^{-1} \text{vec}(\mathbf{P}\mathbf{W}'\mathbf{E}) \quad (54)$$

Proof. Let $\boldsymbol{\Lambda}_{\text{OLS}} := \mathbf{W}'\mathbf{E}(\mathbf{E}'\mathbf{E})^{-1}$ and $\boldsymbol{\lambda}_{\text{OLS}} := \text{vec}(\boldsymbol{\Lambda}_{\text{OLS}})$. By Lemma 1,

$$\begin{aligned} \mathbf{U}_1 &:= (\mathbf{E} \otimes \mathbf{I}_N)' (\mathbf{I}_{T-p} \otimes \mathbf{P}) (\mathbf{E} \otimes \mathbf{I}_N) + \mathbf{U}_0 \\ &= \mathbf{E}'\mathbf{E} \otimes \mathbf{P} + \mathbf{U}_0 \end{aligned}$$

$$\boldsymbol{\lambda}_1 := \mathbf{U}_1^{-1} (\mathbf{E}'\mathbf{E} \otimes \mathbf{P}) \boldsymbol{\lambda}_{\text{OLS}}$$

where

$$\begin{aligned} (\mathbf{E}'\mathbf{E} \otimes \mathbf{P}) \boldsymbol{\lambda}_{\text{OLS}} &= (\mathbf{E}'\mathbf{E} \otimes \mathbf{P}) \text{vec}(\mathbf{W}'\mathbf{E}(\mathbf{E}'\mathbf{E})^{-1}) \\ &= (\mathbf{E}'\mathbf{E} \otimes \mathbf{P}) [(\mathbf{E}'\mathbf{E})^{-1} \otimes \mathbf{W}'] \text{vec}(\mathbf{E}) \\ &= (\mathbf{I}_r \otimes \mathbf{P}\mathbf{W}') \text{vec}(\mathbf{E}) \end{aligned}$$

$$= \text{vec}(\mathbf{P}\mathbf{W}'\mathbf{E})$$

□

A.5 Cointegrating matrix

Write the VECM as for all t ,

$$\Phi(L)(\Delta \mathbf{y}_t - \boldsymbol{\mu}) - \Lambda \boldsymbol{\beta} = -\Lambda \Gamma' [\mathbf{y}_{t-1} - \boldsymbol{\mu}(t-1)] + \mathbf{u}_t \quad (55)$$

Let

$$\mathbf{W}_* := \begin{bmatrix} [\Phi(L)(\Delta \mathbf{y}_{p+1} - \boldsymbol{\mu}) - \Lambda \boldsymbol{\beta}]' \\ \vdots \\ [\Phi(L)(\Delta \mathbf{y}_T - \boldsymbol{\mu}) - \Lambda \boldsymbol{\beta}]' \end{bmatrix}, \quad \mathbf{Z}_* := \begin{bmatrix} -(\mathbf{y}_p - p\boldsymbol{\mu})' \\ \vdots \\ -[\mathbf{y}_{T-1} - (T-1)\boldsymbol{\mu}]' \end{bmatrix}$$

Then we have a normal linear model for $\mathbf{W}_*$ given $\mathbf{Z}_*$ such that

$$\mathbf{W}_* = \mathbf{Z}_* \Gamma \Lambda' + \mathbf{U}$$

$$\mathbf{U} \sim N_{(T-p) \times N}(\mathbf{0}_{(T-p) \times N}, \mathbf{I}_{T-p}, \mathbf{P}^{-1})$$

Let $\boldsymbol{\gamma} := \text{vec}(\Gamma)$. Then we have a normal linear model for $\text{vec}(\mathbf{W}_*)$ such that

$$\text{vec}(\mathbf{W}_*) = (\Lambda \otimes \mathbf{Z}_*) \boldsymbol{\gamma} + \text{vec}(\mathbf{U}) \quad (56)$$

$$\text{vec}(\mathbf{U}) \sim N_{N(T-p)}(\mathbf{0}_{N(T-p)}, \mathbf{P}^{-1} \otimes \mathbf{I}_{T-p}) \quad (57)$$

Theorem 5. Suppose that

$$\boldsymbol{\gamma} | \boldsymbol{\psi}, \Phi, \mathbf{P}, \Lambda \sim N_{Nr}(\mathbf{0}_{Nr}, \mathbf{V}_0^{-1})$$

where $\mathbf{V}_0 := \mathbf{I}_r \otimes \mathbf{H}(\tau_0)$. Then

$$\boldsymbol{\gamma} | \boldsymbol{\psi}, \Phi, \mathbf{P}, \Lambda, \mathbf{W}_*, \mathbf{Z}_* \sim N_{Nr}(\boldsymbol{\gamma}_1, \mathbf{V}_1^{-1}) \quad (58)$$

where

$$\mathbf{V}_1 = \Lambda' \mathbf{P} \Lambda \otimes \mathbf{Z}_*' \mathbf{Z}_* + \mathbf{V}_0 \quad (59)$$

$$\boldsymbol{\gamma}_1 = \mathbf{V}_1^{-1} \text{vec}(\mathbf{Z}_*' \mathbf{W}_* \mathbf{P} \Lambda) \quad (60)$$

Proof. Let $\boldsymbol{\gamma}_{\text{GLS}}$ be the GLS estimator of $\boldsymbol{\gamma}$, i.e.,

$$\begin{aligned} \boldsymbol{\gamma}_{\text{GLS}} &= [(\Lambda \otimes \mathbf{Z}_*)' (\mathbf{P} \otimes \mathbf{I}_{T-p}) (\Lambda \otimes \mathbf{Z}_*)]^{-1} (\Lambda \otimes \mathbf{Z}_*)' (\mathbf{P} \otimes \mathbf{I}_{T-p}) \text{vec}(\mathbf{W}_*) \\ &= (\Lambda' \mathbf{P} \Lambda \otimes \mathbf{Z}_*' \mathbf{Z}_*)^{-1} (\Lambda' \mathbf{P} \otimes \mathbf{Z}_*') \text{vec}(\mathbf{W}_*) \\ &= (\Lambda' \mathbf{P} \Lambda \otimes \mathbf{Z}_*' \mathbf{Z}_*)^{-1} \text{vec}(\mathbf{Z}_*' \mathbf{W}_* \mathbf{P} \Lambda) \end{aligned}$$

By Lemma 1,

$$\begin{aligned} \mathbf{V}_1 &:= (\Lambda \otimes \mathbf{Z}_*)' (\mathbf{P} \otimes \mathbf{I}_{T-p}) (\Lambda \otimes \mathbf{Z}_*) + \mathbf{V}_0 \\ &= \Lambda' \mathbf{P} \Lambda \otimes \mathbf{Z}_*' \mathbf{Z}_* + \mathbf{V}_0 \\ \boldsymbol{\gamma}_1 &:= \mathbf{V}_1^{-1} (\Lambda' \mathbf{P} \Lambda \otimes \mathbf{Z}_*') \boldsymbol{\gamma}_{\text{GLS}} \\ &= \mathbf{V}_1^{-1} \text{vec}(\mathbf{Z}_*' \mathbf{W}_* \mathbf{P} \Lambda) \end{aligned}$$

□

A.6 Tightness hyperparameter

We have

$$\begin{aligned} p(\nu|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Lambda}, \boldsymbol{\Gamma}, \mathbf{Y}) &\propto p(\mathbf{Y}|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Lambda}, \boldsymbol{\Gamma}, \nu)p(\boldsymbol{\psi}|\boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Lambda}, \boldsymbol{\Gamma}, \nu)p(\boldsymbol{\Lambda}, \boldsymbol{\Gamma}|\boldsymbol{\Phi}, \mathbf{P}, \nu)p(\boldsymbol{\Phi}, \mathbf{P}|\nu)p(\nu) \\ &= p(\mathbf{Y}|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Lambda}, \boldsymbol{\Gamma})p(\boldsymbol{\psi}|\boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Lambda}, \boldsymbol{\Gamma})p(\boldsymbol{\Lambda}, \boldsymbol{\Gamma}|\boldsymbol{\Phi}, \mathbf{P})p(\boldsymbol{\Phi}, \mathbf{P}|\nu)p(\nu) \\ &\propto p(\boldsymbol{\Phi}, \mathbf{P}|\nu)p(\nu) \end{aligned}$$

Thus $p(\nu|\boldsymbol{\psi}, \boldsymbol{\Phi}, \mathbf{P}, \boldsymbol{\Lambda}, \boldsymbol{\Gamma}, \mathbf{Y}) = p(\nu|\boldsymbol{\Phi}, \mathbf{P})$.

Theorem 6. Suppose that

$$\nu \sim \text{Gam}\left(\frac{A_0}{2}, \frac{B_0}{2}\right)$$

Then

$$\nu|\boldsymbol{\Phi}, \mathbf{P} \sim \text{Gam}\left(\frac{A_1}{2}, \frac{B_1}{2}\right) \quad (61)$$

where

$$\begin{aligned} A_1 &:= pN^2 + A_0 \\ B_1 &:= \text{tr}(\mathbf{P}(\boldsymbol{\Phi} - \mathbf{M}_0)\mathbf{D}_0(\boldsymbol{\Phi} - \mathbf{M}_0)') + B_0 \end{aligned}$$

Proof. We have

$$\begin{aligned} p(\nu|\boldsymbol{\Phi}, \mathbf{P}) &\propto p(\boldsymbol{\Phi}, \mathbf{P}, \nu) \\ &= p(\boldsymbol{\Phi}|\mathbf{P}, \nu)p(\mathbf{P})p(\nu) \\ &\propto p(\boldsymbol{\Phi}|\mathbf{P}, \nu)p(\nu) \end{aligned}$$

where²⁵

$$p(\boldsymbol{\Phi}|\mathbf{P}, \nu) = \frac{\text{etr}(-\mathbf{P}(\boldsymbol{\Phi} - \mathbf{M}_0)\nu\mathbf{D}_0(\boldsymbol{\Phi} - \mathbf{M}_0)'/2)}{(2\pi)^{pN^2/2} \det(\nu\mathbf{D}_0)^{-N/2} \det(\mathbf{P})^{-pN/2}} \quad (62)$$

$$p(\nu) = \frac{\nu^{A_0/2-1} e^{(B_0/2)\nu}}{\Gamma(A_0/2)(B_0/2)^{-A_0/2}} \quad (63)$$

Hence

$$\begin{aligned} p(\nu|\boldsymbol{\Phi}, \mathbf{P}) &\propto \nu^{pN^2/2} \text{etr}\left(-\frac{\mathbf{P}(\boldsymbol{\Phi} - \mathbf{M}_0)\nu\mathbf{D}_0(\boldsymbol{\Phi} - \mathbf{M}_0)'}{2}\right) \frac{\nu^{A_0/2-1}}{e^{(B_0/2)\nu}} \\ &= \frac{\nu^{(pN^2+A_0)/2-1}}{e^{\{\text{tr}(\mathbf{P}(\boldsymbol{\Phi}-\mathbf{M}_0)\mathbf{D}_0(\boldsymbol{\Phi}-\mathbf{M}_0)')+B_0\}/2}\nu} \end{aligned}$$

□

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²⁵ See Gupta and Nagar (1999) on the pdf of a matrix normal distribution.

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