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Antieigenvalue analysis for continuum mechanics, economics, and number theory

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Abstract: My recent book *Antieigenvalue Analysis*, World-Scientific, 2012, presented the theory of antieigenvalues from its inception in 1966 up to 2010, and its applications within those forty-five years to Numerical Analysis, Wavelets, Statistics, Quantum Mechanics, Finance, and Optimization. Here I am able to offer three further areas of application: Continuum Mechanics, Economics, and Number Theory. In particular, the critical angle of repose in a continuum model of granular materials is shown to be exactly my matrix maximum turning angle of the stress tensor of the material. The important Sharpe ratio of the Capital Asset Pricing Model is now seen in terms of my antieigenvalue theory. Euclid's Formula for Pythagorean triples becomes a special case of my operator trigonometry.

Keywords: Antieigenvalue; granular materials; investment theory; Pythagorean Triples

MSC: 47A63, 35Q70, 91G10, 11E25

1 Introduction

Antieigenvalue analysis [9] is an operator trigonometry concerned with those vectors, called antieigenvectors, which are most-turned by a matrix or a linear operator A. This is in contrast to the conventional eigenvalue analysis, which is concerned with those vectors, called eigenvectors, which are not turned at all by A. Antieigenvalue theory may be usefully thought of as a variational theory, extending the variational Rayleigh-Ritz theory which characterizes eigenvectors, to an enlarged theory also characterizing antieigenvectors.

Two key entities in the antieigenvalue theory are the first antieigenvalue

$$\mu_1 = \cos\phi(A) = \min_{x\neq 0} \frac{\langle Ax, x\rangle}{\|Ax\|\,\|x\|} \tag{1.1}$$

and the related convex minimum

$$\nu_1 = \sin\phi(A) = \min_{\epsilon>0} \|\, \epsilon A - I \,\| \,. \tag{1.2}$$

Here I will specialize the antieigenvalue theory to A, an $n\times n$ symmetric positive definite matrix. One generally has the fundamental relation

$$\cos^2\phi(A) + \sin^2\phi(A) = 1. \tag{1.3}$$

There are two maximally turned first antieigenvectors

$$x_\pm = \left(\frac{\lambda_n}{\lambda_1+\lambda_n}\right)^{\frac{1}{2}} x_1 \pm \left(\frac{\lambda_1}{\lambda_1+\lambda_n}\right)^{\frac{1}{2}} x_n \,, \tag{1.4}$$

where $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ are the eigenvalues of A, and where x_1 is any norm-one eigenvector from the λ_1-eigenspace and x_n is any norm-one eigenvector from the λ_n-eigenspace. The antieigenvectors $x_\pm$ in (1.4)

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have also been normalized to be of norm-one. For $n \times n$ symmetric positive definite A the expressions in (1.1) and (1.2) have useful explicit valuations as

$$\mu_1 = \frac{2\sqrt{\lambda_1 \lambda_n}}{\lambda_1 + \lambda_n}, \quad \nu_1 = \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}. \tag{1.5}$$

For further elaboration of the general antieigenvalue theory and its applications to numerical analysis, wavelets, statistics, quantum mechanics, finance, and optimization, I refer to [9].

In sections 2, 3, and 4 respectively, I will present three new domains of application for antieigenvalue analysis: continuum mechanics, economics, and number theory. New results and insights for each will be obtained. Conclusions and comments are given in section 5.

This paper is an elaboration of my lecture and extended abstract [12] at the 24th IWMS at Haikou, Hainan, China.

2 Continuum Mechanics

My discussion here leans heavily on the recent paper [7] which establishes a continuum mechanical model for the stability of granular material heaps. The paper [7] explores the notion of (maximum) angle of repose for granular materials. On the other hand, my theory of antieigenvalues [9] has as one of its essential ingredients the notion of (maximum) operator turning angle. Here is how to connect the two theories.

Following [7], the equilibrium equations for a granular pile of local slope θ are

$$\partial_x \sigma_{xx} + \partial_z \sigma_{xz} = \rho g \sin\theta, \tag{2.1a}$$

$$\partial_x \sigma_{xz} + \partial_z \sigma_{zz} = \rho g \cos\theta. \tag{2.1b}$$

The stress tensor Σ can be written in singular value decomposition

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}, \tag{2.2}$$

where $\sigma_1 \geqq \sigma_2 > 0$ are the principal stresses and where ψ gives the principal directions. By considering a plane within the material and the normal and tangential stresses upon it in terms of the coefficient of friction of the material and a corresponding angle δ of internal friction, it is deduced in [7] that the largest sustainable angle of repose θ is given by

$$\sin\theta = \frac{\tau}{\sigma}, \text{ where } \sigma = \frac{\sigma_1 + \sigma_2}{2}, \ \tau = \frac{\sigma_1 - \sigma_2}{2}. \tag{2.3}$$

Some assumptions in this modeling of the discrete by the continuous have of course been made. Among those are a linear dependence on the vertical z direction and a stress-free condition at the pile's surface at $z = 0$.

Let us see and recast this continuum mechanical model for stable granular material piles into my antieigenvalue theory. The key is to remember that the unique ϵ for which the minimum in (1.2) is attained in known [9] to be $\epsilon_m = \frac{2}{\sigma_1+\sigma_2}$ for a 2×2 matrix with singular values σ_1 and σ_2. Then straightforward calculations confirm from (2.2) and (2.3) that

$$\sin\phi(\Sigma) = \|\epsilon_m \Sigma - I\| = \frac{\tau}{\sigma}. \tag{2.4}$$

This confirms that my maximum turning angle of the stress tensor matrix Σ gives exactly the largest sustainable angle of repose of the granular material heap. Many physical examples and history of the appropriateness of such models may be found in [7].

Continuing, use of the law of sliding friction and a consideration of normal and tangential stresses plus some nice geometry (see figures in [7]) concludes that for the granular pile to stand up, it must obey the following constraint, stated here in two equivalent forms:

$$\frac{\tau}{\sigma} \leqq \sin\delta \iff \frac{\sigma_1}{\sigma_2} \leqq \frac{1+\sin\delta}{1-\sin\delta}. \tag{2.5}$$

Here δ is the material angle of internal friction defined by $\mu = \tan\delta$ where μ is the coefficient of friction of the material. Note that this μ is from the notation of [7] and is not my first antieigenvalue μ_1. However, as a side-result of the following analysis, it happens that the two are related in terms of the stress operator turning angle by $\mu = \mu_1/\sin\phi(\Sigma)$.

Integrating the equilibrium equations (2.1), assuming a stress free boundary condition at the base of the pile, and assuming a linearized vertical dependency given by $\sigma_{xx} = \lambda\rho g z\cos\theta$, where λ is a linearization proportionality constant, one arrives [7] at the interesting expression

$$\frac{\tau}{\sigma} = \sqrt{1 + 4\frac{\tan^2\theta - \lambda}{(1+\lambda)^2}}\,. \tag{2.6}$$

To find the steepest angle θ which this analysis permits, the right-hand side of (2.6) is minimized with respect to λ, giving the extremum condition

$$\frac{\tau}{\sigma} = \sin\theta\,. \tag{2.7}$$

In particular, this means that the critical angle of repose θ for the granular pile is the angle δ of material internal friction.

This equality in the second equivalent part of (2.5), namely, that

$$\frac{\sigma_1}{\sigma_2} = \frac{1+\sin\delta}{1-\sin\delta}\,, \tag{2.8}$$

is a special instance of a rather general working proposition which I obtained in [8], see also [9, pp. 55–56]. That general rule, or theorem if you like, is that if you encounter some entity γ in any theory which happens to be related to the standard matrix condition number $\kappa = \sigma_1/\sigma_2$ according to

$$\gamma = \frac{\kappa-1}{\kappa+1}\,, \tag{2.9}$$

then there obtains a three-way connection between your theory, the Kantorovich-Wielandt theory treated in [8], and my operator trigonometry [9]. We may apply that working rule here.

To do so, note that since the left side of (2.8) is exactly the condition number κ of the stress tensor Σ, we may solve for $\sin\delta$ to obtain

$$\sin\delta = \frac{\kappa-1}{\kappa+1}\,. \tag{2.10}$$

Thus there obtains the three-way relationship

$$\sin\delta = \cos\beta(\Sigma^{1/2}) = \sin\phi(\Sigma). \tag{2.11}$$

In (2.11) I have used the notation $\beta(A)$ for the Kantorovich-Wielandt condition number angle generally defined by $\cot(\beta/2) = \kappa$. Note that, in particular, just from (2.8) and this working rule of [8] we could have obtained the equality of the granular angle of friction δ to our maximum turning angle $\phi(\Sigma)$ that we obtained above.

Next, let us look at the Law of Sliding Friction utilized in [7]. The requirement there is that

$$|T_T| \leqq \mu T_N, \tag{2.12}$$

where $\mu = \tan\delta$ is the material internal friction constant. Using some results of [7], we may write (2.12) as, with some obvious algebra,

$$\left(\frac{\sigma_1-\sigma_2}{2}\right)|\sin 2(\alpha-\psi)| \leqq \mu\left(\frac{\sigma_1+\sigma_2}{2}\right)\left[1+\sin\phi(\Sigma)\cos 2(\alpha-\psi)\right]. \tag{2.13}$$

Here α is the normal angle to an immersed plane, see [7, Fig. 2.2]. Using above results on equality of angles, (2.13) becomes

$$\sin\phi(\Sigma)\,|\sin 2(\alpha-\psi)| \leqq \frac{\sin\phi(\Sigma)}{\cos\phi(\Sigma)}\left[1+\sin\phi(\Sigma)\cos 2(\alpha-\psi)\right]. \tag{2.14}$$

Assuming the angles are all properly acute so that we can drop the absolute value bars, and canceling out $\sin\phi(\Sigma)$ and using the conventional trigonometric identity $\sin(A-B) = \sin A\cos B - \sin B\cos A$, (2.12) has become

$$\sin[2(\alpha-\psi)-\phi(\Sigma)] \leqq 1. \tag{2.15}$$

Finally, let us note that from (1.4) and (2.2) the antieigenvectors of symmetric stress matrix Σ are

$$x_{\pm} = \left(\frac{\sigma_1}{\sigma_1+\sigma_2}\right)^{1/2} \begin{bmatrix} -\sin\psi \\ \cos\psi \end{bmatrix} \pm \left(\frac{\sigma_2}{\sigma_1+\sigma_2}\right)^{1/2} \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}. \tag{2.16}$$

Thus the two maximally turnable vectors nicely capture the principal stresses along with their principal directions.

Let us summarize our results as follows.

Theorem 2.1. *The (maximum) angle of repose θ, the material angle of friction δ, and the (maximum) turning angle $\phi(\Sigma)$ of the stress tensor, are the same.*

Corollary 2.2. *The Law of Sliding Friction may be reformulated as* (2.15), *now in terms of a plane of normal angle within the material, the principal direction angle ψ of the stress tensor, and the stress tensor maximal operator-trigonometric turning angle $\phi(\Sigma)$.*

Corollary 2.2 is a purely geometrical statement of the Law of Sliding Friction in terms of angles only. The reader may refer to the nice Mohr circle construction in Fig. 2.2 of [7] to see all of the relevant angles as they occur in this analysis of friction of granular heaps. To that figure you may now add our stress tensor maximal turning angle $\phi(\Sigma)$ at both occurrences of the friction angle δ. This brings a kind of new internal dynamics, represented by the antieigenvectors and maximum turning angle $\phi(\Sigma)$, into the previously static picture.

In particular, the maximum value in (2.15) is obtained when the angle $2(\alpha-\psi)-\phi(\Sigma) = \pi/2$. That situation is represented by the vertical dotted line in Fig. 2.2 in [7]. As the angles $\phi(\Sigma)$ and ψ are predetermined by the material stress tensor, this criticality in (2.15) occurs only when the immersed plane of normal angle α has such angle $\alpha = \pi/4 + \psi + \phi(\Sigma)/2$. We could summarize this analysis by saying that this limiting normal angle depends on the eigenvectors (through ψ) and now on the antieigenvectors (through $\phi(\Sigma)$).

3 Economics

I will go into less detail in this section because more detail may be obtained from the forthcoming paper [11]. The main message is that the renowned Sharpe ratio of the Capital Asset Pricing Model (CAPM) may be seen in terms of my antieigenvalue theory. This observation was announced in the book [9, p.182]. The further details are worked out in [11]. Briefly, the fundamental link between the investment theory and my antieigenvalue theory lies in the fact [9, p. 188] that the first antieigenvalue μ_1 in (1.5) may be seen as a ratio of means:

$$\mu_1 = \frac{\text{geometric mean}}{\text{arithmetic mean}}. \tag{3.1}$$

I refer you to the large standard book [14] for the uses of Sharpe's ratio in investment theory. Also I will specifically refer to the excellent book [1] for aspects of the CAPM as it is used in portfolio design theory and in high frequency trading.

The Capital Asset Pricing model assumes the Efficient Market Hypothesis and then tells you to measure the return-to-risk of your portfolio against the market. From the assumption that the full market has optimized the return-to-risk, your Sharpe ratio

$$S = \frac{E[r]}{\sigma[r]}, \tag{3.2}$$

where $E[r]$ is the average return over a number of periods and $\sigma[r]$ is the corresponding standard deviation, will not be greater than that of the whole (e.g, think indexing) market's Sharpe ratio. See especially[1, Fig. 5.2, p. 55], to picture Sharpe ratios as mean-variance slopes. Here I am just dropping the risk-free return rate R_f from numerators, as it is effectively zero these days anyway.

Suppose now we look at the last two years of annualized returns r_1 and r_2. We may form the usual (arithmetic) Sharpe ratio $S_{AM} = \frac{r_1+r_2}{2\sigma}$ and also a (geometric) Sharpe ratio $S_{GM} = \frac{\sqrt{r_1 r_2}}{\sigma}$ and upon dividing the latter by the former we arrive at

$$G = \frac{S_{GM}}{S_{AM}} = \frac{2\sqrt{r_1 r_2}}{r_1 + r_2} \tag{3.3}$$

which is my first antieigenvalue μ_1 as seen from (1.5). Comparison to actual market data [4] favors [11] the use of different variances in S_{AM} and S_{GM} to further sharpen my new return-to-risk trigonometric investment theory. In particular, a new general concept of growth-to-risk angles is proposed [11].

Going beyond [11], as noted there my use of geometric mean denominators relates to currently important economic issues concerning realized volatilities [2]. Perhaps such could be pursued in a later investigation.

Obviously one could set up a more extensive operator-trigonometric theory based upon a whole matrix of annualized returns $r_1, r_2, \ldots, r_n$ from which one could compare financial rewards linked directly to the higher antieigenvalues and corresponding higher antieigenvector and internal critical angles of the general antieigenvalue theory [9].

Separately from [11], in three recent papers [5, 6, 10] we have developed a Time Operator theory for financial markets. Time Operators originated in quantum mechanics and have been adapted to statistical mechanics and to stationary stochastic processes. Essential to our application of Time operators to financial markets are variance estimates. Our theory is worked out for non-stationary Bernoulli processes in [5]. Among five well-known volatility estimators discussed in [16], we found the most suitable for our analysis of the Greek financial market during elections to be the Rogers-Satchel estimator. See [5] and [16] for further details. In [6] we then extend our model to Markov processes with specific examples taken from US GNP data and Dow-Jones closing prices.

As all symmetric and Hermitian matrices have a natural spectral theory and therefore now a natural anti-spectral theory of critical turning angles, such could surely be worked out for Time operator theory as it is now applied to finance and economics. This has already been done for wavelets [9, Chapter 5]. The Time operator Age of a process is a new statistical index which assesses the average level of innovations during the observation period. As such, it is a new measure of the complexity of the market.

4 Number Theory

Now some new connections of the antieigenvalue theory to number theory which I only recently discovered. I was quite surprised as the origins of the two theories are completely different.

Given two arbitrary relatively prime positive integers m and n, with $m > n$, one of them being even, the other odd, then the numbers

$$a = 2mn, \ b = m^2 - n^2, \ c = m^2 + n^2 \tag{4.1}$$

form a primitive Pythagorean triple:

$$a^2 + b^2 = c^2. \tag{4.2}$$

This sufficient condition is also necessary. For more details see [13]. This construction and characterization of Pythagorean triples is often called Euclid's Formula.

To connect that theory to my antieigenvalue theory and its matrix trigonometry, I may now form the matrix (and its similarity class of matrices with the same eigenvalues m^2 and n^2)

$$A = \begin{bmatrix} n^2 & 0 \\ 0 & m^2 \end{bmatrix}. \tag{4.3}$$

Immediately from my matrix operator trigonometry [9] and the expressions (1.5) we have

$$\cos\phi(A) = \frac{2mn}{m^2 + n^2}, \quad \sin\phi(A) = \frac{m^2 - n^2}{m^2 + n^2}. \tag{4.4}$$

Noting also that $\cos\phi(A) = a/c$ and $\sin\phi(A) = b/c$, we have established the following result:

Proposition 4.1. *Euclid's Formula for Pythagorean Triples is a special case of my operator trigonometry.*

We may propose to call these matrices $A_{m,n}$, Pythagorean Triple Matrices. Their maximum turning angles may be called special Pythagorean turning angles $\phi_{m,n}(A)$. Their corresponding normalized Pythagorean antieigenvectors are

$$x_{\pm} = \left(\frac{m^2}{m^2+n^2}\right)^{\frac{1}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm \left(\frac{n^2}{m^2+n^2}\right)^{\frac{1}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{m^2+n^2}} \begin{bmatrix} m \\ \pm n \end{bmatrix}. \tag{4.5}$$

We know of course there are an infinite number of these Pythagorean angles, which are now embedded within my antieigenvalue operator trigonometry. One could build a whole matrix theory of them.

This new connection of my antieigenvalue analysis to the Pythagorean triple number theory may be seen to have other interesting manifestations. Here is another one, couched in the terminology of algebraic geometry. Let $x = \left(\frac{m}{n}, 0\right)$ be a point on the x-axis. It's stereographic projection onto the unit circle becomes, now seen operator-theoretically, the point

$$P = \left(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2}\right) = \left(\cos\phi(A_{m,n}), \ \sin\phi(A_{m,n})\right). \tag{4.6}$$

The stereographic point of view comes from a treatment of spinors and twistors [15].

Here is a precise example. Let $m = 2$ and $n = 1$. Then taking $a = m^2 - n^2 = 3$ as the shorter side, $b = 2mn = 4$ as the longer side, and thus hypotenuse $c = m^2 + n^2 = 5$, the Pythagorean triple matrix

$$A = \begin{bmatrix} n^2 & 0 \\ 0 & m^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \tag{4.7}$$

has from (1.4) the antieigenvector (we ignore the other one, x_-)

$$x_+ = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \cong \begin{bmatrix} 0.894427 \\ 0.447214 \end{bmatrix} \tag{4.8}$$

which is maximally turned by A to

$$Ax_+ = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} \end{bmatrix} \cong \begin{bmatrix} 0.894427 \\ 1.788854 \end{bmatrix} \tag{4.9}$$

with maximal turning angle $\phi(A) \cong 36.869898$ degrees. See similar Figure 1.1 in [9, p. 9].

Taking this example further, in the stereographic projection picture above, if we take $x = \left(\frac{m}{n}, 0\right) = (2, 0)$ on the real axis, its stereographic projection onto the unit circle according to (4.6) is the point $\left(\frac{4}{5}, \frac{3}{5}\right) = (\cos\phi(A), \sin\phi(A))$. We see now that another related angle appears. If you draw the triangle $(0, 0), (0, 1), (2, 0)$, the angle ψ at the vertex $x = (2, 0)$ is easily seen to be determined by either of

$$\cos\psi = \frac{m}{\sqrt{m^2+n^2}} = \frac{2}{\sqrt{5}} \quad \text{or} \quad \sin\psi = \frac{n}{\sqrt{m^2+n^2}} = \frac{1}{\sqrt{5}}. \tag{4.10}$$

from which $\psi \cong 26.56505$ degrees. More interestingly, those (4.10) values are exactly the coordinates of the antieigenvectors $x_{\pm}$ according to (4.5).

A third related operator-theoretic link between the two theories is given by perfect square complex Gaussian integers

$$(m + in)^2 = (m^2 - n^2) + i(2mn). \tag{4.11}$$

If we normalize (4.11) by dividing by $|m + in|^2$ we obtain the complex exponential expression

$$e^{i\phi(A_{m,n})} = \sin\phi(A_{m,n}) + i\cos\phi(A_{m,n}). \tag{4.12}$$

but with the roles of the cosine and sine switched from the usual complex analysis. And of course (4.12) is not an analytic function because those are angles from the operator trigonometry.

We conclude this section with a fourth new connection. The Archimedes method for calculating π is to inscribe and circumscribe the circle with regular n-gons and go to the limit, e.g. see [3, p 338]. To be specific, consider a circle of radius $\frac{1}{2}$ and let a_n be the length of the circumscribing $6 \cdot 2^n$ gon and b_n be the length of the corresponding inscribing $6 \cdot 2^n$ gon. Then one has the well-known iterative relations [3, p. 1]

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}. \tag{4.13}$$

We immediately may now form the ratios

$$\frac{b_{n+1}}{a_{n+1}} = \frac{2\sqrt{a_n b_n}}{a_n + b_n} = \cos\phi(A_n) \tag{4.14}$$

as the first antieigenvalues of the matrix sequence

$$A_n = \begin{bmatrix} b_n & 0 \\ 0 & a_n \end{bmatrix} \to \begin{bmatrix} b_{n+1} & 0 \\ 0 & a_{n+1} \end{bmatrix} \to M(a_0, b_0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.15}$$

Here we have borrowed from [3] the notation and result

$$M(a_0, b_0) = \lim a_n = \lim b_n, \tag{4.16}$$

where a_0 and b_0 are the starting values of the iteration. Because the function M satisfies the relation for $\lambda > 0$

$$M(a_0, b_0) = \frac{1}{\lambda} M(\lambda a_0, \lambda b_0) = M(\frac{a_0 + b_0}{2}, \sqrt{a_0 b_0}), \tag{4.17}$$

we may normalize and recognize that the basic AGM process as it's called in [3], now corresponds to an operator trigonometric process $M(1, \cos\phi(A_n))$ in which the operators A_n slowly "untwist" themselves as their first antieigenvalues $\mu_1^n = \cos\phi(A_n)$ converge to 1 and their operator maximum turning angles converge to zero.

5 Conclusions and Comments

Beyond the continuum model for granular materials shown in Section 2 here to be fundamentally operator-trigonometric, I would expect that the antieigenvalue analysis may be profitably applied to a wide range of stress-strain tensors as they occur within continuum mechanics. If one takes a strength-of-materials course (I did, long ago) or a more general constitutive equations course, one soon perceives that a large number of the partial differential equations describing such elastic or fluid phenomena are derived by starting with a small rectangular material element and "pushing" it to a parallelepiped.

The fundamental connection of my antieigenvalue analysis to the Sharpe ratio of investment theory exposed in Section 3 came to me about twenty years ago but I did not work out the particulars until I decided to present those results at the 22^{nd} IWMS meeting in Toronto in 2013. Fortunately the blind refereeing process for [11] actually brought a very positive review from one of the nation's experts on high frequency trading. I am hopeful that my advocacy of new reward-to-risk growth angles might in the future lead to fruitful actual implementations within financial markets.

The link to Pythagorean triples and number theory given here for the first time was quite unexpected. Let me remind and emphasize that when I first originated the antieigenvalue theory almost fifty years ago, I was coming from semi-group perturbation theory which had led me to a question of when an operator product BA would remain (real) positive, given positive A under multiplicative perturbation by positive B. For rather general semi-group generators A, and bounded B, I found the operator theoretic sufficient condition

$$\sin\phi(B) \leqq \cos\phi(A). \tag{5.1}$$

Then by using variational techniques on the expression (1.1) in conjunction with convexity techniques on the expression (1.2) I found the explicit valuations (1.5) for $n \times n$ symmetric positive definite matrices and the relation (1.3) and the antieigenvectors (1.4). That functional analysis had nothing to do with number theory.

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