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Unified speed estimation of various stabilities

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Abstract: The main topic of this talk is the speed estimation of stability/instability. The word “various” comes with no surprising since there are a lot of different types of stability/instability and each of them has its own natural distance to measure. However, the adjective “unified” is very much unexpected. The talk surveys our recent progress on the topic, made in the past five years or so.

In the next section, we introduce our first unified result: Theorem 1. Then, several extensions or generalizations of Theorem 1 are collected briefly in Section 2.

1 Basic estimates of the first non-trivial eigenvalue

Here is our first stability, the exponential stability in the ergodic case. Given a Markov chain on a countable E with transition probability $P(t) = (p_{ij}(t) : i, j \in E)\,(t \geqslant 0)$, in the irreducible ergodic case, we have a stationary distribution π: $\pi P(t) = \pi$ for all $t \geqslant 0$. Then, we have

$$p_{ij}(t) \to \pi_j \quad \text{as} \quad t \to \infty \quad \text{for all } i, j.$$

We are now looking for the exponential convergence speed (rate) ε:

$$|p_{ij}(t) - \pi_j| \leqslant C_i e^{-\varepsilon t}, \qquad t \geqslant 0,\ i, j \in E.$$

Define the Q-matrix by

$$Q = (q_{ij} : i, j \in E) = \frac{\mathrm{d}}{\mathrm{d}\, t} P(t)\Big|_{t=0} \ \text{(pointwise)}.$$

In the reversible case, we have $\varepsilon_{\max} = \lambda_1$, where λ_1 is the smallest (the first nontrivial) eigenvalue of $-Q$: $Qg = -\lambda g$ for some $g \neq$ constant.

Let us now consider a simpler birth–death Q-matrix on $E = \{0, 1, 2, \dots\}$:

$$Q = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \dots \\ a_1 & -(a_1 + b_1) & b_1 & 0 & \dots \\ 0 & a_2 & -(a_2 + b_2) & b_2 & \dots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix},$$

where $a_k,\ b_k > 0$. Since the sum of each row equals 0, we have $Q\mathbb{1} = \mathbb{0} = 0 \cdot \mathbb{1}$, where $\mathbb{1}$ is the vector having elements 1 everywhere and $\mathbb{0}$ is the zero vector. This means that the Q-matrix has a trivial eigenvalue $\lambda_0 = 0$ with eigenvector $\mathbb{1}$. Our question is what is the next eigenvalue λ_1 of $-Q$?

Actually, the story is much harder than it looks like, as shown in [3, pages 1–3], even for $E = \{0, 1, 2, 3\}$. The reader is urged strongly to have some personal computation or have a look at the pages just mentioned.

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We now show that the story is even much more complicated. Let $E = \{0, 1, \ldots, N\}$ with $N < \infty$ for a moment. Consider the eigenvalue problem:

$$Qg = -\lambda g, \qquad g \neq 0$$

with Dirichlet boundary at 0: $g_0 = 0$ and Neumann boundary at N: $g_N = g_{N+1}$. Using codes 'D' and 'N', we may denote this minimal eigenvalue λ by λ^{DN}. Actually, the DN case is well studied in the history. Obviously, except the DN case, we should have three more cases: ND, DD, and NN. The last one, λ^{NN}, denotes the ergodic rate λ_1 just mentioned above, for which the constraint is not at the endpoints but is having mean zero.

In the non-ergodic case, the symmetric measure μ can not be finite. Hence, the exponential convergence rate is changed to be the exponential decay rate:

$$\begin{aligned}
&\text{ergodic case}: \ |p_{ij}(t) - \pi_j| \leqslant C_i e^{-\varepsilon t}, \ t \geqslant 0, \ \varepsilon_{\max} = \lambda^{\mathrm{NN}};\\
&\text{non-ergodic case}: \ p_{ij}(t) \leqslant C_i e^{-\varepsilon t}, \ t \geqslant 0, \ \varepsilon_{\max} = \lambda^{\#}, \qquad i, j \in E,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{where } \# = \text{DN, ND, or DD}.
\end{aligned}$$

Altogether, there are four cases: NN, DD, DN, and ND.

To state our main result, we need a standard notion. Return to our general state space $E = \{0, 1, \cdots, N\}$, $N \leqslant \infty$. Define

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 b_1 \cdots b_{n-1}}{a_1 a_2 \cdots a_n}, \qquad 1 \leqslant n \leqslant N.$$

For general $N \leqslant \infty$, the principal eigenvalue $\lambda^{\#}$ defined above has to be extended to the largest λ satisfying

$$\sqrt{\lambda}\, \|f\|_{\mu,2} \leqslant \|\partial f\|_{\nu,2} \tag{1}$$

with one of the four boundary conditions, where $\|\cdot\|_{\mu,q}$ denotes the $L^q(\mu)$-norm and

$$\nu_i = \begin{cases} \nu_i^- = \mu_i a_i & i \leqslant \theta \\ \nu_i^+ = \mu_i b_i & \theta \leqslant i < N+1; \end{cases}$$

$$\partial_i f = \begin{cases} (\partial_i f)^- = f_{i-1} - f_i & i \leqslant \theta \\ (\partial_i f)^+ = f_{i+1} - f_i & \theta \leqslant i < N+1 \end{cases}$$

and $\theta \in E$ is a reference point.

The author started to study λ^{NN} in 1988 (cf. [2, 3]), but the following result (the first unified exponential rate estimation) was obtained in 2010 [5] only.

Theorem 1. For the first non-trivial eigenvalue $\lambda^{\#}$ defined above, we have the following unified basic estimates:

$$\left(4\kappa^{\#}\right)^{-1} \leqslant \lambda^{\#} \leqslant \left(\kappa^{\#}\right)^{-1},$$

where

$$\left(\kappa^{\mathrm{NN}}\right)^{-1} = \inf_{n,\, m \in E,\, m<n} \left[\left(\sum_{i=0}^{m} \mu_i\right)^{-1} + \left(\sum_{i=n}^{N} \mu_i\right)^{-1}\right] \left(\sum_{j=m}^{n-1} \frac{1}{\mu_j b_j}\right)^{-1}$$

$$\left(\kappa^{\mathrm{DD}}\right)^{-1} = \inf_{n,\, m \in E,\, m\leqslant n} \left[\left(\sum_{i=0}^{m} \frac{1}{\mu_i a_i}\right)^{-1} + \left(\sum_{i=n}^{N} \frac{1}{\mu_i b_i}\right)^{-1}\right] \left(\sum_{j=m}^{n} \mu_j\right)^{-1}$$

$$\kappa^{\mathrm{DN}} = \sup_{n \in E} \left(\sum_{i=0}^{n} \frac{1}{\mu_i a_i}\right)^{-1} \left(\sum_{j=n}^{N} \mu_j\right)^{-1}$$

$$\kappa^{\mathrm{ND}} = \sup_{n \in E} \left[\left(\sum_{i=0}^{n} \mu_i\right)^{-1} \left(\sum_{j=n}^{N} \frac{1}{\mu_j b_j}\right)^{-1}\right].$$

In particular, $\lambda^{\#} > 0$ iff $\kappa^{\#} < \infty$.

Note that if we define $\hat{\nu}_k = (\mu_k b_k)^{-1}$, and in the DD and DN cases, under the sum $\sum_{k=0}^{m}$, we modify $\hat{\nu}_k$ to be $(\mu_k a_k)^{-1}$ (noting that when $k \in E$, $\mu_k b_k = \mu_{k+1} a_{k+1}$), then the basic estimates given in the theorem can be described completely by two measures μ and $\hat{\nu}$. The upper and lower bounds are the same up to a universal constant 4 only. It is easy to see that the two endpoints 0 and N are symmetric in the constants κ^{NN} and κ^{DD}.

Finally, we mention that the DN and ND cases are known around 1970 in harmonic analysis, our main contribution is for the cases of DD and NN, especially the two isoperimetric constants κ^{NN} and κ^{DD} (come from [5, Corollaries 7.8 and 7.9]). In the proof of the DD and NN cases, three advanced mathematical tools are used and its proof given in [5] consists of five steps. Later, a direct elementary proof was found in [6]. It then leads to the study in the next section.

2 Generalizations

2.1 Bilateral case

Clearly, the birth-death process studied in the last section can be extended to the bilateral one with state space $E = \{i : -M-1 < i < N+1\}$, where $M, N \leqslant \infty$, and with evolution rates: $q_{i,i+1} = b_i$, $q_{i,i-1} = a_i$, and $q_{ij} = 0$ for other $j \neq i$, $i, j \in E$. In this case, the symmetric measure μ is defined as follows.

$$\mu_{\theta+n} = \frac{a_{\theta-1}a_{\theta-2}\cdots a_{\theta+n+1}}{b_\theta b_{\theta-1}\cdots b_{\theta+n}}, \qquad -M-1-\theta < n \leqslant -2,$$
$$\mu_{\theta-1} = \frac{1}{b_\theta b_{\theta-1}}, \qquad \mu_\theta = \frac{1}{a_\theta b_\theta}, \qquad \mu_{\theta+1} = \frac{1}{a_\theta a_{\theta+1}},$$
$$\mu_{\theta+n} = \frac{b_{\theta+1}b_{\theta+2}\cdots b_{\theta+n-1}}{a_\theta a_{\theta+1}\cdots a_{\theta+n}}, \qquad 2 \leqslant n < N+1-\theta.$$

where $\theta \in E$ is a reference point. In this bilateral case, Theorem 1 remains the same. Refer to [5].

2.2 Bilateral Hardy-type inequalities

Obviously, the Poincaré inequalities (1) can be generalized to the following

$$\|f\|_{\mu,\,q} \leqslant A^{\#}\|\partial f\|_{\nu,\,p}, \qquad f \in L^q(\mu) \tag{2}$$

for $p, q \in [1, \infty]$. This and the parallel inequalities with different boundary condition consist of the bilateral Hardy-type inequalities. When $q \geqslant p$, a generalization of Theorem 1 is given in [7] in the continuous context and in [14] in the discrete one.

2.3 Normed linear space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$

In many applications (Sobolev inequalities, logarithmic Sobolev inequalities, Nash inequalities, and so on), the L^q-norm in (2) is not enough. This leads to the extension to a normed linear space $\mathbb{B}$ which is a linear subset of Borel measurable functions on (E, μ) with a specific norm $\|\cdot\|_{\mathbb{B}}$. In other words, instead of (2), we study the following Hardy-type inequalities

$$\||f|^q\|_{\mathbb{B}}^{1/q} \leqslant A_{\mathbb{B}}^{\#}\|\partial f\|_{\nu,\,p}, \qquad f \in \mathbb{B}$$

with different boundary conditions as before. Our result is presented in [1, 7]. For the last two topics, some popular reports are presented in [7–12].

2.4 Birth–death processes with killing

For the remainder of this section, we consider the birth–death processes with killing on $E = \{0, 1, 2, \ldots, N\}$, $N \leqslant \infty$. Its Q-matrix becomes

$$Q^c = \begin{pmatrix} -(b_0 + c_0) & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1 + b_1 + c_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2 + b_2 + c_2) & b_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

with $a_i > 0$, $b_i > 0$, and $c_i \geqslant 0$ for every $i \in E$. Clearly, this is a special type of tridiagonal or Jacobi's matrix. Assume that $c_i \not\equiv 0$ on $(0, N)$, otherwise, we would return to Section 1. Even though the spectral problem becomes much harder than before, since a new sequence of parameter (c_i) is added, we are lucky to obtain a result in parallel to Theorem 1. Refer to [9, 13].

2.5 Discrete spectrum

We say that the matrix Q^c (or its quadratic form) on $L^2(\mu)$ has discrete spectrum if its spectrum consists of only eigenvalues with finite multiplicity. Since an operator on a finite space is compact and hence must have discrete spectrum, we need only consider an infinite state space. Next, since the whole line can be split into two half lines, without loss of generality, we assume that $E = \{0, 1, \ldots\}$. In this subsection, we allow $c_i|_{(0,N-1)} \equiv 0$. This problem is solved completely by [9, Theorem 2.1], based on [13]. From the last cited paper, one finds an interesting story on isospectral operators.

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The references given below are only those the talk is based on. It is regretted that a large number of publications in the active research area is omitted here, otherwise, the list would be too long. For more references on the related sub-topics, the reader is urged to look at the related papers below.

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