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Pentadiagonal Companion Matrices

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Abstract: The class of sparse companion matrices was recently characterized in terms of unit Hessenberg matrices. We determine which sparse companion matrices have the lowest bandwidth, that is, we characterize which sparse companion matrices are permutationally similar to a pentadiagonal matrix and describe how to find the permutation involved. In the process, we determine which of the Fiedler companion matrices are permutationally similar to a pentadiagonal matrix. We also describe how to find a Fiedler factorization, up to transpose, given only its corner entries.

Keywords: companion matrices, pentadiagonal matrices, Fiedler companion matrices, Hessenberg matrices, algorithms, zeros of polynomials

MSC: 15B99, 65F50, 15A18, 15A23, 05C50

1 Introduction

Companion matrices have been used in many contexts, but especially in the context of finding roots of polynomials by using matrix methods to determine the eigenvalues of a companion matrix. There has been much recent work exploring efficient algorithms for finding roots via a companion matrix (see, for example, [1, 2, 5–7]). The structure of pentadiagonal matrices has the potential to be exploited in a fast LR-algorithm for determining roots of monic polynomials (see, for example, [2]). Recent papers (see, for example, [3, 8, 9]) have made particular mention of a pentadiagonal companion matrix introduced by Fiedler [11]. In this paper we characterize various pentadiagonal companion matrices and describe ways to construct these matrices, paying particular focus on the structure of the companion matrices introduced in [10].

Formally, we define a *companion matrix* over a field $\mathbb{C}[a_1, \ldots, a_n]$ to be an n-by-n matrix A with $n^2 - n$ fixed entries and n variable entries $-a_1, -a_2, \ldots, -a_n$ such that the characteristic polynomial of A is $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$. It was shown in [14] that a companion matrix requires at least $2n - 1$ nonzero entries. A companion matrix is called *sparse* if it contains exactly $2n - 1$ nonzero entries. The following three matrices have characteristic polynomial $x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$, and hence are examples of sparse companion matrices:

$$\begin{bmatrix} -a_1 & 1 & 0 & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ -a_3 & 0 & 0 & 1 & 0 \\ -a_4 & 0 & 0 & 0 & 1 \\ -a_5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a_1 & 1 & 0 \\ 0 & -a_3 & -a_2 & 0 & 1 \\ -a_5 & -a_4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -a_2 & -a_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_5 & -a_4 & -a_3 & 0 & 0 \end{bmatrix}. \tag{1}$$

For $1 \le k \le n - 1$ the kth *subdiagonal* of a matrix is the set of positions $\{(i, i - k) : k + 1 \le i \le n\}$. In the case of the three companion matrices in (1), we note that there is exactly one nonzero element on the main

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diagonal and exactly one nonzero element on each subdiagonal. As noted in [10], this is a feature of a sparse companion matrix in unit lower Hessenberg form (see Theorem 1.1).

Let $\mathcal{H}_n$ be the class of unit lower Hessenberg matrices with one nonzero entry on the main diagonal, namely $-a_1$, and one nonzero entry on the kth subdiagonal, namely $-a_{k+1}$, for $1 \le k \le n-1$. Note that each matrix in (1) is in $\mathcal{H}_5$.

We say two matrices M and N are *equivalent* if there exists a permutation matrix P such that $N = P^{-1}MP$ or $N = P^{-1}M^TP$. One particular permutation similarity always changes an upper Hessenberg matrix into a lower Hessenberg matrix: we say that RMR is the *reverse permutation* of M if R is the *reverse permutation matrix*

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}. \tag{2}$$

The matrices in $\mathcal{H}_n$ were characterized in [10] in terms of patterns which uniquely realize each characteristic polynomial; this class includes every sparse companion matrix up to equivalence. Further, the authors in [10] characterized the structure of all sparse companion matrices up to equivalence:

Theorem 1.1. *[10, Theorem 2.4 and Corollary 4.3] If A is a sparse companion matrix, then A is equivalent to a matrix in $\mathcal{H}_n$. Further, for each $A \in \mathcal{H}_n$, A is a companion matrix if and only if for each k with $2 \le k \le n$, the entry $-a_k$ is in the rectangular submatrix whose upper right corner is the placement of entry $-a_1$ on the main diagonal and whose lower left corner is position $(n, 1)$.*

For example, the matrices in (1) are all sparse companion matrices; but for the matrix

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -a_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -a_3 & -a_2 & 0 & 1 \\ -a_5 & -a_4 & 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

while it is in $\mathcal{H}_5$, H is not a companion matrix according to Theorem 1.1 since $-a_2$ is outside the submatrix determined by $-a_1$ and $-a_5$. In fact, the characteristic polynomial of H is $x^5 + a_1x^4 + a_2x^3 + (a_1a_2 + a_3)x^2 + a_4x + a_5$.

We say a matrix has a *pentadiagonal form* if it is equivalent to a pentadiagonal matrix. In Section 2 we characterize which matrices in $\mathcal{H}_n$ have a pentadiagonal form. We observe that there are twelve sparse companion matrices which have a pentadiagonal form, and we characterize these. Hence we have a characterization of the sparse companion matrices of lowest bandwidth (see Remark 2.2). We also describe the permutation matrices P such that P^TAP is pentadiagonal when $A \in \mathcal{H}_n$ has a pentadiagonal form.

One specific class of sparse companion matrices, introduced by Fiedler in [11], were obtained by matrix factorizations. Let

$$A_k = \begin{bmatrix} I_{k-1} & O & O \\ O & C_k & O \\ O & O & I_{n-k-1} \end{bmatrix} \text{ with } C_k = \begin{bmatrix} -a_k & 1 \\ 1 & 0 \end{bmatrix}$$

for $1 \le k \le n-1$ and let A_n be a diagonal matrix with entries $(1, \dots, 1, -a_n)$. Fiedler noted that if $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a permutation of $(1, 2, \dots, n)$, then the product $A_\sigma = A_{\sigma_1}A_{\sigma_2}\cdots A_{\sigma_n}$ is a companion matrix. We say a matrix is a *Fiedler companion matrix* if it is equivalent to one of these products (which we call a *Fielder factorization*). Since each Fiedler companion matrix is sparse, every Fiedler companion matrix is equivalent to a matrix in $\mathcal{H}_n$. Further, as noted in [10], one can recognize a Fiedler companion matrix in $\mathcal{H}_n$ by the fact that the variable entries form a lattice path starting with $-a_n$ in position $(n, 1)$ and $-a_1$ on the diagonal; by *lattice path* we mean that if $-a_k$ is in position (i, j) then $-a_{k-1}$ is in either position $(i, j+1)$ or $(i-1, j)$. There are also sparse companion matrices in $\mathcal{H}_n$ that are not Fiedler (for example, the third matrix in (1) is not a Fiedler companion matrix since the variable entries do not form a lattice path).

Letting $B = A_1A_3A_5\cdots$ and $C = A_2A_4A_6\cdots$, Fiedler noted that the product BC is always pentadiagonal. For example,

$$A_1A_3A_5A_2A_4A_6 = \begin{bmatrix} -a_1 & -a_2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4}$$

In [9], it was noted that there were four different Fiedler products which are pentadiagonal, and they are pairwise transposes. In Section 2, we determine that, up to equivalence, there are 8 Fiedler companion matrices that have a pentadiagonal form (that is, which are equivalent to a pentadiagonal matrix, including via permutation similarity). In Section 3, we describe (Theorem 3.6) how to find the Fiedler factorization of any Fiedler companion matrix, and then describe (Theorem 3.8) the eight Fiedler products which have a pentadiagonal form. As an example, we know from a characterization in [10] that the second matrix in (1) is equivalent to a Fiedler matrix; by Theorem 3.6 in Section 3, we will see that this matrix is equivalent to the pentadiagonal matrix $A = A_5A_3A_1A_2A_4$ (in fact $A = A_1A_3A_5A_2A_4$ since $A_iA_j = A_jA_i$ for Fiedler factors when $|i-j| \neq 1$). In the Appendix, we list the non-equivalent 6-by-6 sparse pentadiagonal companion matrices.

2 Characterization of Matrices in $\mathcal{H}_n$ with a Pentadiagonal Form

In this section we will characterize the matrices in $\mathcal{H}_n$ with a pentadiagonal form. We also explicitly characterize all the pentadiagonal sparse companion matrices, up to equivalence.

One tool that can describe the combinatorial structure of a matrix is its digraph. The *labelled digraph*, $D(M)$, of the matrix $M = [M_{i,j}]$ has vertex set $\{v_1, v_2, \ldots, v_n\}$ and arc set $\{(v_i, v_j) : M_{i,j} \neq 0\}$, with $M_{i,j}$ the label of arc (v_i, v_j). For an arc (v_i, v_j), vertex v_i is called the *tail* of the arc and vertex v_j is called the *head*. Transposing a matrix results in reversing the direction of the arcs of its digraph and any permutation similarity of a matrix results in relabelling (i.e. reordering) the vertices of its digraph. Thus we have the useful fact that M and N are equivalent matrices if and only if $D(M)$ is isomorphic to $D(N)$ or $D(N^T)$.

A *k-cycle* in a digraph is any vertex-disjoint sequence of arcs $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_k}, v_{i_1})$, written $v_{i_1} \to v_{i_2} \to v_{i_3} \to \ldots \to v_{i_k} \to v_{i_1}$. An n-cycle is called a *Hamilton cycle* and a 1-cycle is called a *loop*. The *underlying graph* of a matrix $M = [M_{i,j}]$ (or of a digraph) is a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{\{v_i, v_j\} : M_{i,j} \neq 0\}$. We say that digraph D is an *orientation* of its underlying graph. A *simple graph* is a graph with no edges of the form $\{v_i, v_i\}$. By the definition of $\mathcal{H}_n$ we obtain the following:

Lemma 2.1. *Let C be a matrix with entries in $\mathbb{C}[a_1, \ldots, a_n]$. Then C is equivalent to a matrix in $\mathcal{H}_n$ if and only if $D(C)$ has exactly $2n-1$ arcs and, for each k with $1 \leq k \leq n$, there is exactly one k-cycle in $D(C)$ and this k-cycle has $k-1$ arcs labelled 1 and one arc labelled $-a_k$.*

Proof. The forward direction follows directly from the definition of $\mathcal{H}_n$. For the converse, observe that if $D = D(C)$ has a Hamilton cycle with $n-1$ arcs labelled 1, and if D has exactly $2n-1$ arcs, then each of the remaining arcs of D must be labelled with one of the variables $a_1, \ldots, a_{n-1}$. It follows that every k-cycle shares $k-1$ arcs with the Hamilton cycle, $1 \leq k \leq n-1$, and hence C is equivalent to a matrix in $\mathcal{H}_n$. □

Remark 2.2. By Lemma 2.1, and Theorem 1.1, one can deduce that a sparse companion matrix can not be tridiagonal since the digraph of a tridiagonal matrix has only 1-cycles and 2-cycles but a sparse companion matrix of order $n \geq 3$ must also have a 3-cycle. Thus, by focusing on pentadiagonal matrices, in this paper we characterize the sparse companion matrices with lowest bandwidth.

For a graph G, we use the notation $N(v)$ to denote the set of vertices adjacent to v in G. A *strut* is an undirected graph on $n \geq 4$ vertices with vertex set $V = \{v_1, \ldots, v_n\}$ such that

$$N(v_k) = \begin{cases} \{v_2, v_3\} & \text{if} \quad k = 1, \\ \{v_1, v_3, v_4\} & \text{if} \quad k = 2, \\ \{v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}\} & \text{if} \quad 3 \leq k \leq (n-2), \\ \{v_{n-3}, v_{n-2}, v_n\} & \text{if} \quad k = n-1, \\ \{v_{n-2}, v_{n-1}\} & \text{if} \quad k = n. \end{cases}$$

In particular, note that the underlying simple graph of an n-by-n pentadiagonal matrix is a subgraph of a strut (for $n \geq 4$).

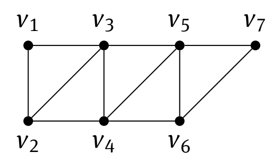

Figure 1: A strut on seven vertices.

Focusing on the strut graph will help us to characterize all matrices in $\mathcal{H}_n$ that have a pentadiagonal form. We first introduce a class of n-by-n pentadiagonal matrices $\mathcal{X}$. Let

$$W \in \left\{ \begin{bmatrix} \Diamond & -a_{n-1} & 1 \\ 0 & \Diamond & 0 \\ \Box & -a_n & \Diamond \end{bmatrix}, \begin{bmatrix} \Diamond & -a_{n-1} & -a_n \\ 0 & \Diamond & \Box \\ 0 & 1 & \Diamond \end{bmatrix} \right\}. \tag{5}$$

We say a matrix X is in $\mathcal{X}$ if, for odd n,

$$X = \left[\begin{array}{cccccccccccc} \Diamond & \Box & 1 & & & & & & & & & \\ 1 & \Diamond & 0 & \Box & & & & & & & & \\ \Box & -a_3 & \Diamond & -a_4 & 1 & & & & & & & \\ & 1 & 0 & \Diamond & 0 & \Box & & & & & & \\ & & \Box & -a_5 & \Diamond & -a_6 & 1 & & & & & \\ & & & 1 & 0 & \Diamond & 0 & \ddots & & & & \\ & & & & \Box & -a_7 & \Diamond & \ddots & \ddots & & & \\ & & & & & 1 & 0 & \ddots & \ddots & \Box & & \\ & & & & & & \Box & \ddots & \ddots & -a_{n-3} & 1 & \\ & & & & & & & \ddots & \ddots & \Diamond & 0 & 0 \\ & & & & & & & & \ddots & -a_{n-2} & \multicolumn{2}{c}{\multirow{2}{*}{\boxed{W}}} \\ & & & & & & & & & 1 & & \end{array} \right],$$

and, for even n,

$$X = \left[\begin{array}{ccccccccccc}
\Diamond & \square & 1 & & & & & & & & \\
1 & \Diamond & 0 & \square & & & & & & & \\
\square & -a_3 & \Diamond & -a_4 & 1 & & & & & & \\
 & 1 & 0 & \Diamond & 0 & \square & & & & & \\
 & & \square & -a_5 & \Diamond & -a_6 & \ddots & & & & \\
 & & & 1 & 0 & \Diamond & \ddots & \ddots & & & \\
 & & & & \square & -a_7 & \ddots & \ddots & 1 & & \\
 & & & & & 1 & \ddots & \ddots & 0 & \square & \\
 & & & & & & \ddots & \ddots & \Diamond & -a_{n-2} & 1 \\
 & & & & & & & \ddots & 0 & \multicolumn{2}{|c|}{} \\
 & & & & & & & & \square & \multicolumn{2}{|c|}{W^T}
\end{array}\right],$$

such that exactly one $\Diamond$ in X is replaced with $-a_1$, the rest with 0, and exactly one $\square$ is replaced with $-a_2$, the rest with 0. We note that the structure of a matrix in $\mathcal{X}$ is a CMV shape (see [4]) when W is the first matrix in (5) and $-a_2$ is in position $(1, 2)$.

Theorem 2.3. *C is equivalent to a matrix in $\mathcal{H}_n$ and has a pentadiagonal form if and only if C is equivalent to a matrix in $\mathcal{X}$.*

Proof. Let C be equivalent to a matrix in $\mathcal{H}_n$, and suppose C is equivalent to a pentadiagonal matrix M. Let $D = D(M)$ be the labelled digraph of M. Then D is isomorphic to $D(C)$ and therefore the underlying graph of D is isomorphic to a subgraph of a strut G on n vertices. Since C is in $\mathcal{H}_n$, D has $2n - 1$ arcs, one of which is a loop and one which forms a 2-cycle. Therefore the underlying graph of D has $2n-3$ edges. Since G has $2n-3$ edges D is merely an orientation of G with two additional arcs: one forming a loop and one forming a 2-cycle.

There is a unique Hamilton cycle in G. Thus we may assume that the Hamilton cycle in D is

$$v_1 \to v_3 \to v_5 \to \cdots \to v_p \to v_q \to v_{q-2} \to v_{q-4} \to \cdots \to v_2 \to v_1 \tag{6}$$

(if the Hamilton cycle in D is reversed, then redefine D as the digraph of M^T) with $(p, q) = (n, n-1)$ if n is odd, and $(p, q) = (n-1, n)$ if n is even. Since C is in $\mathcal{H}_n$ there is an arc $\mathbf{a}$ in D labelled $-a_n$. Since $\mathbf{a}$ is not part of an $(n-1)$-cycle, $\mathbf{a}$ is incident to either v_1 or v_n. If $\mathbf{a}$ is incident to v_1 then replace M with the reverse permutation matrix RM^TR. In other words, we may assume that $\mathbf{a}$ is incident to v_n. Therefore, if n is even, $M_{1,3} = M_{3,5} = \cdots = M_{n-3,n-1} = 1$ and $M_{2,1} = M_{4,2} = M_{6,4} = \cdots = M_{n-2,n-4} = 1$ and $(M_{n-1,n}, M_{n,n-2}) \in \{(1, -a_n), (-a_n, 1)\}$. Likewise if n is odd, $M_{1,3} = M_{3,5} = \cdots = M_{n-4,n-2} = 1$ and $M_{2,1} = M_{4,2} = M_{6,4} = \cdots = M_{n-1,n-3} = 1$ and $(M_{n-2,n}, M_{n,n-1}) \in \{(1, -a_n), (-a_n, 1)\}$.

Consider odd k with $2 < k < n$. If $v_{k-1} \to v_k$ in D, then $v_{k-1} \to v_k \to v_{k+2} \to \cdots \to v_p \to v_q \to v_{q-2} \to \cdots \to v_{k+1} \to v_k$ would be an $(n-k+2)$-cycle in D with one arc labelled $-a_n$, contradicting Lemma 2.1. Therefore $v_k \to v_{k-1}$ in D. Hence $v_k \to v_{k-1} \to v_{k-3} \to \cdots \to v_2 \to v_1 \to v_3 \to \cdots \to v_k$ is a k-cycle in D with all but one arc labelled 1. Thus by Lemma 2.1, $v_k \to v_{k-1}$ is labelled $-a_k$. Thus $M_{k,k-1} = -a_k$.

Similarly, one can show that if k is even with $3 < k < n$, then $M_{k-1,k} = -a_k$. Since D has a loop, $M_{k,k} = -a_1$ for some k with $1 \le k \le n$. By Lemma 2.1, there is a 2-cycle in D with one arc labelled $-a_2$ and the other labelled 1. Thus there exists a pair of distinct indices i, j such that $M_{i,j} = 1$ and $M_{j,i} = -a_2$. Therefore M is equivalent to some X in $\mathcal{X}$.

For the converse, note that if X is a matrix in $\mathcal{X}$ then X is a pentadiagonal matrix with $2n-1$ nonzero entries. Further the digraph $D = D(X)$ has a Hamilton cycle: $v_1 \to v_3 \to v_5 \to \cdots \to v_4 \to v_2 \to v_1$. With the orientation of the arcs, one can check that D has exactly one k-cycle for each k with $1 \le k \le n$ such that this

k-cycle has one arc labelled $-a_k$ and $(k-1)$ arcs labelled 1. Thus by Lemma 2.1, X is equivalent to a matrix in $\mathcal{H}_n$. □

Specific pentadiagonal companion matrices can be identified by considering leading principal submatrices, as in the next two results. Let

$$Y = \begin{bmatrix} \Diamond & \square & 1 & 0 & 0 \\ 1 & \Diamond & 0 & \square & 0 \\ \square & -a_3 & \Diamond & -a_4 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \square & -a_5 & 0 \end{bmatrix} \quad \text{and} \quad Y' = \begin{bmatrix} \Diamond & \square & 1 & 0 & 0 \\ 1 & \Diamond & 0 & \square & 0 \\ \square & -a_3 & \Diamond & -a_4 & -a_5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{7}$$

with one $\Diamond$ in each matrix replaced with $-a_1$ (and the rest with 0) and one $\square$ in the same row or column is replaced with $-a_2$ (and the rest with 0).

Corollary 2.4. *For $n \geq 6$, an n-by-n pentadiagonal matrix M is a sparse companion matrix if and only if M is equivalent to a matrix in $\mathcal{X}$ with leading 5-by-5 principal submatrix Y in (7). A 5-by-5 pentadiagonal matrix M is a sparse companion matrix if only if M is equivalent to Y or Y' in (7).*

Proof. Let M be a pentadiagonal sparse companion matrix. By Theorem 1.1, M is equivalent to a matrix in $\mathcal{H}_n$. Thus by Theorem 2.3, M is equivalent to a matrix X in $\mathcal{X}$. Further, it was noted in [10, Theorem 3.1] that the digraph of any matrix in $\mathcal{H}_n$ is the digraph of a companion matrix if and only if the cycles of the digraph intersect at the loop vertex. In particular, the loop must be on one of the vertices of the three cycle $v_1 \to v_3 \to v_2 \to v_1$ of $D(X)$. It follows that the leading 5-by-5 principal submatrix of X has structure Y for $n \geq 6$. For $n = 5$, the structure of the leading principal 5-by-5 minor is also affected by the placement of $-a_n$ as in (5). Thus the leading principal minor could also be Y' in this case.

Conversely, suppose M is a pentadiagonal matrix equivalent to some matrix X in $\mathcal{X}$ with leading principal submatrix Y (or possibly Y' if $n = 5$). Then by Theorem 2.3, M is equivalent to a matrix in $\mathcal{H}_n$. Further, in the proof of Theorem 2.3, we observed that v_1, v_2 and v_3 are on each k-cycle of the digraph $D(X)$ for $k \geq 3$. Given the structure of Y, it follows that each k-cycle of D intersects the vertex v_1, and so by [10, Theorem 2.4], X and hence M is a sparse companion matrix. □

Let

$$C = \begin{bmatrix} \Diamond & \square & 1 \\ 1 & \Diamond & 0 \\ \square & -a_3 & \Diamond \end{bmatrix}, \tag{8}$$

with exactly one $\Diamond$ replaced with $-a_1$, the rest with 0, and exactly one $\square$ in the same row or column replaced with $-a_2$, the rest with 0.

Corollary 2.5. *A pentadiagonal matrix M is equivalent to a Fiedler companion matrix if and only if M is equivalent to a matrix in $\mathcal{X}$ with leading 3-by-3 principal submatrix C in (8).*

Proof. It was observed in [10, Page 266] that M is equivalent to a Fiedler companion matrix if and only if M is equivalent to a matrix C in $\mathcal{H}_n$ with the variable entries forming a lattice path. In particular, for each k, $1 \leq k \leq n-1$, a_{k+1} is in the same row or column as a_k in such a matrix C. Since the property of being in the same row or column is preserved under permutation similarity and transposition, the result follows from Corollary 2.4. □

Corollary 2.6. *Suppose $n \geq 5$. Up to equivalence, there are $2n(n-1)$ matrices in $\mathcal{H}_n$ that have a pentadiagonal form. Up to equivalence, there are 8 n-by-n Fiedler companion matrices with a pentadiagonal form. Up to equivalence, there are 12 n-by-n sparse companion matrices with a pentadiagonal form if $n > 5$ but only 11 for $n = 5$.*

Proof. For a matrix X in $\mathcal{X}$, there are n possible placements $-a_1$ along the diagonal. Since the $-a_2$ entry in X is always symmetrically opposite a 1, there are $n-1$ possible placements of a_2. Finally, $-a_n$ can be in one of two positions, hence there are $2n(n-1)$ possible matrices in $\mathcal{X}$ and hence by Theorem 2.3, $2n(n-1)$ matrices in $\mathcal{H}_n$ with a pentadiagonal form. Suppose X is equivalent to a Fiedler companion matrix. Thus, by Corollary 2.5, there are four possibilities for the leading 3-by-3 principal submatrix of X. But, since there are two possible positions for $-a_n$, there are 8 possible Fiedler companion matrices with a pentadiagonal form. Suppose X is a sparse companion matrix then, by Corollary 2.4, there are 6 possibilities for the leading 5-by-5 principal submatrix of X for $n \geq 6$. If $n > 5$, then the placement of the $-a_n$ entry in X implies there are 12 sparse companion matrices with a pentadiagonal form. A similar count can be established if $n = 5$, however if $X_{3,5} = -a_n$ then $X_{5,3}$ can not equal $-a_2$, removing one placement of $-a_2$. Hence there are exactly 11 sparse companion matrices with a pentadiagonal form for $n = 5$, namely six of type Y and five of type Y' as described in (7). □

In Theorem 2.9, we describe how to recognize when a matrix in $\mathcal{H}_n$ has a pentadiagonal form. If n is even, let

$$\mathcal{A}_n = \left[\begin{array}{c|ccccc|c} \mathbf{0}^T & & & I_{\frac{n}{2}} & & & O \\ \hline & & & & 0 & 0 & \\ & & & -a_4 & -a_3 & & I_{\frac{n}{2}-1} \\ & & \iddots & \iddots & & & \\ & -a_{n-2} & -a_{n-3} & & & & \\ \cline{7-7} -a_n & -a_{n-1} & & & & & \mathbf{0} \end{array}\right].$$

Otherwise if n is odd, let

$$\mathcal{A}_n = \left[\begin{array}{c|ccccc|c} \mathbf{0}^T & & & I_{\lfloor\frac{n}{2}\rfloor} & & & O \\ \hline & & & & 0 & 0 & \\ & & & & -a_3 & & \\ & & & -a_5 & -a_4 & & I_{\lfloor\frac{n}{2}\rfloor} \\ & & \iddots & \iddots & & & \\ & -a_{n-2} & -a_{n-3} & & & & \\ \cline{7-7} -a_n & -a_{n-1} & & & & & \mathbf{0} \end{array}\right].$$

Likewise, let

$$\mathcal{B}_n = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & \mathcal{A}_{n-2} & & & 0 & 0 \\ & & & & 1 & 0 \\ \hline -a_{n-1} & 0 & \cdots & 0 & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 & 0 \end{array}\right].$$

We say a matrix M is *type* $\mathcal{A}_n$ (or $\mathcal{B}_n$) if every nonzero entry of $\mathcal{A}_n$ (resp. $\mathcal{B}_n$) has the same value in M and M has two additional nonzero entries: one with value $-a_1$ in the main diagonal and one with value $-a_2$ in the first subdiagonal.

Example 2.7. Two matrices in $\mathcal{H}_9$. The first is of type $\mathcal{A}_9$ and the second is of type $\mathcal{B}_9$.

$$\left[\begin{array}{c|cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & -a_3 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & -a_5 & -a_4 & 0 & 0 & 0 & 1 & 0\\
0 & -a_7 & -a_6 & 0 & 0 & 0 & 0 & 0 & 1\\
\cline{6-9}
-a_9 & -a_8 & 0 & 0 & 0 & 0 & 0 & -a_2 & 0
\end{array}\right]
\left[\begin{array}{c|ccc|ccc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & -a_3 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -a_5 & -a_4 & 0 & 0 & 0 & 1 & 0 & 0\\
\cline{5-7}
-a_7 & -a_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
\hline
-a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
-a_9 & 0 & 0 & 0 & 0 & 0 & 0 & -a_2 & 0
\end{array}\right]$$

We can explore the digraph of a matrix A of type $\mathcal{A}_n$ or $\mathcal{B}_n$ to determine that such a matrix has a pentadiagonal form, in particular, its underlying graph is a strut. Therefore the two matrices from Example 2.7 have a pentadiagonal form. We give a formal matrix-theoretic proof in Lemma 2.8. Note that if A is of type $\mathcal{A}_n$, then the Hamilton cycle in $D(A)$ is $v_1 \to v_2 \to v_3 \to \cdots \to v_n \to v_1$. But the Hamilton cycle of the digraph $D(X)$, for a matrix X in $\mathcal{X}$, is described in (6). Accordingly, for $n \geq 2$, we construct the permutation matrix $P_n = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} e_1 & e_3 & e_5 & \cdots & e_p & e_q & e_{q-2} & e_{q-4} & \cdots & e_2 \end{array}\right]$ (with e_k the standard unit vector with a 1 in the kth position). For $n \geq 4$, we let Q_n be the permutation matrix

$$Q_n = \left[\begin{array}{c|cc} O & 0 & 1 \\ & 1 & 0 \\ \hline P_{n-2} & \multicolumn{2}{c}{O} \end{array}\right].$$

Lemma 2.8. *Suppose $n \geq 4$. Then $P_n M P_n^{-1}$ is pentadiagonal if M is of type $\mathcal{A}_n$ and $Q_n W Q_n^{-1}$ is pentadiagonal if W is of type $\mathcal{B}_n$.*

Proof. Let M_n be of type $\mathcal{A}_n$. We show that $P_n M_n P_n^{-1}$ is pentadiagonal by induction. It is straight forward to check that this is the case for $n = 4$ and $n = 5$. Suppose $n \geq 6$. Note that $-a_1$ will remain on the diagonal after permutation similarity. Also, $-a_2$ will remain symmetrically opposite a 1 entry upon permutation similarity; thus if the 1 entries are within the bands of a pentadiagonal matrix after a permutation similarity, then the $-a_2$ entry will be as well. Let $\hat{M}$ be obtained from M by replacing $-a_1$ and $-a_2$ with zero. It is enough to show that $P_n \hat{M}_n P_n^{-1}$ is pentadiagonal.

$$P_n\hat{M}_nP_n^{-1} = \left[\begin{array}{ccc} & e_1^T & \\ \hline & e_n^T & \\ \hline & & \\ 0 & P_{n-2} & 0 \end{array}\right]
\left[\begin{array}{c|cccc|c}
0 & 1 & 0 & \cdots & 0 & 0\\
\hline
0 & & & & & 0\\
\vdots & & & \hat{M}_{n-2} & & \vdots\\
0 & & & & & 0\\
0 & & & & & 1\\
\hline
-a_n & -a_{n-1} & 0 & \cdots & 0 & 0
\end{array}\right]
\left[\begin{array}{c|c|c} & & 0 \\ e_1 & e_n & P_{n-2}^T \\ & & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c|cccc}
0 & 0 & 1 & 0 & \cdots & 0\\
\hline
-a_n & 0 & -a_{n-1} & 0 & \cdots & 0\\
\hline
0 & 0 & & & & \\
0 & 1 & & & & \\
0 & 0 & & P_{n-2}\hat{M}_{n-2}P_{n-2}^T & & \\
\vdots & \vdots & & & & \\
0 & 0 & & & &
\end{array}\right].$$

Thus, by induction, M_n is pentadiagonal. Likewise, suppose W_n is of type $\mathcal{B}_n$. Then

$$Q_n\hat{W}_nQ_n^{-1} = \left[\begin{array}{c|cc} O & 0 & 1 \\ & 1 & 0 \\ \hline P_{n-2} & & O \end{array}\right]\left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & \hat{M}_{n-2} & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & 1 & 0 \\ \hline -a_{n-1} & 0 & \cdots & 0 & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{cc|c} & O & P_{n-2}^T \\ \hline 0 & 1 & \\ 1 & 0 & O \end{array}\right]$$

$$= \left[\begin{array}{cc|cccc} 0 & 0 & -a_n & 0 & \cdots & 0 \\ 1 & 0 & -a_{n-1} & 0 & \cdots & 0 \\ \hline 0 & 0 & & & & \\ 0 & 1 & & & & \\ 0 & 0 & & P_{n-2}\hat{M}_{n-2}P_{n-2}^T & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{array}\right].$$

Since $P_{n-2}M_{n-2}P_{n-2}^T$ is pentadiagonal, $Q_nW_nQ_n^T$ is also pentadiagonal. □

Theorem 2.9. *Suppose C is in $\mathcal{H}_n$. Then C has a pentadiagonal form if and only if either C or RC^TR, with R the reverse permutation matrix, is of type $\mathcal{A}_n$ or type $\mathcal{B}_n$.*

Proof. Let C be a matrix in $\mathcal{H}_n$ that has a pentadiagonal form and let $D = D(C)$. By Theorem 2.3, C is equivalent to a matrix X in $\mathcal{X}$. Let $C' = RC^TR$ for the reverse permutation matrix R. Then C' is also equivalent to X, and in fact, C' is in $\mathcal{H}_n$. Further, if the arcs labelled $-a_k$ and $-a_j$ share a tail (or head) in $D(C)$, then they share a head (resp. tail) in $D(C')$. There are two possible positions for the entry $-a_n$ in X, up to equivalence. For the sake of argument, we will assume n is odd. In one case, the arc labelled $-a_n$ shares a tail with the arc labelled $-a_{n-1}$. The arc labelled $-a_{n-1}$ shares a head with the arc labelled $-a_{n-2}$ and so on, such that the remaining arcs labelled $-a_k$ alternate between sharing a tail or a head with the arc labelled $-a_{k-1}$ for $3 < k \le n$. As such C, or C', is type $\mathcal{A}_n$.

In the other case, the arc labelled $-a_n$ shares a head with the arcs labelled $-a_{n-1}$ and $-a_{n-2}$. The arc labelled $-a_{n-2}$ shares a tail with the arc labelled $-a_{n-3}$ and so the remaining arcs labelled $-a_k$ then also begin to alternate between sharing a tail or a head with the arc labelled $-a_{k-1}$ for $3 < k \le (n-2)$. Thus C or C', is type $\mathcal{B}_n$.

For the converse, note that $-a_1$ will remain on the diagonal after a permutation similarity. Also, $-a_2$ will remain symmetrically opposite a 1 entry upon permutation similarity; thus if the 1 entries are within the bands of a pentadiagonal matrix after a permutation similarity, then the $-a_2$ entry will be as well. The converse then follows from Lemma 2.8. □

The structure of sparse companion matrices described in Theorem 1.1 and the results of Theorem 2.9 could be used as an alternate tool to obtain the number of pentadiagonal sparse companion matrices and the number of pentadiagonal Fiedler companion matrices described in Corollary 2.6. For example, suppose M is a sparse companion matrix in $\mathcal{H}_n$ with $n > 5$ of type $\mathcal{A}_n$, and $M_{i,i-2} = -a_3$ for some i, $3 \le i \le n$. Since all the variable entries of M are in the rectangular submatrix described in Theorem 1.1, there are only three possible placements of $-a_1$: position $(i-2, i-2)$, $(i-1, i-1)$, or (i, i). Suppose $M_{\ell,\ell} = -a_1$ for some $\ell \in \{i-2, i-1, i\}$. There are only two possible locations for $-a_2$ inside the rectangular submatrix: position $(\ell, \ell-1)$ or position $(\ell+1, \ell)$. Hence there are exactly 6 sparse companion matrices corresponding to type $\mathcal{A}_n$ that are pentadiagonal. By Theorem 2.9, the other option to consider is type $\mathcal{B}_n$, but the count works the same way. Thus for $n > 5$, there are 12 pentadiagonal sparse companion matrices up to equivalence.

We end this section by describing an algorithm that starts with a matrix A that is equivalent to a matrix in $\mathcal{H}_n$ and finds the permutation matrix that puts A into lower Hessenberg form. For instance, every Fiedler companion matrix is equivalent to a matrix in $\mathcal{H}_n$, however many Fiedler companion matrices are not in the Hessenberg form of $\mathcal{H}_n$. The method for obtaining a permutation essentially follows the technique of creating a permutation matrix that relabels the vertices in the Hamilton cycle of the digraph of a matrix into consecutive order (as was done for the permutation matrix in Lemma 2.8 and Theorem 2.9). The following algorithm provides the details of how to obtain such a permutation.

Algorithm 2.10. Given A is equivalent to a matrix in $\mathcal{H}_n$, the following algorithm constructs a permutation matrix P such that P^TAP is in $\mathcal{H}_n$:

```
Let P be an n-by-n zero matrix
Let j = 0
for col from 1 to n
   for row from 1 to n
      if A_{row,col} = -a_n then
         j = col
         break both for loops
      endif
   endfor
endfor
P_{j,1} = 1
for k from 2 to n
   for col from 1 to n
      if A_{j,col} = 1 then
         j = col
         break
      endif
   endfor
   P_{j,k} = 1
endfor
return P.
```

Theorem 2.11. *Let A be equivalent to a matrix in $\mathcal{H}_n$. If P is obtained by Algorithm 2.10, then P^TAP is in $\mathcal{H}_n$.*

Proof. Let A be equivalent to matrix in $\mathcal{H}_n$. Considering the digraph structure of a matrix in $\mathcal{H}_n$ (see also [10, p.259]) and since equivalent matrices have the same digraph structure, the digraph $D(A)$ has a Hamilton-cycle $v_{j_1} \to v_{j_2} \to \dots \to v_{j_n} \to v_{j_1}$ for some choice of $(j_1, j_2, \dots, j_n)$, such that $n-1$ corresponding entries of A are labelled 1 and one corresponding entry is labelled $-a_n$. Assume, without loss of generality, that $A_{j_n,j_1} = -a_n$. Hence, by Algorithm 2.10, $P = \left[\, e_{j_1} \;\middle|\; e_{j_2} \;\middle|\; \cdots \;\middle|\; e_{j_n} \,\right]$. Since $-a_1$ is on the main diagonal of A it will still be on the main diagonal of P^TAP. Further, since the digraph $D(A)$ is isomorphic to $D(P^TAP)$ via the mapping $(j_1, j_2, \dots, j_n) \mapsto (1, 2, \dots, n)$, the digraph $D(P^TAP)$ has Hamilton cycle $v_1 \to v_2 \to \dots \to v_n \to v_1$. Thus the superdiagonal of P^TAP is entirely unit entries and $(P^TAP)_{n,1} = -a_n$. By Lemma 2.1, every entry $-a_k$ in A, $2 \le k < n$, corresponds to an arc of a k-cycle in $D(A)$ with $k-1$ of the unit entries. It follows that each a_k, $2 \le k \le n$, is on the $(k-1)$th subdiagonal of P^TAP and hence $P^TAP \in \mathcal{H}_n$. □

3 Factorization of Pentadiagonal Fiedler Companion Matrices

It is interesting to consider the order of the product that results in pentadiagonal Fiedler companion matrices, for instance in [1] the authors provide algorithms that take advantage of the factors of a Fiedler companion matrix. In this section, we tweak definitions from [9] to correspond to the structure of $\mathcal{H}_n$ in order to construct a tool that allows one to factor any matrix that is equivalent to a Fiedler companion matrix by only examining the structure of the matrix. In particular, the factorization depends upon knowing the “corners” of the lattice path of the variable entries in the Hessenberg form $\mathcal{H}_n$. With this tool established, we present the 8 products that result in a Fiedler companion matrix with a pentadiagonal form.

For any matrix M that is equivalent to a Fiedler companion matrix $F = [F_{i,j}]$ in $\mathcal{H}_n$ we say that an entry $F_{i,j}$ is a *corner entry* of the lattice path in F if

1. $i = n$ and $j = 1$,
2. $i = j$, or
3. $F_{i,j}$ is the first or last variable entry in the ith row for some row i with more than one variable entry.

The *ordered list of corner entries of* F is the ordered list $(F_{i_1,j_1}, F_{i_2,j_2}, \dots, F_{i_{t+1},j_{t+1}})$ of all corner entries of F, such that the corner entry F_{i_r,j_r} precedes the corner entry F_{i_s,j_s} if either $i_r > i_s$, or $i_r = i_s$ and $j_r < j_s$. Note that, given the structure of $\mathcal{H}_n$, the first corner entry, F_{i_1,j_1}, is always $F_{n,1} = -a_n$ and the last corner entry, $F_{i_{t+1},j_{t+1}}$, is always $-a_1$.

We observe that the ordered list of corner entries determines the structure of the lattice path in $\mathcal{H}_n$ except for the position of $-a_{n-1}$ which could be in either position $(n-1, 1)$ or $(n, 2)$. But, as noted in [10], the options correspond to two matrices which are equivalent via transpose and reverse permutation similarity.

Example 3.1. The following matrix M in $\mathcal{H}_9$ is equivalent to a Fiedler companion matrix and has the ordered list of corner entries $(M_{9,1}, M_{9,3}, M_{6,3}, M_{6,6}) = (-a_9, -a_7, -a_4, -a_1)$:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-a_4} & -a_3 & -a_2 & \boxed{-a_1} & 1 & 0 & 0 \\ 0 & 0 & -a_5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -a_6 & 0 & 0 & 0 & 0 & 0 & 1 \\ \boxed{-a_9} & -a_8 & \boxed{-a_7} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equivalent matrix RM^TR, with R the reverse permutation matrix, is the other matrix in $\mathcal{H}_9$ with the same corner entries:

$$RM^TR = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-a_1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3 & 0 & 0 & 1 & 0 & 0 \\ \boxed{-a_7} & -a_6 & -a_5 & \boxed{-a_4} & 0 & 0 & 0 & 1 & 0 \\ -a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \boxed{-a_9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that, in contrast to [9], our definition of corner entries is based on the Fiedler structure realized in $\mathcal{H}_n$ which has the variables rising to the right along a lattice path, whereas the form in [9] has a non-contiguous 'staircase' going down to the right. The next example illustrates that our definition of corner entries is consistent with the definition given in [9].

Example 3.2. Consider the following matrix:

$$F = A_5A_2A_1A_3A_4 = \begin{bmatrix} -a_1 & 1 & 0 & 0 & 0 \\ -a_2 & 0 & -a_3 & -a_4 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 \end{bmatrix}.$$

By definition of ordered list of corner entries we need to find an equivalent form of this matrix in $\mathcal{H}_5$. However, recognizing that if an entry $-a_k$ is in the same row (or column) as entry $-a_l$ in any matrix F, then $-a_k$ is in the same row or column as $-a_l$ in any matrix equivalent to F. Thus, we need only consider the submatrix

$$\begin{bmatrix} -a_1 & 0 & 0 \\ -a_2 & -a_3 & -a_4 \\ 0 & 0 & -a_5 \end{bmatrix}$$

of F. For any matrix C in $\mathcal{H}_n$ the entry $-a_n$ is in the bottom left corner of the matrix. Thus if C is equivalent to F, then the 3-by-3 submatrix obtained by taking the last 3 rows and first 3 columns of C is

$$\begin{bmatrix} 0 & 0 & -a_1 \\ -a_4 & -a_3 & -a_2 \\ -a_5 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -a_2 & -a_1 \\ 0 & -a_3 & 0 \\ -a_5 & -a_4 & 0 \end{bmatrix}.$$

In either case, the ordered list of corner entries of F is $(-a_5, -a_4, -a_2, -a_1)$.

Let F be an n-by-n Fiedler companion matrix with ordered list of corner entries $(F_{i_1,j_1}, F_{i_2,j_2}, \ldots, F_{i_{t+1},j_{t+1}})$. The *flight length sequence* of F is the sequence

$$\mathscr{F}(F) := (f_1, f_2, \ldots, f_t)$$

with $f_k = \max\{i_k - i_{k+1}, j_{k+1} - j_k\}$, for $k = 1, \ldots, t$. For instance, the flight length sequence of M from Example 3.1 is $\mathscr{F}(M) = (2, 3, 3)$ and the flight length sequence of F from Example 3.2 is $\mathscr{F}(F) = (1, 2, 1)$.

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a permutation of $(1, 2, \ldots, n)$. Similar to [9], we use the following definitions, adjusting for the fact that the indices of the coefficients of the characteristic polynomial are reversed in [9]:

1. For $i = 1, 2, \ldots, n-1$, we say that σ has a *consecution* at i if i is to the left of $i+1$ in σ and that σ has an *inversion* at i if i is to the right of $i+1$. Note that σ has an inversion at i if and only if A_i is the right of A_{i+1} in the Fiedler factorization A_σ. Equivalently, there is an inversion at i if and only if a_i is in the same column as a_{i+1} in A_σ.
2. The *consecution inversion structure sequence of* σ, denoted CISS(σ), is the sequence $(c_0, i_0, c_1, i_1, \ldots, c_\ell, i_\ell)$, such that σ has c_0 consecutive consecutions at $1, \ldots, c_0$; i_0 consecutive inversions at $c_0+1, c_0+2, \ldots, c_0+i_0$ and so on, up to i_ℓ inversions at $n-i_\ell, \ldots, n-1$. Note that either c_0 or i_ℓ could be zero.
3. The *reduced consecution inversion structure sequence of* σ, denoted RCISS(σ), is the sequence obtained from CISS(σ) after removing any zero entries.

In addition we introduce the following terms.

4. For $0 \le k \le \ell$, we define the *kth consecution subsequence* and the *kth inversion subsequence* of σ, denoted $\mathrm{CS}_k(\sigma)$ and $\mathrm{IS}_k(\sigma)$, as follows:
   - $\mathrm{CS}_0(\sigma) = (2, \ldots, c_0 + 1)$,
   - $\mathrm{IS}_0(\sigma) = (c_0 + 2, c_0 + 3, \ldots, c_0 + i_0 + 1)$, and for $k > 0$,
   - $\mathrm{CS}_k(\sigma) = \left(\sum_{j=0}^{k-1}(c_j + i_j) + 2, \ldots, \sum_{j=0}^{k-1}(c_j + i_j) + c_k + 1\right)$ and
   - $\mathrm{IS}_k(\sigma) = \left(\sum_{j=0}^{k-1}(c_j + i_j) + c_k + 2, \ldots, \sum_{j=0}^{k}(c_j + i_j) + 1\right)$.

Example 3.3. Let $\sigma = (1, 7, 6, 5, 8, 2, 3, 4, 9)$. Then CISS($\sigma$) = (3, 3, 2, 0) and RCISS(σ) = (3, 3, 2). Further, $\mathrm{CS}_0(\sigma) = (2, 3, 4)$, $\mathrm{IS}_0(\sigma) = (5, 6, 7)$, and $\mathrm{CS}_1(\sigma) = (8, 9)$. Note that RCISS($\sigma$) $= (c_0, i_0, c_1) = (3, 3, 2)$ and so $\mathrm{CS}_0(\sigma)$ has length c_0, $\mathrm{IS}_0(\sigma)$ has length i_0, and $\mathrm{CS}_1(\sigma)$ has length c_1.

As noted in [9], the consecution-inversion structure is motivated by the fact that some Fiedler factors commute. In particular, $A_iA_j = A_jA_i$ if $|i-j| \neq 1$. Hence, if CISS(σ_1) = CISS(σ_2) then $A_{\sigma_1} = A_{\sigma_2}$. The next lemma helps us obtain a permutation, and hence a Fiedler factorization, with a given consecution inversion structure.

Given two sequences $a = (a_1, a_2, \ldots, a_l)$ and $b = (b_1, b_2, \ldots, b_m)$ we say that $(a, b) = (a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_m)$. Note that if $a = (\,)$, then $(a, b) = (b, a) = b$. If $a = (a_1, a_2, \ldots, a_l)$ then we define the *reverse* of a by $\overleftarrow{a} = (a_l, a_{l-1}, \ldots, a_2, a_1)$.

Lemma 3.4. *Suppose* $\mathrm{CISS}(\rho) = (c_0, i_0, c_1, i_1, \dots, c_t, i_t)$ *is the consecution inversion structure sequence of some permutation* ρ *of* $(1, \dots, n)$. *If*

$$\sigma = \left(\overleftarrow{\mathrm{IS}}_t(\rho), \overleftarrow{\mathrm{IS}}_{t-1}(\rho), \cdots, \overleftarrow{\mathrm{IS}}_1(\rho), \overleftarrow{\mathrm{IS}}_0(\rho), (1), \mathrm{CS}_0(\rho), \mathrm{CS}_1(\rho), \cdots, \mathrm{CS}_{t-1}(\rho), \mathrm{CS}_t(\rho)\right)$$

then $\mathrm{CISS}(\sigma) = \mathrm{CISS}(\rho)$, *and hence* $A_\rho = A_\sigma$.

Proof. We need to show that $\mathrm{CISS}(\sigma) = (c_0, i_0, c_1, i_1, \dots, c_t, i_t)$. In ρ, the values $2, 3, \dots, c_0 + 1$ are all to the right of 1. Since $\mathrm{CS}_0(\rho)$ is to the right of 1 in σ, there are *at least* c_0 consecutive consecutions at 1 in σ. Since $c_0 + 2$ is in $\mathrm{IS}_0(\rho)$ so $c_0 + 2$ is to the left of $c_0 + 1$ in σ. Thus σ has *exactly* c_0 consecutive consecutions at 1. The values $c_0 + 3, c_0 + 4, \dots, c_0 + i_0 + 1$ are all to the right of $c_0 + 2$ in $\mathrm{IS}_0(\rho)$ and hence are all to the left of $c_0 + 2$ in $\overleftarrow{\mathrm{IS}}_0(\rho)$. Thus σ has *at least* i_0 consecutive inversions at $c_0 + 1$. Since $c_0 + i_0 + 2$ is in $\mathrm{CS}_1(\rho)$, $c_0 + i_0 + 2$ is to the right of $c_0 + i_0 + 1$ in σ, thus there are *exactly* i_0 consecutive inversions at $c_0 + 1$ in σ. Let $1 \le k \le t$. Then $\sum_{j=0}^{k-1}(c_j + i_j) + 3, \sum_{j=0}^{k-1}(c_j + i_j) + 4, \dots, \sum_{j=0}^{k-1}(c_j + i_j) + c_k + 1$ are all to the right of $\sum_{j=0}^{k-1}(c_j + i_j) + 2$ in $\mathrm{CS}_k(\rho)$. Thus σ has *at least* $c_k - 1$ consecutive consecutions at $\sum_{j=0}^{k-1}(c_j + i_j) + 2$. But $\sum_{j=0}^{k-1}(c_j + i_j) + 1$ is in $\mathrm{IS}_{k-1}(\rho)$ and $\sum_{j=0}^{k-1}(c_j + i_j) + 2$ is in $\mathrm{CS}_k(\rho)$. Therefore $\sum_{j=0}^{k-1}(c_j + i_j) + 2$ is to the right of $\sum_{j=0}^{k-1}(c_j + i_j) + 1$ in σ. Note that for $k < t$, $\sum_{j=0}^{k-1}(c_j + i_j) + c_k + 2$ is in $\mathrm{IS}_k(\rho)$ and hence is to the left of $\sum_{j=0}^{k-1}(c_j + i_j) + c_k + 1$ in σ, therefore σ has *exactly* c_k consecutive consecutions at $\sum_{j=0}^{k-1}(c_j + i_j) + 1$. Similarly $\sum_{j=0}^{k-1}(c_j + i_j) + c_k + 3, \sum_{j=0}^{k-1}(c_j + i_j) + c_k + 4, \dots, \sum_{j=0}^{k}(c_j + i_j) + 1$ are all to the right of $\sum_{j=0}^{k-1}(c_j + i_j) + c_k + 2$ in $\mathrm{IS}_k(\rho)$ and hence are to the left of $\sum_{j=0}^{k-1}(c_j + i_j) + c_k + 2$ in σ. For $k < t$, $\sum_{j=0}^{k}(c_j + i_j) + 2$ is in $\mathrm{CS}_{k+1}(\rho)$ and hence is to the right of $\sum_{j=0}^{k}(c_j + i_j) + 1$ in σ. Thus σ has exactly i_k consecutive inversions at $\sum_{j=0}^{k-1}(c_j + i_j) + c_k + 1$. Therefore $\mathrm{CISS}(\sigma) = \mathrm{CISS}(\rho)$. □

Example 3.5. Suppose $C = (2, 4, 1, 2)$ is the CISS of some unknown permutation ρ. We will construct a permutation σ such that $\mathrm{CISS}(\sigma) = C$. Now, since $c_0 = 2$, we have $\mathrm{CS}_0(\rho) = (2, 3)$; since $i_0 = 4$, $\mathrm{IS}_0(\rho) = (4, 5, 6, 7)$; since $c_1 = 1$, $\mathrm{CS}_1(\rho) = (8)$; and since $i_1 = 2$, $\mathrm{IS}_1(\rho) = (9, 10)$. Therefore, by Lemma 3.4, $\sigma = \left(\overleftarrow{\mathrm{IS}}_1(\rho), \overleftarrow{\mathrm{IS}}_0(\rho), (1), \mathrm{CS}_0(\rho), \mathrm{CS}_1(\rho)\right) = (10, 9, 7, 6, 5, 4, 1, 2, 3, 8)$. Thus the consecution inversion structure sequence C describes the ordering of a Fiedler product: $A_\sigma = A_{10}A_9A_7A_6A_5A_4A_1A_2A_3A_8$.

It was observed in [9] that if a Fiedler factorization A_σ is reversed, then one obtains the transpose A_σ^T. It follows that if $\mathrm{CISS}(\sigma) = (0, b_1, b_2, \dots, b_t)$ then $\mathrm{RCISS}(\sigma) = (b_1, b_2, \dots, b_t)$ is the consecution inversion structure sequence for the transpose of A_σ.

In the next result, we observe that the flights can be used to describe a Fiedler factorization, up to equivalence. Let M be a matrix with ordered list of corner entries $(M_{i_1,j_1}, M_{i_2,j_2}, \dots, M_{i_t,j_t})$. If $(M_{i_1,j_1}, M_{i_2,j_2}, \dots, M_{i_t,j_t}) = (-a_{k_1}, -a_{k_2}, \dots, -a_{k_t})$, then the *ith flight indices* of M, denoted fl_i, is the sequence $(k_i, k_i - 1, \dots, k_{i+1} + 1)$ for all $1 \le i < t$ and $fl_t = (k_t) = (1)$. The *flight indices* of M is the t-tuple $[fl_1, fl_2, \dots, fl_{t-1}, (1)]$. Note that, for $1 \le i < t$, the sequence fl_i has length f_i where $(f_1, f_2, \dots, f_{t-1})$ is the flight length sequence of M. Since the flight indices are completely determined by the corner entries, this theorem demonstrates how to factor a Fiedler companion matrix, up to equivalence, given only its corner entries.

Theorem 3.6. *Let* M *in* $\mathcal{H}_n$ *be equivalent to a Fiedler companion matrix. If* M *has flight indices* $[fl_1, \dots, fl_{2t}, (1)]$ *(allowing* fl_{2t} *to be empty depending on parity) then* M *is equivalent to* $A_{\sigma_1} \cdots A_{\sigma_n}$ *with*

$$\sigma = \left(fl_1, fl_3, \cdots, fl_{2t-3}, fl_{2t-1}, (1), \overleftarrow{fl}_{2t}, \overleftarrow{fl}_{2t-2}, \cdots, \overleftarrow{fl}_4, \overleftarrow{fl}_2\right).$$

Proof. Let M in $\mathcal{H}_n$ be equivalent to a Fiedler companion matrix with flight indices $[fl_1, fl_2, \dots, fl_{2t}, (1)]$ for some integer t. Without loss of generality, assume fl_{2t} is nonempty. Let $\mathscr{F}(M) = (f_1, f_2, \dots, f_{2t})$ be the flight length sequence of M. In [9, Theorem 5.11], it was noted that $\mathscr{F}(M) = \overleftarrow{\mathrm{RCISS}}(\rho)$ for some permutation ρ. Thus $\mathrm{CISS}(\rho) = (f_{2t}, f_{2t-1}, \dots)$ or $\mathrm{CISS}(\rho) = (0, f_{2t}, f_{2t-1}, \dots)$. We may assume the former since the matrices of the two cases are transpose equivalent. Thus $fl_{2k} = \overleftarrow{\mathrm{CS}}_{t-k}(\rho)$ and $fl_{2k-1} = \overleftarrow{\mathrm{IS}}_{t-k}(\rho)$ for all integers $1 \le k \le t$. Therefore, by Lemma 3.4, M is equivalent to A_σ with $\sigma = \left(fl_1, fl_3, \cdots, fl_{2t-3}, fl_{2t-1}, (1), \overleftarrow{fl}_{2t}, \overleftarrow{fl}_{2t-2}, \cdots, \overleftarrow{fl}_4, \overleftarrow{fl}_2\right)$. □

Example 3.7. Consider the matrix M from Example 3.1. The flight indices of M are $[(9,8),(7,6,5),(4,3,2),(1)]$. Therefore, by Theorem 3.6, M is equivalent to the Fiedler factorization

$$A_\sigma = (A_9A_8)(A_4A_3A_2)(A_1)(A_5A_6A_7) = \begin{bmatrix} -a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & 0 & 0 & -a_5 & -a_6 & -a_7 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_9 & 0 & 0 \end{bmatrix}.$$

To describe the equivalence, we can find a permutation matrix P via Algorithm 2.10:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $P^TA_\sigma P$ is in $\mathcal{H}_9$. In this case $P^TA_\sigma P$ is not quite M: the lattice path goes up and over, instead of over and up (that is, $-a_8$ is in the same column as $-a_9$ instead of the same row). But this means that its transpose is the reverse permutation of M. In particular, using R from (2), $M = RP^TA_\sigma^TPR$.

Corollary 2.6 and Theorem 3.6 together allow us to characterize all 8 possibilities of $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that $A_{\sigma_1} \cdots A_{\sigma_n}$ has a pentadiagonal form. Since A_1 and A_3 commute there are only 4 choices of β in the following theorem statement that produce unique matrices $A_{\sigma_1} \cdots A_{\sigma_n}$. Hence there are only 8 unique matrices produced by the choice of σ in Theorem 3.8.

Theorem 3.8. *Let $n \geq 6$. Let $(p,s) = (n-2, n-1)$ if n is even, and $(p,s) = (n-1, n-2)$ if n is odd. Suppose A is a Fiedler matrix with a pentadiagonal form. Then A is equivalent to A_σ with*

$$\sigma = \left(n, (p, p-2, \ldots, 6, 4), \beta, (5, 7, \ldots, s-2, s)\right) \text{ or } \sigma = \left(n, (s, s-2, \ldots, 7, 5), \overleftarrow{\beta}, (4, 6, \ldots, p-2, p)\right)$$

for some permutation β of $(1,2,3)$.

Proof. Let F be a matrix in $\mathcal{H}_n$ equivalent to a Fiedler companion matrix with a pentadiagonal form. Suppose $F_{i,j} = -a_3$ for some integers $1 \leq i, j \leq n$. Then by definition of $\mathcal{H}_n$ and Theorem 2.9, $2 < i \leq n-1$ and $2 \leq j < n-1$. Consider the 3-by-3 submatrix, denoted B, obtained by taking rows $(i, i-1, i-2)$ and columns $(j, j+1, j+2)$. Since the variable entries of F form a lattice path, there are four possibilities for B:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -a_1 & 1 \\ -a_3 & -a_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -a_2 & -a_1 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix},$$

noting that a_4 is in position $(i, j-1)$ if n is even, and position $(i+1, j)$ if n is odd. Note that the flight lengths of any matrix of type $\mathcal{A}_n$ are of the form $(1, 1, \ldots, 1, g)$, whereas those of type $\mathcal{B}_n$ are of the form $(2, 1, 1, \ldots, 1, g)$, with g one of (3), (2,1), (1,1,1) or (1,2). Thus, the result follows from Theorem 3.6. □

Corollary 2.6 implies that the eight patterns found in Theorem 3.8 are nonequivalent and hence characterize the Fiedler matrices that have a pentadiagonal form. It is noted in [9, Example 2.2] that there are exactly four pentadiagonal Fiedler factorizations, but the four are pairwise transposes, so there are only two up to equivalence. These two Fielder companion matrices correspond to the choice of $\beta = (2, 1, 3)$ in Theorem 3.8. The other six nonequivalent Fiedler factorizations that have a pentadiagonal form require a permutation similarity to obtain a pentadiagonal matrix. As can be seen from the proof, those permutations σ in Therorem 3.8 having $(n, n-1)$ as the first two entries will produce a pentadiagonal matrix equivalent to a matrix of type $\mathcal{B}_n$, and otherwise, the pentadiagonal matrix will be equivalent to a matrix of type $\mathcal{A}_n$. Note also that each permutation in Theorem 3.8 produces a matrix with $-a_{n-1}$ in the same column as $-a_n$. To obtain the equivalent Fiedler matrix with $-a_{n-1}$ in the same row as $-a_n$, one can take $\overleftarrow{\sigma}$ to obtain the transpose matrix.

4 Conluding Remarks

We have characterized, and counted, the pentadiagonal matrices that are equivalent to a matrix in $\mathcal{H}_n$. As such we have characterized the structure of all sparse pentadiagonal companion matrices, up to equivalence, and provided the explicit factorization of those which are equivalent to Fiedler pentadiagonal matrices. Further we have provided a tool that allows one to find the Fiedler factorization of any Fiedler companion matrix. For any matrix M equivalent to a matrix in $\mathcal{H}_n$, we have provided an algorithm to determine the permutation necessary to bring M into the lower-Hessenberg form of the matrices in $\mathcal{H}_n$ and, if M has a pentadiagonal form, we have also presented the permutation necessary to bring M into a pentadiagonal form. We illustrate some of the tools developed with an example.

Example 4.1. Let C be any matrix in $\mathcal{H}_n$. It was observed in [10] that $C = E_1 Q^T A Q E_2$ for some Fiedler companion matrix $A = A_{i_1} A_{i_2} \cdots A_{i_n}$, some permutation matrix Q, and some products of elementary matrices E_1 and E_2. The elementary matrices can be obtained by pivoting on the unit entries in the superdiagonal of C and are easy to choose. In fact, E_1 and E_2 can always be chosen so that $E_2 = E_1^{-1}$ since a Fiedler matrix in Hessenberg form can be obtained from C by shifting subdiagonal entries along their subdiagonal. In particular, any single shift of $-a_k$ from position (i, j) to $(i-1, j-1)$ can be obtained by an elementary row operation of multiplying row $j-1$ by a_k and adding to row i, and likewise, the column operation of multiplying column i by $-a_k$ and adding to row $j-1$. (It is necessary to note that row $j-1$ and column i of C have only one nonzero entry, namely 1, since, by Theorem 1.1, all the variable entries must be contained within a rectangle determined by $-a_1$, and $-a_1$ must be in one of the positions $(j, j), (j+1, j+1), \ldots, (i-1, i-1)$ in C.)

For instance, consider the matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -a_4 & -a_3 & 0 & -a_1 & 1 \\ -a_6 & -a_5 & 0 & 0 & -a_2 & 0 \end{bmatrix}$$

in $\mathcal{H}_6$. There are multiple ways to use elementary row and column operations to transform C into a Fiedler companion matrix. One option is to take the entry $-a_2$ from position $(6, 5)$ to position $(5, 4)$. This can be obtained by two elementary operations: adding a multiple of row 4 to row 6 and adding a multiple of column 6 to column 4. If we let E_1 be the corresponding elementary matrix, then $C = E_1 F E_1^{-1}$ for a Fiedler matrix F with flight indices $[(6), (5), (4, 3, 2), (1)]$. Using $\overleftarrow{\sigma}$ from Theorem 3.6 (since a_6 and a_5 are in the same row), we get $F = Q^T A_5 A_1 A_2 A_3 A_4 A_6 Q$ for some permutation matrix Q. Algorithm 2.10 provides us one way to determine the permutation matrix Q.

Theorem 3.6 allows us to factor any matrix in $\mathcal{H}_n$, not just those equivalent to Fiedler companion matrices, by first using elementary matrices to put the matrix into an equivalent Fiedler matrix. In this example, $C =$

$E_1 Q^T A_5 A_1 A_2 A_3 A_4 A_6 Q E_1^{-1}$ with

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Such a factorization can be done for any matrix in $\mathcal{H}_n$. In this case, C is of type $\mathcal{A}_6$ and so, by Theorem 2.9, C has a pentadiagonal form. By Lemma 2.8, $P_6 C P_6^T$ is pentadiagonal for $P_6 = \left[\; e_1 \mid e_3 \mid e_5 \mid e_6 \mid e_4 \mid e_2 \;\right]$. Thus $P_6 E_1 Q^T A_5 A_1 A_2 A_3 A_4 A_6 Q E_1^{-1} P_6^T$ is a factorization of a pentadiagonal matrix equivalent to C.

Remark 4.2. Note that our definition of companion matrix requires that each polynomial coefficient appears exactly once in the matrix. Part of the reason for that was our focus on the sparse companion matrices as introduced in [10]. In other contexts it may be worth exploring matrices that allow the coefficients to appear more often since, for example, there are linearizations of matrix polynomials in which some coefficients appear more than once (see e.g. [13, 15]). In [12], sparse matrices whose entries are rational functions in the coefficients are considered. One class named in [12] is the class of *generalized companion matrices* derived from the class $\mathcal{H}_n$. For example the matrix H from (3) gives rise to the generalized companion matrix

$$H' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -a_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & a_1 a_2 - a_3 & -a_2 & 0 & 1 \\ -a_5 & -a_4 & 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

with characteristic polynomial $x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$. Pentadiagonal results in this paper apply to the generalized companion matrices defined in [12] since we characterize all the matrices in $\mathcal{H}_n$ which are equivalent to a pentadiagonal matrix, not just the companion matrices. For example, applying Lemma 2.8 to H, we see that H' is equivalent to the pentadiagonal matrix

$$P_6 H' P_6^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -a_5 & 0 & -a_4 & 0 & 0 \\ 0 & 0 & -a_1 & 0 & 1 \\ 0 & 1 & a_1 a_2 - a_3 & 0 & -a_2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

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Appendix: The 6-by-6 Sparse Pentadiagonal Companion Matrices

This Appendix lists all 6-by-6 sparse pentadiagonal companion matrices up to equivalence. Both a pentadiagonal form X and a Hessenberg form H are presented. The permutation that takes the matrix from Hessenberg form to pentadiagonal form is presented as one of two options: $P = RP_6 = [e_6|e_4|e_2|e_1|e_3|e_5]$ or $Q = RQ_6 = [e_4|e_2|e_1|e_3|e_5|e_6]$, depending if the Hessenberg matrix is type $\mathcal{A}_6$ or $\mathcal{B}_6$, respectively. In particular, either $PHP^T = X$ or $QHQ^T = X$ as indicated. Note that using $P = P_6$ and $Q = Q_6$ would also give a pentadiagonal matrix, but we have chosen to get X in $\mathcal{X}$. Non-Fiedler companion matrices are indicated with

NF and if the companion matrix is a Fiedler companion matrix, then the corresponding permutation σ for a Fiedler factorization A_σ is indicated.

Pentadiagonal Form X	Hessenberg Form H	Type	σ
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -a_1 & 0 & -a_2 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -a_2 & -a_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -a_4 & -a_3 & 0 & 0 & 1 \\ -a_6 & -a_5 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{A}_6$	NF
$\begin{bmatrix} 0 & -a_2 & 1 & 0 & 0 & 0 \\ 1 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 1 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \\ 0 & -a_4 & -a_3 & 0 & 0 & 1 \\ -a_6 & -a_5 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{A}_6$	$(5, 3, 2, 1, 4, 6)$
$\begin{bmatrix} -a_1 & -a_2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a_2 & -a_1 & 1 & 0 \\ 0 & -a_4 & -a_3 & 0 & 0 & 1 \\ -a_6 & -a_5 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{A}_6$	$(5, 3, 1, 2, 4, 6)$
$\begin{bmatrix} -a_1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_1 & 1 & 0 \\ 0 & -a_4 & -a_3 & -a_2 & 0 & 1 \\ -a_6 & -a_5 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{A}_6$	$(5, 2, 1, 3, 4, 6)$
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & -a_3 & -a_1 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -a_4 & -a_3 & -a_2 & -a_1 & 1 \\ -a_6 & -a_5 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{A}_6$	$(5, 1, 2, 3, 4, 6)$
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & -a_1 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & -a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -a_4 & -a_3 & 0 & -a_1 & 1 \\ -a_6 & -a_5 & 0 & 0 & -a_2 & 0 \end{bmatrix}$	$\mathcal{A}_6$	NF
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -a_1 & 0 & -a_2 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & 1 \\ 0 & 0 & 0 & -a_6 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_2 & -a_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -a_4 & -a_3 & 0 & 0 & 1 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 1 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{B}_6$	NF
$\begin{bmatrix} 0 & -a_2 & 1 & 0 & 0 & 0 \\ 1 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & 1 \\ 0 & 0 & 0 & -a_6 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -a_1 & 1 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 1 & 0 & 0 \\ -a_4 & -a_3 & 0 & 0 & 1 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 1 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{B}_6$	$(6, 5, 3, 2, 1, 4)$
$\begin{bmatrix} -a_1 & -a_2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & 1 \\ 0 & 0 & 0 & -a_6 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -a_2 & -a_1 & 1 & 0 & 0 \\ -a_4 & -a_3 & 0 & 0 & 1 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 1 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{B}_6$	$(6, 5, 3, 1, 2, 4)$

Pentadiagonal Form X	Hessenberg Form H	Type	σ
$\begin{bmatrix} -a_1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & -a_3 & 0 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & 1 \\ 0 & 0 & 0 & -a_6 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 1 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & 0 & 1 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 1 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{B}_6$	$(6, 5, 2, 1, 3, 4)$
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & -a_3 & -a_1 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 & 0 & 1 \\ 0 & 0 & 0 & -a_6 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 & 1 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 1 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{B}_6$	$(6, 5, 1, 2, 3, 4)$
$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & -a_1 & -a_4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & -a_5 & 0 & 1 \\ 0 & 0 & 0 & -a_6 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -a_4 & -a_3 & 0 & -a_1 & 1 & 0 \\ -a_5 & 0 & 0 & -a_2 & 0 & 1 \\ -a_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\mathcal{B}_6$	NF

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