

Research Article

Open Access

R. B. Bapat and Sivaramakrishnan Sivasubramanian*

The Smith normal form of product distance matrices

DOI 10.1515/spma-2016-0005

Received June 15, 2015; accepted November 10, 2015

Abstract: Let $G = (V, E)$ be a connected graph with 2-connected blocks H_1, H_2, \dots, H_r . Motivated by the exponential distance matrix, Bapat and Sivasubramanian in [4] defined its product distance matrix D_G and showed that $\det D_G$ only depends on $\det D_{H_i}$ for $1 \leq i \leq r$ and not on the manner in which its blocks are connected. In this work, when distances are symmetric, we generalize this result to the Smith Normal Form of D_G and give an explicit formula for the invariant factors of D_G .

1 Introduction

For a positive integer n , denote the set $\{1, 2, \dots, n\}$ as $[n]$. Let G be a connected graph with vertex set $V(G) = [n]$. A *block* of G is a maximally connected subgraph without a cut-vertex. In this work, for a graph G , we will look at symmetric functions $\eta : V(G) \times V(G) \rightarrow R$ where R is a commutative principal ideal domain (PID henceforth). A product distance on G is a function $\eta : V(G) \times V(G) \rightarrow R$, that satisfies the following three conditions:

1. $\eta(i, i) = 1$ for all $i \in [n]$.
2. $\eta(i, j) = \eta(j, i)$ for all $i, j \in V$ and
3. if $i, j \in V(G)$ are vertices such that every path from i to j passes through the cut-vertex k , then $\eta(i, j) = \eta(i, k)\eta(k, j)$.

Thus, we essentially have the freedom to assign distances within each block subject to symmetry and diagonal entries being 1. Once these distances are fixed, distances across blocks will follow from the third rule above. We sometimes denote $\eta(i, j)$ equivalently as $\eta_{i,j}$, especially when we form a matrix of the distances. Let G have blocks H_1, H_2, \dots, H_r . Let $\eta(\cdot, \cdot)$ be a product distance on G and let $D_G = (\eta_{i,j})_{1 \leq i, j \leq n}$ be the corresponding distance matrix.

The definition of *product distances* is motivated by a concrete example: the exponential distance matrix ED_G of a connected graph. Given a connected graph G on the vertex set $[n]$, let the distance between two vertices $i, j \in V(G)$ be denoted $d_{i,j}$. That is, $d_{i,j}$ is the length of the minimum length path from i to j in G . Define the $n \times n$ matrix $ED_G = (q^{d_{i,j}})_{1 \leq i, j \leq n}$ as the exponential distance matrix where q is an indeterminate and $q^0 = 1$. It can be readily checked that $\eta(i, j) = q^{d_{i,j}}$ is a product distance.

A large family of product distances can be obtained from geodetic distances as follows. Let $G = (V, E, w)$ be a graph with weights $w : E \rightarrow \mathbb{R}^+$ on its edges. A function $d : V \times V \rightarrow \mathbb{R}$ is defined to be *graph geodetic* if for $i, j, k \in V$, the condition $d(i, j) + d(j, k) = d(i, k)$ holds iff every path in G from i to k passes through j . It is easy to see that the usual weighted graph distance is graph-geodetic. Klein and Randić in [9] showed that the resistance distance is also graph-geodetic. Chebotarev in [5] has constructed several graph geodetic distances parametrised by a real variable α . He showed that at boundary values of α , his distance coincides

*Corresponding Author: Sivaramakrishnan Sivasubramanian: Department of Mathematics, Indian Institute of Technology, Bombay, Mumbai 400 076, India, E-mail: krishnan@math.iitb.ac.in

R. B. Bapat: Stat-Math Unit, Indian Statistical Institute, Delhi, 7-SJS Marg, New Delhi 110 016, India, E-mail: rbb@isid.ac.in



with the usual shortest path distance and the resistance distance. In [6], Chebotarev constructed more graph geodesic distances from positive functions $f : V \times V \rightarrow \mathbb{R}$ which satisfy the “transition inequality”.

Let $d : V \times V \rightarrow \mathbb{R}$ be a graph-geodesic distance and consider a new function $e : V \times V \rightarrow \mathbb{R}$ defined by $e(x, y) = q^{d(x,y)}$, where q is an indeterminate. Then, $e(x, y)$ becomes a product distance by virtue of the graph-geodesic property. Our results are applicable to this large class of distances.

If $\eta(\cdot, \cdot)$ is a product distance on G and if G has blocks H_1, H_2, \dots, H_r , then, each H_i is a graph in its own right and thus has an induced product distance matrix D_{H_i} obtained by restricting η to vertices in H_i . Indeed, these are the distances within vertices of a block which induce the product-distance on G . If the graph G is clear from the context, we abridge D_G to D .

If D is a matrix whose entries form a product distance on G , Bapat and Sivasubramanian [4] showed that $\det D$ only depends on $\det D_{H_i}$ for individual blocks H_i of G and not on the manner in which the H_i 's are connected. Their result is true in a more general asymmetric distance case. We state below, a symmetric version of their result.

Theorem 1. ([4, Theorem 4]) *Let G be a connected graph with blocks H_i , $1 \leq i \leq r$ and product distance matrix D_G . For each such i , let the distance matrix of each H_i be D_{H_i} . Then,*

$$\det D_G = \prod_{i=1}^r \det D_{H_i}.$$

In particular, $\det D_G$ is independent of the manner in which the blocks H_i of G are connected. In this paper, we work with matrices M over a PID R . In this case, every finite subset $S \subseteq R$ naturally has a greatest common divisor (gcd henceforth). The determinant of an $n \times n$ matrix M with entries from a PID clearly equals the gcd of all $n \times n$ minors of M (as there is only one such minor).

Thus, Theorem 1 can be alternatively stated as “the gcd of $n \times n$ minors of D_G is independent of the manner in which its blocks are connected.” Each $k \times k$ minor of M is an element of the PID R and hence, we can talk of the gcd of $k \times k$ minors, with gcd being taken over all the $\binom{n}{k}^2$ choices. If R is a PID, for a multiset $T = \{x_1, x_2, \dots, x_t\} \subseteq R$, the gcd of the elements of T will be denoted as $\gcd(x_1, \dots, x_t)$ or as $\gcd_{x \in T} x$. In the above expression and throughout this paper, when we write $\gcd_{x \in T} x$, we remove those $x \in T$ that are zero and consider the gcd() only over the non-zero elements of T .

In this work, we extend the above gcd interpretation of Theorem 1 to all $k \times k$ minors of D_G . Our main result is Theorem 3 where we give an explicit formula for the gcd of $k \times k$ minors of D_G as a function of the gcd of smaller minors of the product distance matrix D_{H_i} of blocks H_i of G . Since the gcd of $k \times k$ minors occurs in the Smith Normal Form (SNF henceforth) of D_G , our results have implications for the SNF of D_G (see Corollary 9).

Shiu [11] has shown some results about the SNF of exponential distance matrices arising from hyperplane arrangements. We are not aware of any other results similar to ours in the literature.

2 The main result

Let D_G be the product distance matrix of a graph G and let H_1, H_2, \dots, H_r be the blocks of G . Let G have n vertices and similarly, let H_i have n_i vertices for all i . Clearly $n = \sum_{i=1}^r n_i - (r - 1)$.

Recall that D_{H_i} is the distance matrix of H_i . For $1 \leq i \leq r$ and for $1 \leq k \leq n_i$, let the gcd of $k \times k$ minors of D_{H_i} be denoted as $g_{i,k-1}$ where the first index is the block number and the second index denotes size minus one. The reason for the second parameter being size minus one will be clear after we see Corollary 7. Thus, for $1 \leq i \leq r$, we have $g_{i,0}, g_{i,1}, \dots, g_{i,n_i-1}$. For $1 \leq i \leq r$, and for values $j \geq n_i$ define $g_{i,j} = 0$. It is easy to note for all i , that $g_{i,0} = 1$ as each diagonal entry of D_{H_i} is a 1×1 matrix which equals 1.

Recall r is the number of blocks of G and for positive integers s satisfying $1 \leq s \leq (\sum_{i=1}^r n_i) - r$, define T_s to be the set of ordered integral solutions to the equation $s = s_1 + s_2 + \dots + s_r$ where $0 \leq s_i < n_i$. Here, ordered

means that the order of the elements (s_1, s_2, \dots, s_r) is important. We will index elements of T_s by ordered tuples (s_1, s_2, \dots, s_r) . Since the graph G is fixed, its number of blocks r is also fixed and hence, the number of summands, r is tacitly obvious. We thus denote the above solution set as T_s instead of the more precise $T_{s,r}$.

Definition 2. With the notation $g_{i,k}$ above, define for $1 \leq s \leq (\sum_{i=1}^r n_i) - r$,

$$g_s = \gcd_{(s_1, s_2, \dots, s_r) \in T_s} \left(\prod_{i=1}^r g_{i, s_i} \right). \tag{1}$$

With this notation, our main result is the following.

Theorem 3. Let D_G be the product distance matrix of a connected graph G with blocks H_i for $1 \leq i \leq r$ where H_i has n_i vertices. For $1 \leq i \leq r$, let the gcd of $k \times k$ minors of D_{H_i} be $g_{i,k-1}$ where $1 \leq k \leq n_i$. Then, the gcd of 1×1 minors of D_G is 1 and for $2 \leq s \leq (\sum_{i=1}^r n_i) - (r - 1)$ the gcd of $s \times s$ minors of D_G is g_{s-1} , where g_s is defined by (1).

In order to find the gcd of $k \times k$ minors of a matrix A , we look at equivalent matrices B defined as follows. Two $n \times n$ matrices A, B are said to be equivalent, denoted $B \sim A$ if there exist $n \times n$ matrices U, V with both $\det U$ and $\det V$ being units in the ring R and with $B = UAV$. If $R = \mathbb{Z}$, then, we require $\det U = \det V = \pm 1$. We will use elementary row and column operations on matrices. These are the non-multiplicative elementary operations (that is we do not multiply a row or column by a scalar). It is well known that such elementary operations can be accomplished by premultiplying or postmultiplying by matrices whose determinants are ± 1 . It follows from the Binet-Cauchy theorem that if $B \sim A$, then, the gcd of $k \times k$ minors of A equals the gcd of $k \times k$ minors of B for all $1 \leq k \leq n$. (See [10, Theorem II.8].)

2.1 Proof of Theorem 3

We will calculate the gcd of $s \times s$ minors of D_G by getting an equivalent matrix M_r which is a direct sum of several diagonal blocks K_i . We will know the gcd of $k \times k$ minors of each direct summand K_i for all $1 \leq k \leq |K_i|$. From this, we will get the gcd of $s \times s$ minors of D_G .

We need two results, one on getting the gcd of $s \times s$ minors of a direct-sum matrix when we know the gcds of minors of its direct-sum constituents. Secondly, we need to get the gcd of $k \times k$ minors of each direct summand K_i - this will be done inductively. The first point is addressed by the following. We now change our notation slightly and denote gcds of $k \times k$ minors by a_k, b_k and so on. See Remark 10 later for an explanation for this change.

Lemma 4. Let M be an $n \times n$ square matrix over a PID R and let $M = A \oplus B$ be a direct sum of two square matrices A, B , where A is an $s \times s$ matrix and B is an $(n - s) \times (n - s)$ matrix. That is, $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Let the gcd of $k \times k$ minors of A be denoted a_k for $1 \leq k \leq s$ and likewise, let b_k denote the gcd of $k \times k$ minors of B for $1 \leq k \leq n - s$. Define $a_0 = b_0 = 1$. Let the gcd of $k \times k$ minors of M be denoted m_k for $1 \leq k \leq n$. For an integer $1 \leq k \leq n$, let T_k be the set of ordered integral solutions (x, y) to the equation $k = x + y$, where $0 \leq x \leq s$ and $0 \leq y \leq n - s$. Then, $m_k = \gcd_{(x,y) \in T_k} a_x b_y$.

Proof. Let Y be a $k \times k$ submatrix of M . Then, Y is obtained by choosing $P, Q \subset [s]$, where P is a set of chosen rows and Q is a set of chosen columns, and choosing $L, N \subset [n] - [s]$, where L is a set of chosen rows and N is a set of chosen columns. Clearly, $|P| + |L| = k$ and $|Q| + |N| = k$. That is, the submatrices $A[P, Q]$ and $B[L, N]$ are chosen and so $Y = A[P, Q] \oplus B[L, N]$ can be written as a direct sum. If $|L| + |Q| < k$, then $|P| + |N| > k$ and so there will exist a zero submatrix (induced on the rows indexed by P and columns indexed by N) of order $|P| \times |N|$ where $|P| + |N| > k$. For any $k \times k$ matrix $Y = (y_{i,j})_{1 \leq i, j \leq k}$, if there exists $P, N \subseteq [k]$

with $|P| + |N| > k$ and with $Y[P, N] = 0$, then we claim that $\det Y = 0$. We will show a stronger statement that all terms in the Laplace expansion of $\det Y$ will be zero. Without loss of generality, assume that the zero submatrix of Y is formed on the rows $P = \{1, 2, \dots, |P|\}$ and the columns $N = \{1, 2, \dots, |N|\}$. We claim that all permutations $\pi \in \mathfrak{S}_k$ satisfy $y_{i, \pi_i} = 0$ for some $i \in [k]$. To see this, suppose this does not happen. Then, we have $\{\pi_1, \pi_2, \dots, \pi_{|P|}\} \cap \{1, 2, \dots, |N|\} = \emptyset$, which contradicts our assumption that $|P| + |N| > k$. (This proof is very similar to the proof of the Frobenius-König theorem [3, Theorem 2.1.4].)

Thus, if we want $\det Y \neq 0$, we must have $|L| + |Q| \geq k$. A similar argument shows that we must have $|P| + |N| \geq k$ if we want $\det Y \neq 0$. Hence, if $\det Y \neq 0$, we must have $|L| + |Q| = k = |P| + |N|$. We already know $|L| + |P| = k = |Q| + |N|$. Thus, we infer that $|P| = |Q|$ and $|L| = |N|$. That is, only square submatrices of the direct summands can be chosen if we want $\det Y \neq 0$. Thus the determinant of any non-singular $k \times k$ submatrix of M equals the product of determinants of non-singular submatrices M_1 of A and M_2 of B . Hence the gcd m_k of $k \times k$ minors of M equals $\gcd_{(x,y) \in T_k} a_x b_y$. The proof is complete. \square

More generally, if M is the direct sum of r diagonal blocks D_1, D_2, \dots, D_r , then it is clear that we only need to modify the definition of T_k to contain r ordered summands adding upto k . We state this generalisation without proof as a corollary. We just remark that a proof by induction on the number of summands is straightforward.

For $1 \leq i \leq r$, let D_i be an $n_i \times n_i$ matrix, and for $1 \leq k \leq n_i$, let the gcd of $k \times k$ minors of D_i be denoted as $g_{i,k}$. For $1 \leq i \leq r$, define $g_{i,0} = 1$. Define T_s to be the set of ordered integral solutions to the equation $s = s_1 + s_2 + \dots + s_r$ where $0 \leq s_i \leq n_i$. This definition is similar, but slightly different from Definition 2. See Remark 10 later for an explanation.

Definition 5. With the notation $g_{i,k}$ above, define for $1 \leq s \leq \sum_{i=1}^r n_i$,

$$g_s = \gcd_{(s_1, s_2, \dots, s_r) \in T_s} \left(\prod_{i=1}^r g_{i, s_i} \right). \quad (2)$$

Corollary 6. Let $M = \bigoplus_{i=1}^r D_i$ be an $n \times n$ matrix where D_i has dimension $n_i \times n_i$. Then, for $1 \leq s \leq n$, the gcd of $s \times s$ minors of M is g_s .

We now move on to a useful corollary of Lemma 4, where we find the gcd of $k \times k$ minors when one direct summand is 1. This corollary will be useful in the proof of Theorem 3.

Corollary 7. Let A be an $n \times n$ matrix and for $1 \leq k \leq n$, let the gcd of $k \times k$ minors of A be denoted a_k . Let M be a matrix obtained by performing elementary row and column operations on A such that M is the direct sum of a 1×1 matrix 1 and an $(n-1) \times (n-1)$ matrix B . That is, $M = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$. For $1 \leq k \leq n-1$, let b_k be the gcd of $k \times k$ minors of B . Then, $b_k = a_{k+1}$ for $1 \leq k \leq n-1$.

Proof. As M is obtained by performing row/column operations on A , $M \sim A$. Thus, the gcd of $(k+1) \times (k+1)$ minors of A equals the gcd of $(k+1) \times (k+1)$ minors of M . For $1 \leq k \leq n$, denote the gcd of $k \times k$ minors of M as m_k . Then, for $0 \leq k \leq n-1$, we have $a_{k+1} = m_{k+1}$. By Lemma 4, since M is a direct-sum of a 1×1 matrix and an $(n-1) \times (n-1)$ matrix, $m_{k+1} = \gcd(b_k, b_{k+1}) = b_k$, where the last equality follows as each $(k+1) \times (k+1)$ minor of B is a linear combination of $k \times k$ minors of B . The proof is complete. \square

The above lemma says that the gcd of 1×1 minors of B equals the gcd of 2×2 minors of A ; the gcd of 2×2 minors of B equals the gcd of 3×3 minors of A and so on. We are now in a position to prove Theorem 3. As the proof is algorithmic, we illustrate our proof later in Example 11 for clarity.

Remark 8. Corollary 7 will be repeatedly used in this work and thus at several places, we will denote the gcds of $k \times k$ minors of matrices with index $k-1$. (That is, as a_{k-1} and so on).

Proof. (Of Theorem 3) Among the blocks H_j of G , define a leaf-block as one with exactly one cut-vertex. Clearly, leaf-blocks exist and let H_1 be a leaf-block of G connected via cut-vertex cv_1 . Let $L_0 = G$ and let $L_1 = L_0 -$

$(H_1 - \{cv_1\})$. Let the vertices of L_1 be $\{u_1, u_2, \dots, u_p\}$ where $u_p = cv_1$. Let the rows and columns of D_G be written in the order u_1, u_2, \dots, u_p , followed by the vertices of $H_1 - \{cv_1\}$ in some order w_2, w_3, \dots, w_{n_1} .

For an $n \times n$ matrix M , denote its i -th row for $1 \leq i \leq n$ as Row_i and its j -th column for $1 \leq j \leq n$ as Col_j . For vertices $u, v \in H_1$, recall the distance between them is $\eta(u, v)$. Obtain a matrix M_1 equivalent to D_G as follows: perform the elementary row operations $\text{Row}_{w_i} = \text{Row}_{w_i} - \eta(cv_1, w_i)\text{Row}_{cv_1}$ for $2 \leq i \leq n_1$ and then perform the elementary column operations $\text{Col}_{w_i} = \text{Col}_{w_i} - \eta(cv_1, w_i)\text{Col}_{cv_1}$ for $2 \leq i \leq n_1$. The resulting matrix $M_1 = ((M_1)_{u,v})$ will have $(M_1)_{u,w} = 0$ for $u \in L_1, w \in H_1 - \{cv_1\}$ (likewise for $(M_1)_{w,u}$). Thus, M_1 can be written as the following direct sum. $M_1 = \begin{pmatrix} D_{L_1} & 0 \\ 0 & K_1 \end{pmatrix}$, where K_1 is an $(n_1 - 1) \times (n_1 - 1)$ sized matrix with rows and columns indexed by w_2, \dots, w_{n_1} with the (w_r, w_s) -th entry being $\eta(w_r, w_s) - \eta(w_r, cv_1)\eta(cv_1, w_s)$.

Recall that $g_{1,k-1}$ is the gcd of $k \times k$ minors of D_{H_1} for $1 \leq k \leq n_1 - 1$ and let b_k be the gcd of $k \times k$ minors of K_1 for $1 \leq k \leq n_1 - 1$. By performing the same row/column operations as before, we get that D_{H_1} is equivalent to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & K_1 \end{pmatrix}$ and hence by Corollary 7, it follows that

$$\text{for } 1 \leq k < n_1, \text{ we have } b_k = g_{1,k+1}. \tag{3}$$

Upto equivalence, we have decomposed D_G into a direct sum of K_1 and D_{L_1} and we know the gcd of $k \times k$ minors of K_1 for all k . If we also know the gcd of $t \times t$ minors of D_{L_1} , then we can get the gcd of all minors of the direct sum of D_{L_1} and K_1 using Lemma 4.

Towards doing this, we iterate this process, just that we now work with the graph L_1 . Pick a leaf block H_2 of L_1 (H_2 need not necessarily be a leaf-block of G). Let H_2 be connected via cut-vertex cv_2 to L_1 . Define $L_2 = L_1 - (H_2 - \{cv_2\})$. We may assume (by multiplying by a permutation matrix if necessary) that the rows and columns of D_{L_1} are listed with vertices of $L_2 - \{cv_2\}$ in some order, followed by cv_2 and then the vertices of $H_2 - \{cv_2\}$ in some order. Denote the vertices of $H_2 - \{cv_2\}$ as z_2, z_3, \dots, z_{n_2} . Obtain a matrix equivalent to D_{H_2} as follows: perform the elementary row operations $\text{Row}_{z_i} = \text{Row}_{z_i} - \eta(cv_2, z_i)\text{Row}_{cv_2}$ for $2 \leq i \leq n_2$ and then perform the elementary column operations $\text{Col}_{z_i} = \text{Col}_{z_i} - \eta(cv_2, z_i)\text{Col}_{cv_2}$ for $2 \leq i \leq n_2$. We note that though it appears that we perform row and column operations on D_{L_1} , we can actually perform operations on D_G to get an equivalent matrix M_2 to M_1 (and hence equivalent to D_G) where D_{L_1} is broken into a direct

sum of K_2 , a square matrix with dimension $n_2 - 1$ and D_{L_2} . ie we get $M_2 = \begin{pmatrix} D_{L_2} & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_1 \end{pmatrix}$. Thus, we

get an equivalent matrix with one more direct summand. Iterate this procedure till we have a graph with only one block H_r . From our process, it is clear that we get one direct summand for each 2-connected block of G . We stop iterating when we have one block H_r of G . In this case, the leaf-block is H_r itself and denote its vertices as $1, 2, \dots, n_r$. Treat 1 as a cut-vertex connecting H_r to the empty graph and perform the same row and column operations for all vertices of $H_r - \{1\}$. This will result in a decomposition of D_{H_r} as a direct sum of K_r of dimension $(n_r - 1) \times (n_r - 1)$ and a 1×1 matrix consisting of the entry 1. Thus, we get a matrix M_r ,

equivalent to D_G , of the form $M_r = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & K_r & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & K_2 & 0 \\ 0 & 0 & \cdots & 0 & K_1 \end{pmatrix}$.

In this direct-sum decomposition, each square block K_i has size $n_i - 1$ for $1 \leq i \leq r$ and we have a 1×1 block K_0 with entry 1. We apply Corollary 6 to get the gcd of $k \times k$ minors of the direct sum $\bigoplus_{i=1}^r K_i$ and combine with Corollary 7 to complete the proof of the theorem. \square

A simple corollary of Theorem 3 is the following.

Corollary 9. *The SNF of D_G is independent of the manner in which its blocks H_i are connected.*

Proof. By Theorem 3, the gcd h_k of $k \times k$ minors for $1 \leq k \leq n$ of D_G is independent of the manner in which the blocks H_i are connected. Set $h_0 = 1$. The invariant factors s_k for $1 \leq k \leq n$ of D_G are $s_k = \frac{h_k}{h_{k-1}}$. Clearly the h_k 's only depend on the blocks of G and as taking gcds is also independent of the ordering of the blocks, the h_k 's are independent of the manner in which the blocks H_i are connected. Thus, so are the s_k 's, completing the proof. \square

Remark 10. The reason for denoting the gcd of $k \times k$ minors of D_{H_i} as $g_{i,k-1}$ stems from the proof of Theorem 3 and Corollary 7.

Example 11. We illustrate the inductive argument occurring in the proof of Theorem 3 on an example graph, shown in Figure 1) where we use the exponential distance matrix. Let D_G be the exponential distance matrix ED_G of $L_0 = G$.

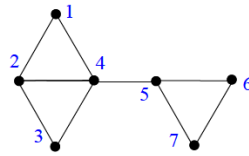


Figure 1: An example of a graph.

Let H_1 be the subgraph on the vertices $\{5, 6, 7\}$ with cut vertex $cv_1 = 5$ and let $L_1 = G - (H_1 - \{cv_1\})$.

Clearly, $ED_G = \begin{pmatrix} 1 & q & q^2 & q & q^2 & q^3 & q^3 \\ q & 1 & q & q & q^2 & q^3 & q^3 \\ q^2 & q & 1 & q & q^2 & q^3 & q^3 \\ q & q & q & 1 & q & q^2 & q^2 \\ q^2 & q^2 & q^2 & q & 1 & q & q \\ q^3 & q^3 & q^3 & q^2 & q & 1 & q \\ q^3 & q^3 & q^3 & q^2 & q & q & 1 \end{pmatrix}.$

Let the i -th row of ED_G for $1 \leq i \leq n$ be denoted Row_i and the i -th column of ED_G be denoted Col_i respectively.

After performing the row operations $Row_i = Row_i - q \cdot Row_5$ for $i = 6, 7$ and then performing the column operations $Col_i = Col_i - q \cdot Col_5$ for $i = 6, 7$, we get an equivalent matrix $M_1 = \begin{pmatrix} ED_{L_1} & 0 \\ 0 & K_1 \end{pmatrix}$, where we

have $K_1 = \begin{pmatrix} 1 - q^2 & q - q^2 \\ q - q^2 & 1 - q^2 \end{pmatrix}$. Note that K_1 is a 2×2 matrix as H_1 has 3 vertices. Next, if we perform the

row operation $Row_5 = Row_5 - q \cdot Row_4$ and the column operation $Col_5 = Col_5 - q \cdot Col_4$ on H_2 (the subgraph induced on $\{4, 5\}$) with $cv_2 = 4$, we get an equivalent matrix $M_2 = \begin{pmatrix} ED_{L_2} & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_1 \end{pmatrix}$, where $K_2 = (1 - q^2)$

(i.e. K_2 is a 1×1 matrix). We are now left with H_3 , the induced subgraph on $\{1, 2, 3, 4\}$. Choosing $cv_3 = 1$ and performing the row operations $Row_2 = Row_2 - q \cdot Row_1$; $Row_3 = Row_3 - q^2 \cdot Row_1$ and $Row_4 = Row_4 - q \cdot Row_1$ and then the column operations $Col_2 = Col_2 - q \cdot Col_1$; $Col_3 = Col_3 - q^2 \cdot Col_1$ and $Col_4 = Col_4 - q \cdot Col_1$, we get the following equivalent matrix

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & K_3 & 0 & 0 \\ 0 & 0 & K_2 & 0 \\ 0 & 0 & 0 & K_1 \end{pmatrix}, \text{ where } K_3 = \begin{pmatrix} 1 - q^2 & q - q^3 & q - q^2 \\ q - q^3 & 1 - q^4 & q - q^3 \\ q - q^2 & q - q^3 & 1 - q^2 \end{pmatrix}.$$

Thus, we get $ED_G \sim M_3$. We know the gcd of $k \times k$ minors in each K_i by Corollary 7 and using Corollary 6, we can get the gcd of $s \times s$ minors of ED_G .

3 Special Cases

In this section, we consider some instances of product distance matrices and their inverses and draw corollaries on their SNFs. We first consider exponential distance matrices ED_G of connected graphs G defined in Section 1. Clearly, they are examples of a product distance and so Theorem 3 is applicable to them.

3.1 Exponential distance matrix of Trees

Firstly consider the case when G is a tree T . In this case, each block of G is the complete graph on two vertices. Hence, in ED_H , for each block H of G , the gcd of 1×1 minors is 1 and the gcd of 2×2 minors is $(q^2 - 1)$. In this case, as $n_i = 2$, we will get an *equivalent diagonal* matrix.

Thus if T is a tree on n vertices, by Theorem 3, we infer that the gcd of $k \times k$ minors of ED_G is $g_{k-1} = (q^2 - 1)^{k-1}$. Hence, we get the invariant factors s_k of ED_T to be $s_1 = 1, s_k = (q^2 - 1)$ for $1 < k \leq n$. This is a reproof of the following fact: the invariant factors of ED_T only depend on n , the number of vertices of T and is independent of the structure of the tree T . That is, the SNF of ED_T is the equivalent diagonal matrix $M = 1 \oplus \underbrace{(q^2 - 1) \oplus \dots \oplus (q^2 - 1)}_{(n-1) \text{ times}}$.

By using Jacobi’s Theorem we can get the gcds of $k \times k$ minors of inverses of the exponential distance matrices of trees. We cover this in the next few lines. Bapat, Lal and Pati [2, Proposition 3.3] showed that the inverse of the exponential distance matrix ED_T of a tree T is upto scalar, the q -analogue of T ’s laplacian \mathcal{L}_q which is defined as $\mathcal{L}_q = I - qA + (D - I)q^2$. Here, q is a variable, A is the adjacency matrix of T and D a diagonal matrix with the (i, i) -th entry being the degree of vertex i .

Theorem 12 (Bapat, Lal and Pati). *Let T be a tree with exponential distance matrix ED_T and let \mathcal{L}_q be the q -analogue of T ’s laplacian matrix. Then $ED_T^{-1} = \frac{1}{1 - q^2} \mathcal{L}_q$.*

By Jacobi’s Theorem [7, Section 4.2], we know that for invertible matrices M , its minors are related to complementary minors of M^{-1} . From this, we see that the gcd b_k of $k \times k$ minors of \mathcal{L}_q is 1 for $1 \leq k < n$ and that $b_n = q^2 - 1$.

3.2 Exponential distance matrices when all blocks are K_t ’s

We consider exponential distance matrices of graphs G , in the case when all blocks of G are the complete graph K_t on t vertices. For concreteness, we assume $t = 3$, though our results are applicable for larger t as well (see the paragraph before Corollary 13, below).

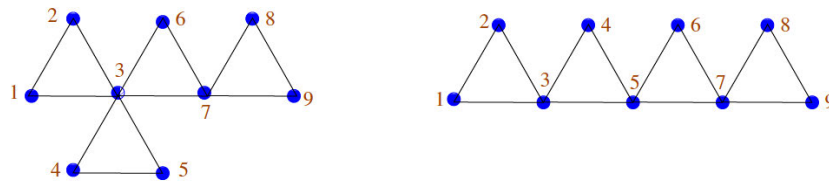


Figure 2: Two examples of graphs with only K_3 as blocks.

Let G be a graph with blocks being K_3 ’s. Two examples are given in Figure 2. It can be checked that the gcd of $k \times k$ minors of $ED(K_3)$ is as given in the following table.

k	gcd of $k \times k$ minors
1	1
2	$q - 1$
3	$(2q + 1)(q - 1)^2$

In general when we consider ED_{K_t} , it can be shown that the gcd of $k \times k$ blocks is $(q - 1)^{k-1}$ for $1 \leq k < t$ and that the gcd of $t \times t$ blocks is $(1 + (t - 1)q)(q - 1)^{t-1}$. We can derive the following corollary of Theorem 3.

Corollary 13. *Let G be a graph with r blocks, each of which is a K_3 and let ED_G be its exponential distance matrix. Let the gcd of $s \times s$ minors of ED_G be denoted g_{s-1} . Then, for $1 \leq s \leq r + 1$, we have $g_s = (q - 1)^{s-1}$ and for $r + 2 \leq s \leq 2r + 1$, we have $g_s = (2q + 1)^{s-r-1}(q - 1)^{s-1}$.*

Proof. Since each block is a K_3 , we have in the notation of Theorem 3 that $g_{i,0} = 1$, $g_{i,1} = q - 1$ and $g_{i,2} = (2q + 1)(q - 1)^2$ for each $1 \leq i \leq r$. It is easy to see that the number of vertices in G is $n = 2r + 1$. By Theorem 3, when $2 \leq s \leq n$, the gcd of $s \times s$ minors of ED_G is g_{s-1} . To calculate g_{s-1} , when $2 \leq s \leq r$, note that $\gcd(\prod_{i=1}^r g_{i,s_i})$ over choices $(s_1, s_2, \dots, s_r) \in T_{s-1}$ is attained when each $s_i = 0$ or 1 (ie the choice $s_i = 2$ for some i will result in a larger product and hence not be chosen). Thus in this case, the gcd of $s \times s$ minors of ED_G is $(q - 1)^{s-1}$. When $r + 1 \leq s \leq n$, we will have all $s_i \geq 1$ and some $s - r - 1$ indices j with $s_j = 2$. In this case, the gcd will be $(2q + 1)^{s-r-1}(q - 1)^{s-1}$, completing the proof. \square

The above corollary states that if G has r blocks, then the topmost r largest-sized minors have one form as their gcd and the remaining smaller sized minors all have another form for their gcd. This is true when all blocks of G are K_p 's as well (i.e. Corollary 13 can be generalised to this case). A similar proof can be given, though we omit it. We note that all blocks are required to be K_t for the above corollary. If G has two blocks, a K_3 and a K_4 , then it can be checked that only the topmost gcd is different (as opposed to the two topmost gcds as asserted by Corollary 13.)

For graphs that are not necessarily trees by Theorem 1, ED_G is invertible provided each of its blocks are. In this case, Bapat and Sivasubramanian [4, Theorem 6] give an explicit inverse of ED_G . It would be interesting to get a list of 2-connected graphs for which the SNF of its exponential distance matrix is known.

Similarly, the SNF of exponential distance matrices of block graphs (defined as graphs all of whose blocks are cliques) can be determined. The formulae for these are not as attractive as in the case when G has all blocks being K_t with the same t and so we leave things here.

3.3 Exponential version of scaled resistance matrix

As mentioned in Section 1, exponential versions of resistance distances are also product distances. Let G have two blocks B and F where $B = C_4$ is the four-cycle and $F = C_5$ is the five-cycle (see Figure 3). Bapat in [1, Equation 4] has shown that the resistance distance between vertices i, j in the n -cycle is given by $r_{i,j} = (j - i)(n - j + i)/n$.

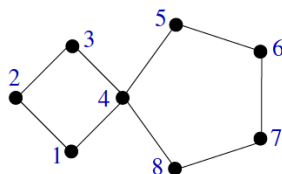


Figure 3: Graphs with distances given by scaled resistance distance.

Among vertices $i, j \in V(B)$, let $r_{i,j}$ be its resistance distance. Let κ_B denote the number of spanning trees of B . For vertices $i, j \in V(B)$, define its distance by $d_{i,j} = q^{\kappa_B r_{i,j}}$, where $r_{i,j}$ is the resistance distance between i and j in B , and q is an indeterminate. Similarly, let κ_F denote the number of spanning trees of F and define the distance between two vertices u, v of F by $d_{u,v} = q^{\kappa_F r_{u,v}}$ where $r_{u,v}$ is the resistance distance in F . For concreteness, we give the distance matrices of B and F .

$$D_B = \begin{pmatrix} 1 & q^3 & q^4 & q^3 \\ q^3 & 1 & q^3 & q^4 \\ q^4 & q^3 & 1 & q^3 \\ q^3 & q^4 & q^3 & 1 \end{pmatrix} \quad D_F = \begin{pmatrix} 1 & q^4 & q^6 & q^6 & q^4 \\ q^4 & 1 & q^4 & q^6 & q^6 \\ q^6 & q^4 & 1 & q^4 & q^6 \\ q^6 & q^6 & q^4 & 1 & q^4 \\ q^4 & q^6 & q^6 & q^4 & 1 \end{pmatrix}$$

Let B and F have a common cut vertex c . For vertices $i \in V(B), u \in V(F)$, define $d_{i,u} = d_{i,c} \times d_{c,u}$. Let D_G be the matrix of $d_{i,j}$ for $i, j \in G$. Let b_k, f_k and g_k be the gcd of $k \times k$ minors of D_B, D_F and D_G respectively. A table of b_k, f_k and g_k for $1 \leq k \leq 8$ is as follows. Here, $p = q^3 - q^2 - q - 1, r = q^3 + q^2 - q + 1, u = q^8 - q^6 - 2q^4 - 2q^2 - 1$ and $w = 2q^6 + 2q^4 + 1$. It is easy to check that Theorem 3 agrees with computed gcd g_k .

k	b_k	f_k	g_k
1	1	1	1
2	$q^2 - 1$	$q^2 - 1$	$q^2 - 1$
3	$(q^2 - 1)^2(1 + q^2)$	$(q^2 - 1)^2$	$(q^2 - 1)^2$
4	$(q^2 - 1)^3(1 + q^2)^2 pr$	$(q^2 - 1)^3 u$	$(q^2 - 1)^3$
5	0	$(q^2 - 1)^4 w u^2$	$(q^2 - 1)^4$
6	0	0	$(q^2 - 1)^5$
7	0	0	$(q^2 - 1)^6(1 + q^2)u$
8	0	0	$(q^2 - 1)^7(1 + q^2)^2 pr w u^2$

Acknowledgement: We sincerely thank the referees for several invaluable suggestions that have resulted in a vastly improved exposition of this work. Some Theorems in this work were as conjectures, tested using the computer package ‘‘Sage’’ [8]. We thank the authors of Sage for generously releasing their software as an open-source package.

The first author gratefully acknowledges the support of the JC Bose Fellowship, Department of Science and Technology, Government of India. The second author thanks the Stat-Math unit of ISI-Delhi for their hospitality during several trips. He also acknowledges support from project grant Spons/MA/KS-1/2015 given by IIT Bombay.

References

- [1] Bapat R. B. Resistance matrix and q-laplacian of a unicyclic graph. In Ramanujan Mathematical Society Lecture Notes Series, 7, Proceedings of ICDM 2006, Ed. R. Balakrishnan and C.E. Veni Madhavan (2008), pp. 63–72.
- [2] Bapat R. B., Lal A. K., Pati S. A q-analogue of the distance matrix of a tree. Linear Algebra and its Applications 416 (2006), 799–814.
- [3] Bapat R. B., Raghavan T. E. S. Nonnegative Matrices and Applications. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1997.
- [4] Bapat R. B., Sivasubramanian S. Product Distance Matrix of a Graph and Squared Distance Matrix of a Tree. Applicable Analysis and Discrete Mathematics 7 (2013), 285–301.
- [5] Chebotarev P. A class of graph-geodetic distances generalizing the shortest-path and the resistance distances. Discrete Applied Math 159, Issue 5, (2011), 295–302. [2]
- [6] Chebotarev P. The graph bottleneck identity. Advances in Applied Mathematics 47, Issue 3, (2011), 403–413.

- [7] Dealba L. M. Determinants and Eigenvalues. In Handbook of Linear Algebra, L. Hogben, Ed. Chapman & Hall CRC Press, 2007, ch. 4.
- [8] Developers T. S. Sage Mathematics Software (Version 3.1.1), 2008. <http://www.sagemath.org>.
- [9] Klein D. J., Randić M. Resistance distance. Journal of Mathematical Chemistry 12, Issue 1, (1993), 81–95.
- [10] Newman M. Integral Matrices. Academic Press, 1972.
- [11] Shiu W. C. Invariant factors of graphs associated with hyperplane arrangements. Discrete Mathematics 288 (2004), 135–148.