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Nonlinear maps preserving Lie products on triangular algebras

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Abstract: In this paper we prove that every bijection preserving Lie products from a triangular algebra onto a normal triangular algebra is additive modulo centre. As an application, we described the form of bijections preserving Lie products on nest algebras and block upper triangular matrix algebras.

Keywords: Preserver; Lie product; Triangular algebra; Nest algebra

MSC: 16W25; 15A78; 47L35

1 Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Recall that a left $\mathcal{A}$-module $\mathcal{M}$ is faithful if $a \in \mathcal{A}$ and $a\mathcal{M} = 0$ imply that $a = 0$. The $\mathcal{R}$-algebra

$$\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra.

Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and let $Z(\mathcal{U})$ be its centre. We define two natural projections $\pi_{\mathcal{A}} : \mathcal{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{U} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}} : \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mapsto a \text{ and } \pi_{\mathcal{B}} : \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mapsto b.$$

If $\pi_{\mathcal{A}}(Z(\mathcal{U})) = Z(\mathcal{A})$, $\pi_{\mathcal{B}}(Z(\mathcal{U})) = Z(\mathcal{B})$ and $a\mathcal{M}b = 0$ imply that $a = 0$ or $b = 0$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is called a normal triangular algebra. The most important examples of normal triangular algebras are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras. Cheung [6,7] described commuting maps and Lie derivations of these algebras. Benkovič and Eremita [2] studied commuting traces of bilinear maps and Lie isomorphisms of triangular algebras. Benkovič [3] investigated biderivations of triangular algebras. Wong [19] treated Jordan isomorphisms of triangular algebras, while Zhang and Yu [20,21] studied Jordan derivations and nonlinear Lie derivations.

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over a commutative ring $\mathcal{R}$. A map $\varphi : \mathcal{A} \to \mathcal{B}$ is called additive modulo centre if $\varphi(x + y) - \varphi(x) - \varphi(y) \in Z(\mathcal{B})$ for all $x, y \in \mathcal{A}$. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a map (without the additivity assumption). We say that ϕ preserves Lie products if $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathcal{A}$, where $[x, y] = xy - yx$ is the Lie product of x and y. This kind of maps is closely related to Lie homomorphisms (linear maps preserving Lie products) and commutativity preserving maps (maps preserving zero Lie products), which have been studied

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by many authors. For example, see [1,4,5,12-17,22] and their references. Recently, Šemrl [18] characterized bijective maps preserving Lie products on $\mathcal{B}(X)$, the algebra of all bounded linear operators on a Banach space X. Zhang [23] considered bijective maps preserving Lie products between two factor von Neumann algebras. Zhang and Yu [24] studied nonlinear $\star$-Lie derivations on factor von Neumann algebras. In this paper we will investigate bijective maps preserving Lie products between two triangular algebras.

2 Maps preserving Lie products on triangular algebras

Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and let $Z(\mathcal{U})$ be its centre. It follows from [6, Proposition 3] that

$$Z(\mathcal{U}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb \text{ for all } m \in \mathcal{M} \right\},$$

and that there exists a unique algebra isomorphism $\tau : \pi_{\mathcal{A}}(Z(\mathcal{U})) \to \pi_{\mathcal{B}}(Z(\mathcal{U}))$ such that $am = m\tau(a)$ for all $m \in \mathcal{M}$.

Let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be identities of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and let 1 be the identity of the triangular algebra $\mathcal{U}$. Throughout this section we shall use following notation

$$e_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = 1 - e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$$

and

$$\mathcal{U}_{ij} = e_i \mathcal{U} e_j \text{ for } 1 \le i \le j \le 2.$$

It is clear that the triangular algebra $\mathcal{U}$ may be represented as

$$\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{22}.$$

Here $\mathcal{U}_{11}$ and $\mathcal{U}_{22}$ are subalgebras of $\mathcal{U}$ isomorphic to $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\mathcal{U}_{12} \subseteq \mathcal{U}$ is a $(\mathcal{U}_{11}, \mathcal{U}_{22})$-bimodule isomorphic to the bimodule $\mathcal{M}$.

In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{V} = \text{Tri}(\mathcal{C}, \mathcal{N}, \mathcal{D})$ be a normal triangular algebra. Suppose that $\phi : \mathcal{U} \to \mathcal{V}$ is a bijection satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathcal{U}$. Then ϕ is additive modulo centre.*

To prove theorem 2.1, we need some lemmas.

Lemma 2.1. *Let $\mathcal{V} = \text{Tri}(\mathcal{C}, \mathcal{N}, \mathcal{D})$ be a normal triangular algebra and let $v_0 \in \mathcal{V}$. Then $[v_0, [v_0, v]] = [v_0, v]$ for all $v \in \mathcal{V}$ if and only if there exist an element $z_0 \in Z(\mathcal{V})$ and an invertible element $u \in \mathcal{V}$ such that one of the following holds:*

(a) *There is an idempotent $e_0 \in \mathcal{C}$ with $(1_{\mathcal{C}} - e_0)\mathcal{C}e_0 = 0$ such that*

$$uv_0u^{-1} = \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix} + z_0;$$

(b) *There is an idempotent $e_0 \in \mathcal{D}$ with $(1_{\mathcal{D}} - e_0)\mathcal{D}e_0 = 0$ such that*

$$uv_0u^{-1} = \begin{pmatrix} 1_{\mathcal{C}} & 0 \\ 0 & e_0 \end{pmatrix} + z_0.$$

Proof. If $v_0 \in \mathcal{V}$ satisfies (a) or (b), it is easy to check that

$$[v_0, [v_0, v]] = [v_0, v] \tag{1}$$

for all $v \in \mathcal{V}$. Conversely, let $v_0 = \begin{pmatrix} c_0 & n_0 \\ 0 & d_0 \end{pmatrix}$ and $v = \begin{pmatrix} c & n \\ 0 & d \end{pmatrix}$. It follows from Eq. (1) that

$$[c_0, [c_0, c]] = [c_0, c], \quad [d_0, [d_0, d]] = [d_0, d] \tag{2}$$

for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, and that

$$H(c, n, d) = c_0 H(c, n, d) - H(c, n, d)d_0 + n_0[d_0, d] - [c_0, c]n_0 \tag{3}$$

for all $(c, n, d) \in \mathcal{C} \times \mathcal{N} \times \mathcal{D}$, where $H(c, n, d) = c_0 n - n d_0 + n_0 d - c n_0$. Taking $c = 0$ and $d = 0$ in Eq. (3), we have

$$c_0 n - n d_0 = c_0^2 n - 2c_0 n d_0 + n d_0^2 \tag{4}$$

for all $n \in \mathcal{N}$. Taking $c = 0$, $n = 0$ and $d = 1_{\mathcal{D}}$ in Eq. (3), we have

$$n_0 = c_0 n_0 - n_0 d_0. \tag{5}$$

Replacing n by cn in Eq. (4), we have

$$c_0 c n - c n d_0 = c_0^2 c n - 2c_0 c n d_0 + c n d_0^2. \tag{6}$$

On the other hand, we have from Eq. (4) that

$$c c_0 n - c n d_0 = c c_0^2 n - 2c c_0 n d_0 + c n d_0^2. \tag{7}$$

It follows from Eqs. (6) and (7) that

$$2[c_0, c]n d_0 = [c_0^2 - c_0, c]n. \tag{8}$$

Replacing n by nd in Eq. (8), we have

$$2[c_0, c]n d d_0 = [c_0^2 - c_0, c]nd. \tag{9}$$

On the other hand, we have from Eq. (8) that

$$2[c_0, c]n d_0 d = [c_0^2 - c_0, c]nd. \tag{10}$$

It follows from Eqs. (9) and (10) that

$$[c_0, c]n[d_0, d] = 0$$

for all $(c, n, d) \in \mathcal{C} \times \mathcal{N} \times \mathcal{D}$. This implies that $c_0 \in Z(\mathcal{C})$ or $d_0 \in Z(\mathcal{D})$.

If $d_0 \in Z(\mathcal{D}) = \pi_{\mathcal{D}}(Z(\mathcal{V}))$, then $z_0 = \begin{pmatrix} \tau^{-1}(d_0) & 0 \\ 0 & d_0 \end{pmatrix} \in Z(\mathcal{V})$. Write $e_0 = c_0 - \tau^{-1}(d_0)$. It follows that $v_0 - z_0 = \begin{pmatrix} e_0 & n_0 \\ 0 & 0 \end{pmatrix}$ satisfies Eq. (1), and so by Eq. (4), we get that $(e_0^2 - e_0)n = 0$ for all $n \in \mathcal{N}$. This and Eq. (2) imply that e_0 is an idempotent of $\mathcal{C}$ and $(1_{\mathcal{C}} - e_0)\mathcal{C}e_0 = 0$. Let $u = \begin{pmatrix} 1_{\mathcal{C}} & n_0 \\ 0 & 1_{\mathcal{D}} \end{pmatrix}$. It is clear that u is invertible in $\mathcal{V}$ and $u^{-1} = \begin{pmatrix} 1_{\mathcal{C}} & -n_0 \\ 0 & 1_{\mathcal{D}} \end{pmatrix}$. Then by Eq. (5),

$$\begin{aligned} u v_0 u^{-1} &= \begin{pmatrix} 1_{\mathcal{C}} & n_0 \\ 0 & 1_{\mathcal{D}} \end{pmatrix} \begin{pmatrix} e_0 & n_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{C}} & -n_0 \\ 0 & 1_{\mathcal{D}} \end{pmatrix} + z_0 \\ &= \begin{pmatrix} e_0 & \tau^{-1}(d_0)n_0 - n_0 d_0 \\ 0 & 0 \end{pmatrix} + z_0 = \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix} + z_0. \end{aligned}$$

If $c_0 \in Z(\mathcal{C}) = \pi_{\mathcal{C}}(Z(\mathcal{V}))$, then $z_0 = \begin{pmatrix} c_0 - 1_{\mathcal{C}} & 0 \\ 0 & \tau(c_0) - 1_{\mathcal{D}} \end{pmatrix} \in Z(\mathcal{V})$. Write $e_0 = 1_{\mathcal{D}} + d_0 - \tau(c_0)$. It follows that $v_0 - z_0 = \begin{pmatrix} 1_{\mathcal{C}} & n_0 \\ 0 & e_0 \end{pmatrix}$ satisfies Eq. (1), and so by Eq. (4), we obtain that $n(e_0^2 - e_0) = 0$ for all $n \in \mathcal{N}$. This and Eq. (2) imply that e_0 is an idempotent of $\mathcal{D}$ and $(1_{\mathcal{D}} - e_0)\mathcal{D}e_0 = 0$. By Eq. (5), we have

$$\begin{aligned} uv_0u^{-1} &= \begin{pmatrix} 1_{\mathcal{C}} & n_0 \\ 0 & 1_{\mathcal{D}} \end{pmatrix}\begin{pmatrix} 1_{\mathcal{C}} & n_0 \\ 0 & e_0 \end{pmatrix}\begin{pmatrix} 1_{\mathcal{C}} & -n_0 \\ 0 & 1_{\mathcal{D}} \end{pmatrix} + z_0 \\ &= \begin{pmatrix} 1_{\mathcal{C}} & c_0n_0 - n_0\tau(c_0) \\ 0 & e_0 \end{pmatrix} + z_0 = \begin{pmatrix} 1_{\mathcal{C}} & 0 \\ 0 & e_0 \end{pmatrix} + z_0. \end{aligned}$$

The proof is completed. □

Lemma 2.2. Let $\mathcal{U}$ and $\mathcal{V}$ be two algebras and $\varphi : \mathcal{U} \to \mathcal{V}$ be a bijection preserving Lie products. Then $\varphi(0) = 0$, $\varphi(Z(\mathcal{U})) = Z(\mathcal{V})$ and $\varphi(u) + \varphi(z - u) \in Z(\mathcal{V})$ for all $u \in \mathcal{U}$ and $z \in Z(\mathcal{U})$.

Proof. It is easy to verify that $\varphi(0) = 0$ and $\varphi(Z(\mathcal{U})) = Z(\mathcal{V})$. Let $u \in \mathcal{U}$ and $z \in Z(\mathcal{U})$. It follows from the fact $[u, y] = [y, z - u]$ that

$$[\varphi(u), \varphi(y)] = [\varphi(y), \varphi(z - u)] = -[\varphi(z - u), \varphi(y)]$$

for all $y \in \mathcal{U}$. The surjectivity of φ implies that $\varphi(u) + \varphi(z - u) \in Z(\mathcal{V})$. The proof is completed. □

Next we assume that $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular algebra and $\mathcal{V} = \text{Tri}(\mathcal{C}, \mathcal{N}, \mathcal{D})$ is a normal triangular algebra, and that $\phi : \mathcal{U} \to \mathcal{V}$ is a bijection preserving Lie products.

Remark 2.1. It follows from the fact $[e_1, [e_1, u]] = [e_1, u]$ for all $u \in \mathcal{U}$ that $[\phi(e_1), [\phi(e_1), \phi(u)]] = [\phi(e_1), \phi(u)]$, and so by Lemmas 2.1, there exist a nontrivial idempotent $f_1 = \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix}$ or $f_1 = \begin{pmatrix} 1_{\mathcal{C}} & 0 \\ 0 & e_0 \end{pmatrix}$ in $\mathcal{V}$, an invertible element v in $\mathcal{V}$ and an element $z_1 \in Z(\mathcal{V})$ such that $v\phi(e_1)v^{-1} = z_1 + f_1$. Hence we have from Lemma 2.2 that $v\phi(e_2)v^{-1} = z_2 + f_2$ for some $z_2 \in Z(\mathcal{V})$ and $f_2 + f_1 = 1$ the identity of $\mathcal{V}$. Therefore, without loss of generality, we can assume that $\phi(e_i) = z_i + f_i$, $i = 1, 2$. From Lemma 2.1, it is easy to check that $f_2\mathcal{V}f_1 = 0$. Write $\mathcal{V}_{ij} = f_i\mathcal{V}f_j$ for $1 \le i \le j \le 2$. Then $\mathcal{V} = \mathcal{V}_{11} + \mathcal{V}_{12} + \mathcal{V}_{22}$ and we have the following lemma.

Lemma 2.3. $\phi(\mathcal{U}_{12}) = \mathcal{V}_{12}$.

Proof. It follows from $x = [[e_1, x], e_2]$ for all $x \in \mathcal{U}_{12}$ that

$$\phi(x) = [[f_1, \phi(x)], f_2] = f_1\phi(x)f_2 \in \mathcal{V}_{12}.$$

Hence $\phi(\mathcal{U}_{12}) \subseteq \mathcal{V}_{12}$. Applying the same argument to ϕ^{-1}, we can obtain the reverse inclusion and equality follows. The proof is completed. □

Lemma 2.4. *$\mathcal{V}_{12}$ is a faithful left $\mathcal{V}_{11}$-module and also a faithful right $\mathcal{V}_{22}$-module.*

Proof. Let $a \in \mathcal{V}_{11}$ with $an = 0$ for all $n \in \mathcal{V}_{12}$. Then

$$[\phi^{-1}(a), \phi^{-1}(n)] = \phi^{-1}([a, n]) = \phi^{-1}(0) = 0.$$

It follows from Lemma 2.3 that $[\phi^{-1}(a), m] = 0$ for all $m \in \mathcal{U}_{12}$, and so by [21, Lemma 2.1] we have $\phi^{-1}(a) \in \mathcal{U}_{12} + Z(\mathcal{U})$. By Lemmas 2.2 and 2.3, then

$$a = \phi(\phi^{-1}(a)) \in \phi(\mathcal{U}_{12} + Z(\mathcal{U})) \subseteq \mathcal{V}_{12} + Z(\mathcal{V}).$$

This yields that $a \in Z(\mathcal{V})$. From Remark 2.1, we see that

$$f_1 = \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } f_1 = \begin{pmatrix} 1_{\mathcal{C}} & 0 \\ 0 & e_0 \end{pmatrix}.$$

If $f_1 = \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix}$, then $a = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ for some $c \in \mathcal{C}$. Since $a \in Z(\mathcal{V})$, we have $c = \tau^{-1}(0) = 0$. Hence $a = 0$. If $f_1 = \begin{pmatrix} 1_{\mathcal{C}} & 0 \\ 0 & e_0 \end{pmatrix}$, then

$$\mathcal{V}_{12} = \begin{pmatrix} 0 & \mathcal{N}(1_{\mathcal{D}} - e_0) \\ 0 & e_0\mathcal{D}(1_{\mathcal{D}} - e_0) \end{pmatrix}.$$

Since $a \in Z(\mathcal{V})$, we have $a = \begin{pmatrix} c & 0 \\ 0 & \tau(c) \end{pmatrix}$ for some $c \in \mathcal{C}$. It follows from $a\mathcal{V}_{12} = 0$ that $c\mathcal{N}(1_{\mathcal{D}} - e_0) = 0$. This implies that $c = 0$. Hence $a = 0$. Thus, $\mathcal{V}_{12}$ is a faithful left $\mathcal{V}_{11}$-module. Similarly, we can show that $\mathcal{V}_{12}$ is a faithful right $\mathcal{V}_{22}$-module. The proof is completed. □

Lemma 2.5. *Let $v \in \mathcal{V}$. Then $v \in \mathcal{V}_{12} + Z(\mathcal{V})$ if and only if $[v, n] = 0$ for all $n \in \mathcal{V}_{12}$.*

Proof. If $v \in \mathcal{V}_{12} + Z(\mathcal{V})$, it is clear that $[v, n] = 0$ for all $n \in \mathcal{V}_{12}$. Conversely, if $[v, n] = 0$ for all $n \in \mathcal{V}_{12}$, then

$$f_1vf_1n = nf_2vf_2 \tag{11}$$

for all $n \in \mathcal{V}_{12}$. It follows from Lemma 2.4 that

$$f_1vf_1 \in Z(\mathcal{V}_{11}) \text{ and } f_2vf_2 \in Z(\mathcal{V}_{22}). \tag{12}$$

Let $u \in \mathcal{V}$ be any element, then $u = a + n + b$ where $a \in \mathcal{V}_{11}$, $n \in \mathcal{V}_{12}$ and $b \in \mathcal{V}_{22}$. It follows from Eqs. (11) and (12) that

$$(f_1vf_1 + f_2vf_2)u = u(f_1vf_1 + f_2vf_2).$$

This implies that $f_1vf_1 + f_2vf_2 \in Z(\mathcal{V})$. Hence

$$v = f_1vf_2 + (f_1vf_1 + f_2vf_2) \in \mathcal{V}_{12} + Z(\mathcal{V}).$$

The proof is completed. □

Lemma 2.6. (a) *$\phi(a + m) - \phi(a) - \phi(m) \in Z(\mathcal{V})$ for all $a \in \mathcal{U}_{11}$ and $m \in \mathcal{U}_{12}$;*
(b) *$\phi(m + b) - \phi(m) - \phi(b) \in Z(\mathcal{V})$ for all $m \in \mathcal{U}_{12}$ and $b \in \mathcal{U}_{22}$;*
(c) *$\phi(m + n) = \phi(m) + \phi(n)$ for all $m, n \in \mathcal{U}_{12}$;*
(d) *$\phi(a + c) - \phi(a) - \phi(c) \in Z(\mathcal{V})$ for all $a, c \in \mathcal{U}_{11}$;*
(e) *$\phi(b + d) - \phi(b) - \phi(d) \in Z(\mathcal{V})$ for all $b, d \in \mathcal{U}_{22}$;*
(f) *$\phi(a + b) - \phi(a) - \phi(b) \in Z(\mathcal{V})$ for all $a \in \mathcal{U}_{11}$ and $b \in \mathcal{U}_{22}$.*

Proof. (a) Let $a \in \mathcal{U}_{11}$ and $m, n \in \mathcal{U}_{12}$. It follows from the fact $[a + m, n] = [a, n]$ that

$$[\phi(a + m) - \phi(a), \phi(n)] = 0$$

for all $n \in \mathcal{U}_{12}$. By Lemmas 2.3 and 2.5, then $\phi(a + m) - \phi(a) \in \mathcal{V}_{12} + Z(\mathcal{V})$. This implies that

$$\phi(a + m) - \phi(a) - f_1(\phi(a + m) - \phi(a))f_2 \in Z(\mathcal{V}). \tag{13}$$

Since $[[e_1, a + m], e_2] = m$ and $[[e_1, a], e_2] = 0$, we have $f_1\phi(a + m)f_2 = \phi(m)$ and $f_1\phi(a)f_2 = 0$. Hence by Eq. (13),

$$\phi(a + m) - \phi(a) - \phi(m) \in Z(\mathcal{V}).$$

Similarly, we can show that (b) holds.

(c) Let $m, n \in \mathcal{U}_{12}$. Then $m + n = [e_1 + m, e_2 + n]$, and so by the conclusions of (a) and (b) we have

$$\phi(m + n) = [\phi(e_1 + m), \phi(e_2 + n)] = [f_1 + \phi(m), f_2 + \phi(n)].$$

This and Lemma 2.3 imply that $\phi(m + n) = \phi(m) + \phi(n)$ for all $m, n \in \mathcal{U}_{12}$.

(d) Let $a, c \in \mathcal{U}_{11}$ and $m \in \mathcal{U}_{12}$. Then $[a+c, m] = am + cm$, and so by the conclusion of (c) we have

$$\begin{aligned}[\phi(a+c), \phi(m)] &= \phi(am) + \phi(cm) = \phi([a,m]) + \phi([c,m]) \\ &= [\phi(a) + \phi(c), \phi(m)].\end{aligned}$$

Hence $[\phi(a+c) - \phi(a) - \phi(c), \phi(m)] = 0$ for all $m \in \mathcal{U}_{12}$. It follows from Lemmas 2.3 and 2.5 that

$$\phi(a+c) - \phi(a) - \phi(c) \in \mathcal{V}_{12} + Z(\mathcal{V}). \tag{14}$$

From the fact $[[e_1, \mathcal{U}_{11}], e_2] = 0$, we see that $f_1\phi(\mathcal{U}_{11})f_2 = 0$. Thus,

$$f_1(\phi(a+c) - \phi(a) - \phi(c))f_2 = 0.$$

This together with Eq. (14) gives us that

$$\phi(a+c) - \phi(a) - \phi(c) \in Z(\mathcal{V}).$$

Similarly, we can show that (e) holds.

(f) Let $a \in \mathcal{U}_{11}$ and $b \in \mathcal{U}_{22}$. Then $[a+b, m] = am - mb$ for all $m \in \mathcal{U}_{12}$, and so by (c) we have

$$\begin{aligned}[\phi(a+b), \phi(m)] &= \phi(am) + \phi(-mb) = \phi([a,m]) + \phi([b,m]) \\ &= [\phi(a) + \phi(b), \phi(m)].\end{aligned}$$

That is, $[\phi(a+b) - \phi(a) - \phi(b), \phi(m)] = 0$ for all $m \in \mathcal{U}_{12}$. It follows from Lemmas 2.3 and 2.5 that

$$\phi(a+b) - \phi(a) - \phi(b) \in Z(\mathcal{V}).$$

The proof is completed. □

Lemma 2.7. *$\phi(a+m+b) - \phi(a) - \phi(m) - \phi(b) \in Z(\mathcal{V})$ for all $a \in \mathcal{U}_{11}$, $m \in \mathcal{U}_{12}$ and $b \in \mathcal{U}_{22}$.*

Proof. Let $a \in \mathcal{U}_{11}$, $m \in \mathcal{U}_{12}$ and $b \in \mathcal{U}_{22}$. Then $[a+m+b, n] = [a+b, n]$ for all $n \in \mathcal{U}_{12}$, and so by Lemma 2.6(f) we have

$$[\phi(a+m+b), \phi(n)] = [\phi(a+b), \phi(n)] = [\phi(a) + \phi(b), \phi(n)].$$

It follows from Lemmas 2.3, 2.5 and the fact $f_1(\phi(a) + \phi(b))f_2 = 0$ that

$$\phi(a+m+b) - \phi(a) - \phi(b) - f_1\phi(a+m+b)f_2 \in Z(\mathcal{V}). \tag{15}$$

Since $m = [[e_1, a+m+b], e_2]$, we have $\phi(m) = f_1\phi(a+m+b)f_2$, and so by Eq. (15)

$$\phi(a+m+b) - \phi(a) - \phi(m) - \phi(b) \in Z(\mathcal{V}).$$

The proof is completed. □

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $x, y \in \mathcal{U}$. Then $x = a + m + b$ and $y = c + n + d$ for some $a, c \in \mathcal{U}_{11}$, $m, n \in \mathcal{U}_{12}$ and $b, d \in \mathcal{U}_{22}$. By Lemma 2.7, we have

$$\phi(x+y) - (\phi(a+c) + \phi(m+n) + \phi(b+d)) \in Z(\mathcal{V})$$

and

$$\phi(x) + \phi(y) - (\phi(a) + \phi(m) + \phi(b)) - (\phi(c) + \phi(n) + \phi(d)) \in Z(\mathcal{V}).$$

This together with Lemma 2.6 gives us that $\phi(x+y) - \phi(x) - \phi(y) \in Z(\mathcal{V})$ for all $x, y \in \mathcal{U}$. The proof is completed. □

3 Maps preserving Lie products on nest algebras

Let $\mathcal{H}$ be a complex separable Hilbert space. Recall that a nest $\mathcal{M}$ on $\mathcal{H}$ is a totally ordered family of orthogonal projections of $\mathcal{B}(\mathcal{H})$ which is closed in the strong operator topology, and which includes 0 and I. Clearly, $\mathcal{M}^{\perp} = \{I - P : P \in \mathcal{M}\}$ is also a nest on $\mathcal{H}$. A nest is said to be nontrivial if it contains at least one nontrivial projection. The nest algebra associated to a nest $\mathcal{M}$, denoted by $\tau(\mathcal{M})$, is the set

$$\tau(\mathcal{M}) = \{T \in \mathcal{B}(\mathcal{H}) : PTP = TP \text{ for all } P \in \mathcal{M}\}.$$

Let $Q \in \mathcal{M} \cup \mathcal{M}^{\perp}$ be a projection. Then $\mathcal{M}|_Q = \{PQ : P \in \mathcal{M}\}$ is a nest on $Q\mathcal{H}$. It is easy to check that $\tau(\mathcal{M}|_Q) = Q\tau(\mathcal{M})Q$ and $\tau(\mathcal{M}^{\perp}) = \tau(\mathcal{M})^{*}$. Let $\mathcal{E}(\mathcal{M})$ denote the set of all idempotents of $\tau(\mathcal{M})$. It is well-known that the linear span of $\mathcal{E}(\mathcal{M})$ is weakly dense in $\tau(\mathcal{M})$ and the commutant of $\tau(\mathcal{M})$ is $\mathbb{C}I$. For $E \in \mathcal{E}(\mathcal{M})$, we denote by $[E]$ the orthogonal projection from $\mathcal{H}$ onto the range of E. Let $E_1, E_2 \in \mathcal{E}(\mathcal{M})$. We say that $E_1 \leq E_2$ if $E_1E_2 = E_2E_1 = E_1$. We say that $E_1 < E_2$ if $E_1 \leq E_2$ and $E_1 \neq E_2$. We also refer the reader to [8] for the theory of nest algebras.

In this section, we will use Theorem 2.1 to prove the following result.

Theorem 3.1. *Let $\mathcal{M}$, $\mathcal{N}$ be two nests on a complex separable Hilbert space $\mathcal{H}$ and $\phi : \tau(\mathcal{M}) \to \tau(\mathcal{N})$ be a bijection preserving Lie products. Then there is a map $h : \tau(\mathcal{M}) \to \mathbb{C}I$ with $h([A, B]) = 0$ for all $A, B \in \tau(\mathcal{M})$ such that one of the following holds:*

(a) *There exists an additive isomorphism $\varphi : \tau(\mathcal{M}) \to \tau(\mathcal{N})$ such that $\phi(A) = \varphi(A) + h(A)$ for all $A \in \tau(\mathcal{M})$.*

(b) *There exists an additive anti-isomorphism $\varphi : \tau(\mathcal{M}) \to \tau(\mathcal{N})$ such that $\phi(A) = -\varphi(A) + h(A)$ for all $A \in \tau(\mathcal{M})$.*

To prove Theorem 3.1, we need some lemmas. The following lemmas can be found in [11].

Lemma 3.1. *Let $\mathcal{M}$ be a nest and $A \in \tau(\mathcal{M})$. Then*

(a) *A is the sum of a scalar and an idempotent if and only if $[A, [A, [A, X]]] = [A, X]$ for all $X \in \tau(\mathcal{M})$.*

(b) *A is the sum of a scalar and an idempotent whose range belongs to $\mathcal{M}$ if and only if $[A, [A, X]] = [A, X]$ for all $X \in \tau(\mathcal{M})$.*

Next we assume that $\mathcal{M}, \mathcal{N}$ are nontrivial nests and $\phi : \tau(\mathcal{M}) \to \tau(\mathcal{N})$ is a bijective map preserving Lie products.

Now we chose a nontrivial projection P_1 in $\mathcal{M}$ and set $P_2 = I - P_1$. It follows from Lemma 3.1(b) that there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ and idempotents $F_1, F_2 \in \tau(\mathcal{N})$ with $[F_1] \in \mathcal{N}$ and $F_1 + F_2 = I$ such that $\phi(P_i) = \lambda_i I + F_i$, $i = 1, 2$. Let $Q_1 = [F_1]$, $Q_2 = I - Q_1$ and $S = I + Q_1F_1Q_2$. Then S is invertible in $\tau(\mathcal{N})$ and $SF_iS^{-1} = Q_i$, $i = 1, 2$. Therefore, without loss of generality, we can assume that $\phi(P_i) = \lambda_i I + Q_i$, $i = 1, 2$. Write $\mathcal{A}_{ij} = P_i\tau(\mathcal{M})P_j$ and $\mathcal{B}_{ij} = Q_i\tau(\mathcal{N})Q_j$, $1 \leq i \leq j \leq 2$. It is obvious that $\tau(\mathcal{M}) = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{22}$ and $\tau(\mathcal{N}) = \mathcal{B}_{11} + \mathcal{B}_{12} + \mathcal{B}_{22}$.

Lemma 3.2. *If $E = E^2 \in \tau(\mathcal{M})$, then $\phi(E) = \lambda_E I + F$ where $\lambda_E \in \mathbb{C}$ and $F \in \tau(\mathcal{N})$ is an idempotent. Furthermore, if E is nontrivial, then both the scalar λ_E and the idempotent F are uniquely determined.*

Proof. Since $E = E^2 \in \tau(\mathcal{M})$, we have $[E, [E, [E, X]]] = [E, X]$ for all $X \in \tau(\mathcal{M})$. Then $[\phi(E), [\phi(E), [\phi(E), \phi(X)]]] = [\phi(E), \phi(X)]$, and so by Lemma 3.1(a), we have $\phi(E) = \lambda_E I + F$ for some scalar $\lambda_E \in \mathbb{C}$ and idempotent $F \in \tau(\mathcal{N})$. If E is a nontrivial idempotent, then F is a nontrivial idempotent. Otherwise, $F = 0$ or $F = I$. Then $\phi(E) \in \mathbb{C}I$, and so by Lemma 2.2, $E = \lambda I$ for some $\lambda \in \mathbb{C}$. Then the fact $E = E^2$ shows that $E = 0$ or $E = I$, which is a contradiction. If E is a nontrivial idempotent and $\phi(E) = \lambda_E I + F = \alpha_E I + G$ where $\lambda_E, \alpha_E \in \mathbb{C}$ and F, G are idempotents of $\tau(\mathcal{N})$, then F, G are nontrivial idempotents and $(\lambda_E - \alpha_E)I + F = G$. Thus,

$$\{\lambda_E - \alpha_E, \lambda_E - \alpha_E + 1\} = \sigma((\lambda_E - \alpha_E)I + F) = \sigma(G) = \{0, 1\}.$$

This yields that $\lambda_E = \alpha_E$, and so $F = G$. Hence both the scalar λ_E and the idempotent F are uniquely determined. The proof is completed. □

Now we define a map

$$\tilde{\phi} : \mathcal{E}(\mathcal{M}) \setminus \{0, I\} \to \mathcal{E}(\mathcal{N}) \setminus \{0, I\}$$

by $\tilde{\phi}(E) = \phi(E) - \lambda_E I$. It follows from Lemma 3.2 that $\tilde{\phi}$ is well-defined. We claim that $\tilde{\phi}$ is a bijective map. If $\tilde{\phi}(E_1) = \tilde{\phi}(E_2)$ for some $E_1, E_2 \in \mathcal{E}(\mathcal{M}) \setminus \{0, I\}$, then $\phi(E_1) - \lambda_{E_1} I = \phi(E_2) - \lambda_{E_2} I$. It follows that

$$\phi([E_1, X]) = [\phi(E_1), \phi(X)] = [\phi(E_2), \phi(X)] = \phi([E_2, X])$$

for all $X \in \tau(\mathcal{M})$, and so by the injectivity of ϕ, we have $[E_1, X] = [E_2, X]$ for all $X \in \tau(\mathcal{M})$. This implies that $E_1 = E_2 + \lambda I$ for some $\lambda \in \mathbb{C}$. Hence

$$\{0, 1\} = \sigma(E_1) = \sigma(E_2 + \lambda I) = \{\lambda, \lambda + 1\}.$$

Thus $\lambda = 0$, and so $\tilde{\phi}$ is injective. Let $F \in \mathcal{E}(\mathcal{N}) \setminus \{0, I\}$. By Lemma 3.2, there exist $\alpha_F \in \mathbb{C}$ and $E \in \mathcal{E}(\mathcal{M}) \setminus \{0, I\}$ such that $\phi^{-1}(F) = \alpha_F I + E$. It follows that $F = \phi(\alpha_F I + E)$. From Lemma 2.2, we see that

$$\phi(E) - F = \phi(E) - \phi(\alpha_F I + E) \in \mathbb{C}I.$$

Then there exists $\lambda_E \in \mathbb{C}$ such that $\phi(E) = \lambda_E I + F$. This implies that $\tilde{\phi}$ is surjective. We conclude that $\tilde{\phi}$ is a bijective map.

Lemma 3.3. *Let $E_1, E_2 \in \mathcal{E}(\mathcal{M}) \setminus \{0, I\}$ with $E_1 < E_2$, and let $F_i = \tilde{\phi}(E_i)$, $i = 1, 2$. Then either $F_1 < F_2$ or $F_2 < F_1$.*

Proof. Firstly, we observe that F_1 and F_2 commute since $[E_1, E_2] = 0$. The fact $E_2 - E_1 \notin \mathbb{C}I$ implies $F_2 - F_1 \notin \mathbb{C}I$, and in particular, $F_1 \neq F_2$. Since $E_2 - E_1$ is an idempotent in $\tau(\mathcal{M})$, we have from Lemma 3.2 and Theorem 2.1 that

$$F_2 - F_1 \in \mathbb{C}I + \mathcal{E}(\mathcal{N}),$$

and so $\sigma(F_2 - F_1) = \{\lambda, \lambda + 1\}$ for some $\lambda \in \mathbb{C}$. We may chose a Hamel basis that diagonalizes F_1 and F_2 simultaneously. Now if F_1 and F_2 are not comparable, then $\{1, -1\} \subseteq \sigma(F_2 - F_1) = \{\lambda, \lambda + 1\}$ for some $\lambda \in \mathbb{C}$, which is impossible. The proof is completed. □

Lemma 3.4. *Let $E_1, E_2, E_3 \in \mathcal{E}(\mathcal{M}) \setminus \{0, I\}$ with $E_1 < E_2 < E_3$, and let $F_i = \tilde{\phi}(E_i)$, $i = 1, 2, 3$.*

(a) *If $F_1 < F_2$, then $F_1 < F_2 < F_3$.*

(b) *If $F_1 > F_2$, then $F_1 > F_2 > F_3$.*

Consequently, the map $\tilde{\phi}$ is either order preserving or order reversing.

Proof. (a) The idempotents F_1, F_2 and F_3 are distinct and mutually comparable by Lemma 3.3. Since $E_1 + E_3 - E_2 \in \mathcal{E}(\mathcal{M})$, we have from Lemma 3.2 and Theorem 2.1 that $F_1 + F_3 - F_2 \in \mathbb{C}I + \mathcal{E}(\mathcal{N})$, and so $\sigma(F_1 + F_3 - F_2) = \{\lambda, \lambda + 1\}$ for some $\lambda \in \mathbb{C}$. If $F_1 < F_3 < F_2$ or $F_3 < F_1 < F_2$, then $\sigma(F_1 + F_3 - F_2) = \{-1, 0, 1\}$. This contradiction implies that (a) holds. Similarly, we can show that (b) is valid. The proof is completed. □

Remark 3.1. By Lemma 3.4, we refer to ϕ itself as being order preserving or order reversing according as $\tilde{\phi}$ is order preserving or order reversing respectively. We extend the definition of $\tilde{\phi}$ to all of $\mathcal{E}(\mathcal{M})$ by $\tilde{\phi}(0) = 0$, $\tilde{\phi}(I) = I$ if ϕ is order preserving, and $\tilde{\phi}(0) = I$, $\tilde{\phi}(I) = 0$ if ϕ is order reversing. Now we can reduce our discussions to the case where ϕ is order preserving. Indeed if ϕ is order reversing, we may consider the bijective map $\eta : \tau(\mathcal{M}^{\perp}) \to \tau(\mathcal{N})$ defined by $\eta(A) = -\phi(A^*)$. It is easy to verify that $\eta([A, B]) = [\eta(A), \eta(B)]$ for all $A, B \in \tau(\mathcal{M}^{\perp})$, and that η is order preserving.

Next we assume that ϕ is order preserving. Recall that $\mathcal{A}_{ij} = P_i \tau(\mathcal{M}) P_j$, $\mathcal{B}_{ij} = Q_i \tau(\mathcal{N}) Q_j$ and $Q_i = \tilde{\phi}(P_i)$, $1 \le i \le j \le 2$.

Lemma 3.5. *$Q_2 \phi(\mathcal{A}_{11}) Q_2 \subseteq \mathbb{C}Q_2$ and $Q_1 \phi(\mathcal{A}_{22}) Q_1 \subseteq \mathbb{C}Q_1$.*

Proof. Let $F \in \mathcal{B}_{22}$ be any idempotent. Then $Q_1 + F$ is an idempotent of $\tau(\mathcal{N})$ and $Q_1 + F \ge Q_1 = \tilde{\phi}(P_1)$. Since ϕ is order preserving, there exists an idempotent $E \in \tau(\mathcal{M})$ with $E \ge P_1$ such that $\phi(E) \in \mathbb{C}I + (Q_1 + F)$. Let $A \in \mathcal{A}_{11}$, it follows from the fact $[A, E] = 0$ that

$$[\phi(A), Q_1 + F] = 0.$$

This together with the fact $[\phi(A), Q_1] = 0$ yields that $[\phi(A), F] = 0$, and so

$$[Q_2 \phi(A) Q_2, F] = 0$$

for all idempotents F in the nest algebra $\mathcal{B}_{22} = \tau(\mathcal{N}|_{Q_2})$. Since the linear span of all idempotents of a nest algebra is weakly dense in the nest algebra, we have $Q_2\phi(A)Q_2 \in \mathbb{C}Q_2$ for all $A \in \mathcal{A}_{11}$. Similarly, we can show that $Q_1\phi(\mathcal{A}_{22})Q_1 \subseteq \mathbb{C}Q_1$. The proof is completed. □

Remark 3.2. By Lemma 3.5, we see that $\phi(\mathcal{A}_{ii}) + \mathbb{C}I = \mathcal{B}_{ii} + \mathbb{C}I$, $i = 1, 2$. For each $X_i \in \mathcal{A}_{ii}$, $i = 1, 2$, we define $f_i(X_i)$ to be the scalar that appears in $Q_j\phi(X_i)Q_j$ $(j \neq i)$, that is,

$$f_1(X_1)Q_2 = Q_2\phi(X_1)Q_2 \quad \text{and} \quad f_2(X_2)Q_1 = Q_1\phi(X_2)Q_1,$$

and set $\varphi_i(X_i) = \phi(X_i) - f_i(X_i)I$, $i = 1, 2$. Then φ_i is a bijective map from $\mathcal{A}_{ii}$ onto $\mathcal{B}_{ii}$. Indeed if $\varphi_i(A_i) = \varphi_i(B_i)$ for some $A_i, B_i \in \mathcal{A}_{ii}$, we have for any $X \in \tau(\mathcal{M})$,

$$\begin{aligned}\phi([A_i, X]) &= [\phi(A_i), \phi(X)] = [\varphi_i(A_i), \phi(X)] = [\varphi_i(B_i), \phi(X)] \\ &= [\phi(B_i), \phi(X)] = \phi([B_i, X]).\end{aligned}$$

The injectivity of ϕ implies that $[A_i, X] = [B_i, X]$ for all $X \in \tau(\mathcal{M})$. Then $A_i - B_i = \alpha_i I$ for some $\alpha_i \in \mathbb{C}$, and so $\alpha_i = 0$. This implies that φ_i is injective. Since $\phi(\mathcal{A}_{ii}) + \mathbb{C}I = \mathcal{B}_{ii} + \mathbb{C}I$, we have for every $T_i \in \mathcal{B}_{ii}$ there exist $S_i \in \mathcal{A}_{ii}$ and $\alpha_i \in \mathbb{C}$ such that $\phi(S_i) + \alpha_i I = T_i$. It follows that

$$\alpha_i Q_j = -Q_j\phi(S_i)Q_j = -f_i(S_i)Q_j \ (j \neq i).$$

Then $\alpha_i = -f_i(S_i)$, and so $\varphi_i(S_i) = \phi(S_i) - f_i(S_i)I = T_i$. This shows that φ_i is surjective. We conclude that φ_i is a bijective map from $\mathcal{A}_{ii}$ onto $\mathcal{B}_{ii}$, $i = 1, 2$.

Lemma 3.6. *Let φ_i be as in Remark 3.2, then $\varphi_i(X_iY_i) = \varphi_i(X_i)\varphi_i(Y_i)$ for all $X_i, Y_i \in \mathcal{A}_{ii}$, $i = 1, 2$.*

Proof. Let $X_1 \in \mathcal{A}_{11}$ and $T \in \mathcal{A}_{12}$. Then

$$\phi(X_1T) = \phi([X_1, T]) = [\phi(X_1), \phi(T)] = \varphi_1(X_1)\phi(T).$$

Thus if $X_1, Y_1 \in \mathcal{A}_{11}$, we have

$$\varphi_1(X_1Y_1)\phi(T) = \phi(X_1Y_1T) = \varphi_1(X_1)\phi(Y_1T) = \varphi_1(X_1)\varphi_1(Y_1)\phi(T)$$

for all $T \in \mathcal{A}_{12}$. Hence $\varphi_1(X_1Y_1) = \varphi_1(X_1)\varphi_1(Y_1)$ for all $X_1, Y_1 \in \mathcal{A}_{11}$. Similarly, we can show that

$$\varphi_2(X_2Y_2) = \varphi_2(X_2)\varphi_2(Y_2)$$

for all $X_2, Y_2 \in \mathcal{A}_{22}$. The proof is completed. □

Remark 3.3. Let f_i, φ_i be as in Remak 3.2. For each $X \in \tau(\mathcal{M})$, we define $f(X) = f_1(P_1XP_1) + f_2(P_2XP_2)$. By Theorem 2.1, we see that

$$\phi(X) - \sum_{1 \le i \le j \le 2} \phi(P_iXP_j) \in \mathbb{C}I$$

for all $X \in \tau(\mathcal{M})$. Now we define a function $g : \mathcal{M} \to \mathbb{C}$ by

$$g(X)I = \phi(X) - \sum_{1 \le i \le j \le 2} \phi(P_iXP_j),$$

and set $\varphi(X) = \phi(X) - h(X)I$ where $h = g + f$. Then $\varphi|_{\mathcal{A}_{ii}} = \varphi_i$, $i = 1, 2$.

Lemma 3.7. *Let φ be as in Remark 3.3, then φ is an additive bijection.*

Proof. By the definition of φ, we see that $\varphi(P_iXP_j) = Q_i\varphi(X)Q_j$ for all $X \in \tau(\mathcal{M})$ and $1 \le i \le j \le 2$. Thus

$$\varphi(A_{11} + A_{12} + A_{22}) = \varphi(A_{11}) + \varphi(A_{12}) + \varphi(A_{22}) \tag{16}$$

for all $A_{ij} \in \mathcal{A}_{ij}$ and $1 \le i \le j \le 2$. Since $h(\mathcal{A}_{12}) = f(\mathcal{A}_{12}) = 0$, we have $\varphi(A_{12}) = \phi(A_{12})$ and so by Lemma 2.6(c),

$$\varphi(A_{12} + B_{12}) = \varphi(A_{12}) + \varphi(B_{12}) \tag{17}$$

for all $A_{12}, B_{12} \in \mathcal{A}_{12}$. By Theorem 2.1 and the definition of φ, we have for any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$,

$$\varphi(A_{ii} + B_{ii}) - \varphi(A_{ii}) - \varphi(B_{ii}) = \phi(A_{ii} + B_{ii}) - \phi(A_{ii}) - \phi(B_{ii}) - h(A_{ii} + B_{ii}) + h(A_{ii}) + h(B_{ii}) \in \mathbb{C}I.$$

This and the fact $\varphi(A_{ii} + B_{ii}) - \varphi(A_{ii}) - \varphi(B_{ii}) \in \mathcal{B}_{ii}$ give us that

$$\varphi(A_{ii} + B_{ii}) = \varphi(A_{ii}) + \varphi(B_{ii}). \tag{18}$$

From Eqs. (16)-(18), we get that $\varphi(A + B) = \varphi(A) + \varphi(B)$ for all $A, B \in \tau(\mathcal{M})$.

Since $\varphi|_{\mathcal{A}_{ii}} = \varphi_i$ and φ_i is bijective, it follows that φ maps $\mathcal{A}_{ii}$ onto $\mathcal{B}_{ii}$, $i = 1, 2$. By Lemma 2.3, we have $\varphi(\mathcal{A}_{12}) = \phi(\mathcal{A}_{12}) = \mathcal{B}_{12}$. Thus φ is surjective. Now if $\ker\varphi \neq \{0\}$, let $X \in \ker\varphi$ with $X \neq 0$, then $\phi(X) \in \mathbb{C}I$, and so $X = \lambda I$ for some nonzero scalar $\lambda \in \mathbb{C}$. By Lemma 3.6 and the additivity of φ, we can obtain that $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathcal{A}_{11} \oplus \mathcal{A}_{22}$. In particular, we have for any $A \in \mathcal{A}_{11} \oplus \mathcal{A}_{22}$,

$$\varphi(A) = \varphi(\lambda^{-1}A)\varphi(\lambda I) = 0.$$

Then $\phi(A) \in \mathbb{C}I$, and so $A \in \mathbb{C}I$ for all $A \in \mathcal{A}_{11} \oplus \mathcal{A}_{22}$. This is a contradiction. We conclude that φ is an additive bijection. The proof is completed. □

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1.. When $\mathcal{M}$ is a trivial nest, it follows that

$$\tau(\mathcal{M}) = \tau(\mathcal{N}) = \mathcal{B}(\mathcal{H}),$$

and so by the result of [22], the conclusion is valid. Next we assume that $\mathcal{M}$ is a nontrivial nest. Then by Lemma 3.4, ϕ is either order preserving or order reversing.

If ϕ is order preserving, it follows from Lemma 3.7 that $\phi = \varphi + h$, where $\varphi : \tau(\mathcal{M}) \to \tau(\mathcal{N})$ is an additive bijection and $h : \tau(\mathcal{M}) \to \mathbb{C}I$ is a map. Now let $X, Y \in \tau(\mathcal{M})$, then $X = X_{11} + X_{12} + X_{22}$ and $Y = Y_{11} + Y_{12} + Y_{22}$ for some $X_{ij}, Y_{ij} \in \mathcal{A}_{ij}$. Since $\varphi|_{\mathcal{A}_{ii}} = \varphi_i$ and $\varphi(\mathcal{A}_{12}) = \phi(\mathcal{A}_{12}) = \mathcal{B}_{12}$, we have from Lemma 3.6 that

$$\varphi(XY_{11}) = \varphi_1(X_{11}Y_{11}) = \varphi(X_{11})\varphi(Y_{11}) = \varphi(X)\varphi(Y_{11}),$$

$$\varphi(XY_{12}) = \phi([X_{11}, Y_{12}]) = \varphi(X_{11})\varphi(Y_{12}) = \varphi(X)\varphi(Y_{12}),$$

and

$$\begin{aligned}\varphi(XY_{22}) &= \varphi(X_{12}Y_{22}) + \varphi(X_{22}Y_{22}) = \phi([X_{12}, Y_{22}]) + \varphi(X_{22})\varphi(Y_{22}) \\ &= \varphi(X_{12})\varphi(Y_{22}) + \varphi(X_{22})\varphi(Y_{22}) = \varphi(X)\varphi(Y_{22}).\end{aligned}$$

This shows that $\varphi(XY) = \varphi(X)\varphi(Y)$ for all $X, Y \in \tau(\mathcal{M})$. Hence φ is an additive isomorphism from $\tau(\mathcal{M})$ onto $\tau(\mathcal{N})$.

If ϕ is order reversing, then the map $\eta : \tau(\mathcal{M}^{\perp}) \to \tau(\mathcal{N})$ defined by $\eta(X) = -\phi(X^*)$ is order preserving. By the above case, we see that $\eta = \rho - h$ where ρ is an additive isomorphism from $\tau(\mathcal{M}^{\perp})$ onto $\tau(\mathcal{N})$. For each $X \in \tau(\mathcal{M})$, we define $\varphi(X) = \rho(X^*)$. It is clear that φ is an additive anti-isomorphism from $\tau(\mathcal{M})$ onto $\tau(\mathcal{N})$ and $\phi(X) = -\varphi(X) + h(X)$ for all $X \in \mathcal{M}$. The proof is completed. □

As a consequence of Theorem 3.1 and [9, Theorem 2.1], we have the following corollary.

Corollary 3.1. *Let $\mathcal{M}, \mathcal{N}$ be nests on an infinite dimensional complex separable Hilbert space $\mathcal{H}$ and $\phi : \tau(\mathcal{M}) \to \tau(\mathcal{N})$ be a bijection preserving Lie products. Then there exist a bounded invertible linear or conjugate-linear operator $T : \mathcal{H} \to \mathcal{H}$ and a map $h : \tau(\mathcal{M}) \to \mathbb{C}I$ with $h([A, B]) = 0$ for all $A, B \in \tau(\mathcal{M})$ such that either $\phi(A) = TAT^{-1} + h(A)$ for all $A \in \tau(\mathcal{M})$, or $\phi(A) = -TA^*T^{-1} + h(A)$ for all $A \in \tau(\mathcal{M})$.*

Now we treat maps preserving Lie products on block upper triangular matrix algebras. Let $M_n(\mathbb{F})$ be the algebra of all $n \times n$ matrices over real or complex field $\mathbb{F}$. For every finite sequence of positive integers

$n_1, n_2, \dots, n_k$ satisfying $n_1 + n_2 + \cdots + n_k = n$, we associate an algebra consisting of all $n \times n$ matrices of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}$$

where A_{ij} is an $n_i \times n_j$ matrix. We will call such an algebra a block upper triangular algebra in $M_n(\mathbb{F})$. It is clear that every nontrivial nest algebra on a finite dimensional space is isomorphic to a block upper triangular matrix algebra. From Theorem 3.1 and the result of [10, Theorem 4.6], we have the following corollary.

Corollary 3.2. *Let $\mathcal{A}, \mathcal{B} \subseteq M_n(\mathbb{F})$ be block upper triangular matrix algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a bijection preserving Lie products. Then there exist an invertible matrix $T \in M_n(\mathbb{F})$, an additive automorphism $\alpha : \mathbb{F} \to \mathbb{F}$ and a map $h : \mathcal{A} \to \mathbb{F}$ with $h([X, Y]) = 0$ for all $X, Y \in \mathcal{A}$ such that either $\phi(A) = TA_\alpha T^{-1} + h(A)I_n$ for all $A = (a_{ij}) \in \mathcal{A}$, or $\phi(A) = -TA_\alpha^t T^{-1} + h(A)I_n$ for all $A = (a_{ij}) \in \mathcal{A}$, where $A_\alpha = (\alpha(a_{ij}))$, I_n is the identity of $M_n(\mathbb{F})$ and t stands for the transpose.*

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