DE GRUYTER OPEN Spec. Matrices 2016; 4:73–79

Research Article Open Access

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Orthogonal diagonalization for complex skew-persymmetric anti-tridiagonal Hankel matrices

DOI 10.1515/spma-2016-0008
Received May 13, 2015; accepted November 12, 2015

Abstract: In this paper, we obtain an eigenvalue decomposition for any complex skew-persymmetric anti-tridiagonal Hankel matrix where the eigenvector matrix is orthogonal.

Keywords: Anti-tridiagonal matrices; Hankel matrices; orthogonal diagonalization; skew-persymmetric matrices.

1 Introduction

An $n \times n$ complex anti-tridiagonal Hankel matrix is a matrix of the form

$$\operatorname{antitridiag}_n(a_1, a_0, a_{-1}) := \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_1 & a_0 \\ \vdots & & \iddots & a_1 & a_0 & a_{-1} \\ \vdots & \iddots & a_1 & a_0 & a_{-1} & 0 \\ 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ a_1 & \iddots & \iddots & \iddots & & \vdots \\ a_0 & a_{-1} & 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad n \in \mathbb{N}, \quad a_1, a_0, -a_1 \in \mathbb{C},$$

where $\mathbb{N}$ is the set of natural numbers (that is, the set of positive integers) and $\mathbb{C}$ denotes the set of (finite) complex numbers.

In [1, Theorem 4] an orthogonal diagonalization for any real persymmetric anti-tridiagonal Hankel matrix was presented. The following result, which was given in [2, Theorem 5 and Lemma 1], is a generalization of [1, Theorem 4] to complex matrices ([3] is another recent reference where the eigenvalues of certain persymmetric anti-tridiagonal Hankel matrices are given).

Theorem 1. *Let $n \in \mathbb{N}$ and $a_1, a_0 \in \mathbb{C}$. Then*

$$\operatorname{antitridiag}_n(a_1, a_0, a_1) = Y_n \operatorname{diag}(\tau_1, \ldots, \tau_n) Y_n^{-1},$$

where $\operatorname{diag}(\tau_1, \ldots, \tau_n) := (\tau_j \delta_{j,k})_{j,k=1}^n$, *with δ being the Kronecker delta and*

$$\tau_j = (-1)^{j+1} \left(a_0 + 2a_1 \cos \frac{j\pi}{n+1} \right), \qquad j \in \{1, \ldots, n\},$$

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and Y_n is the $n \times n$ real symmetric orthogonal matrix whose entries are given by

$$[Y_n]_{j,k} = \sqrt{\frac{2}{n+1}} \sin \frac{jk\pi}{n+1}, \qquad j, k \in \{1, \ldots, n\}.$$

In the present paper we give an orthogonal diagonalization for any complex skew-persymmetric anti-tridiagonal Hankel matrix, i.e., for any matrix of the form $\mathrm{antitridiag}_n(a, 0, -a)$ with $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

2 Orthogonal diagonalization

We first give an orthogonal diagonalization for any complex skew-persymmetric anti-tridiagonal Hankel matrix of odd order.

Theorem 2. *Consider $a \in \mathbb{C}$ and $n = 2p + 1$ with $p \in \mathbb{N} \cup \{0\}$. Let Y_n be as in Theorem 1. Then*

$$\mathrm{antitridiag}_n(a, 0, -a) = U_n \mathrm{diag}(\lambda_1, \ldots, \lambda_n) U_n^{-1},$$

where

$$\lambda_j = -2a \cos \frac{j\pi}{n+1}, \qquad j \in \{1, \ldots, n\},$$

and U_n is the $n \times n$ real orthogonal matrix given by

$$[U_n]_{j,k} = s_{j,k} [Y_n]_{j,k}, \qquad j, k \in \{1, \ldots, n\},$$

with

$$s_{j,k} = \begin{cases} (-1)^{\frac{j+1}{2}} & \text{if } j \text{ is odd}, \\ (-1)^{\frac{j+1+n+2k}{2}} & \text{if } j \text{ is even}, \end{cases} \qquad j, k \in \{1, \ldots, n\}. \tag{1}$$

Proof. See A. □

We now give an orthogonal diagonalization for any complex skew-persymmetric anti-tridiagonal Hankel matrix of even order.

Theorem 3. *Consider $a \in \mathbb{C}$ and $n = 2p$ with $p \in \mathbb{N}$. Let Y_n be as in Theorem 1. Then*

$$\mathrm{antitridiag}_n(a, 0, -a) = U_n \mathrm{diag}(\lambda_1, \ldots, \lambda_n) U_n^{-1},$$

where

$$\lambda_j = -2a \cos \frac{j\pi}{n+1}, \qquad j \in \{1, \ldots, n\},$$

and U_n is the $n \times n$ real orthogonal matrix given by

$$[U_n]_{j,k} = \sqrt{2} s_{j,k} [Y_n]_{j,k}, \qquad j, k \in \{1, \ldots, n\},$$

with

$$s_{j,k} = \begin{cases} \frac{(-1)^{\frac{j-1}{2}} + (-1)^{\frac{n+2k+j+1}{2}}}{2} & \text{if } j \text{ is odd}, \\ \frac{(-1)^{\frac{j}{2}} + (-1)^{\frac{n+2k+j}{2}}}{2} & \text{if } j \text{ is even}, \end{cases} \qquad j, k \in \{1, \ldots, n\}. \tag{2}$$

Proof. See B. □

In [4, 5] Rimas gave a diagonalization for the matrix $\mathrm{antitridiag}_n(1, 0, -1)$ of the form

$$\mathrm{antitridiag}_n(1, 0, -1) = W_n \mathrm{diag}\left(-2\cos\frac{\pi}{n+1}, -2\cos\frac{2\pi}{n+1}, \ldots, -2\cos\frac{n\pi}{n+1}\right) W_n^{-1}, \tag{3}$$

where the eigenvector matrix W_n is real and not orthogonal. Although W_n is not orthogonal it satisfies

$$[W_n^\top W_n]_{j,k} = 0$$

whenever $j,k \in \{1,\dots,n\}$ and $j \neq k$, because eigenvectors of a real symmetric matrix corresponding to different eigenvalues are orthogonal (see, e.g., [6, p. 548]).

From (3) we have

$$\text{antitridiag}_n(a,0,-a) = W_n \text{diag}\,(\lambda_1,\dots,\lambda_n)\, W_n^{-1} \tag{4}$$

for all $a \in \mathbb{C}$. Thus, the difference between the diagonalization in (4) and the one given in Theorems 2 and 3 for the matrix $\text{antitridiag}_n(a,0,-a)$ lies in the eigenvector matrix. Since the algebraic multiplicity of λ_k is 1 with $a \neq 0$, its geometric multiplicity is also 1 (see, e.g., [7, Section 4.12]), and consequently, there exists $r_k \in \mathbb{R}$ such that

$$[U_n]_{1:n,k} = r_k [W_n]_{1:n,k}, \qquad k \in \{1,\dots,n\},$$

where $[U_n]_{1:n,k}$ and $[W_n]_{1:n,k}$ denote the kth column of the matrix U_n and W_n, respectively. Therefore,

$$1 = [I_n]_{k,k} = [U_n^\top U_n]_{k,k} = [U_n]_{1:n,k}^\top [U_n]_{1:n,k} = r_k^2 [W_n]_{1:n,k}^\top [W_n]_{1:n,k} = r_k^2 \|[W_n]_{1:n,k}\|_2^2, \qquad k \in \{1,\dots,n\},$$

and hence,

$$|r_k| = \frac{1}{\|[W_n]_{1:n,k}\|_2}, \qquad k \in \{1,\dots,n\},$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Thus,

$$[U_n]_{1:n,k} \in \left\{ \frac{[W_n]_{1:n,k}}{\|[W_n]_{1:n,k}\|_2}, -\frac{[W_n]_{1:n,k}}{\|[W_n]_{1:n,k}\|_2} \right\}, \qquad k \in \{1,\dots,n\}. \tag{5}$$

Observe that to prove (5) is another way to prove Theorems 2 and 3.

We finish by mentioning that Theorems 2 and 3 are useful to obtain the natural powers of the considered matrix $\text{antitridiag}_n(a,0,-a)$:

$$\left(\text{antitridiag}_n(a,0,-a)\right)^q = U_n \text{diag}(\lambda_1{}^q,\dots,\lambda_n{}^q) U_n^\top, \qquad q \in \mathbb{N}.$$

The natural powers of complex skew-persymmetric anti-tridiagonal Hankel matrices were also studied in [4, 5, 8, 9]. In [8] the expression for those powers was obtained by using the eigenvalue decomposition of the complex tridiagonal Toeplitz matrices presented in [10].

Acknowledgement: The authors thank the two anonymous reviewers for their helpful comments.

This work was supported in part by the Spanish Ministry of Economy and Competitiveness through the RACHEL project (TEC2013-47141-C4-2-R).

A Proof of Theorem 2

We divide the proof into two steps. In the first step we prove that U_n is an orthogonal matrix, or equivalently, that $U_n^\top U_n = I_n$, where $\top$ denotes transpose and I_n is the $n \times n$ identity matrix. For convenience we rewrite (1) as

$$s_{j,k} = (-1)^{\lceil \frac{j}{2} \rceil} \left((-1)^{\frac{1+n+2k}{2}}\right)^{j+1}, \qquad j,k \in \{1,\dots,n\},$$

where $\lceil x \rceil$ denotes the smallest integer not less than x. Applying the basic trigonometric formula $-2\sin x \sin y = \cos(x+y) - \cos(x-y)$ (see, e.g., [11, p. 97]) yields

$$\left[U_n^\top U_n\right]_{j,k} = \sum_{h=1}^{n} \left[U_n^\top\right]_{j,h} [U_n]_{h,k} = \sum_{h=1}^{n} [U_n]_{h,j} [U_n]_{h,k}$$

$$= \sum_{h=1}^{n}(-1)^{\lceil \frac{h}{2}\rceil}\left((-1)^{\frac{1+n+2j}{2}}\right)^{h+1}[Y_n]_{h,j}(-1)^{\lceil \frac{h}{2}\rceil}\left((-1)^{\frac{1+n+2k}{2}}\right)^{h+1}[Y_n]_{h,k}$$
$$= \frac{2}{n+1}\sum_{h=1}^{n}\left((-1)^{j+k}\right)^{h+1}\sin\frac{hj\pi}{n+1}\sin\frac{hk\pi}{n+1}$$
$$= \frac{2}{n+1}\sum_{h=1}^{n}((-1)^{j+k})^{h+1}\left[-\frac{1}{2}\left(\cos\frac{(j+k)h\pi}{n+1}-\cos\frac{(j-k)h\pi}{n+1}\right)\right]$$
$$= \frac{(-1)^{j+k+1}}{n+1}\sum_{h=1}^{n}((-1)^{j+k})^{h}\left[\cos\frac{(j+k)h\pi}{n+1}-\cos\frac{(j-k)h\pi}{n+1}\right], \qquad j,k\in\{1,\dots,n\}.$$

Consequently, using

$$\sum_{h=1}^{n}\cos\frac{mh\pi}{n+1} = \begin{cases} 0 & \text{if } m \text{ is odd},\\ n & \text{if } m\in\Omega,\\ -1 & \text{if } m\in 2\mathbb{N}\setminus\Omega, \end{cases} \tag{6}$$

and

$$\sum_{h=1}^{n}(-1)^{h}\cos\frac{mh\pi}{n+1} = \begin{cases} 0 & \text{if } n+1+m \text{ is odd},\\ n & \text{if } n+1+m\in\Omega \text{ and } |n+1-m|\in\Omega,\\ \frac{n-1}{2} & \text{if } n+1+m\in\Omega \text{ and } |n+1-m|\in 2\mathbb{N}\setminus\Omega,\\ \frac{n-1}{2} & \text{if } n+1+m\in 2\mathbb{N}\setminus\Omega \text{ and } |n+1-m|\in\Omega,\\ -1 & \text{if } n+1+m\in 2\mathbb{N}\setminus\Omega \text{ and } |n+1-m|\in 2\mathbb{N}\setminus\Omega, \end{cases} \tag{7}$$

where $m\in\mathbb{N}\cup\{0\}$ and $\Omega=\{2(n+1)k : k\in\mathbb{N}\cup\{0\}\}$ (see [1, Eqs. (5) and (7)]), we obtain

$$\left[U_n^{\top}U_n\right]_{j,k} = \begin{cases} \frac{-1}{n+1}[-1-n]=1 & \text{if } j=k,\\ \frac{-1}{n+1}[-1-(-1)]=0 & \text{if } j\neq k \text{ and } j+k \text{ is even},\\ \frac{1}{n+1}[0-0]=0 & \text{if } j\neq k \text{ and } j+k \text{ is odd}, \end{cases}$$
$$= [I_n]_{j,k}, \qquad j,k\in\{1,\dots,n\}.$$

In the second step we prove that $\text{antitridiag}_n(a,0,-a)=U_nD_nU_n^{-1}$, or equivalently, $\text{antitridiag}_n(a,0,-a)=U_nD_nU_n^{\top}$, where $D_n=\text{diag}(\lambda_1,\dots,\lambda_n)$. Applying the basic trigonometric formulas $-2\sin x\sin y=\cos(x+y)-\cos(x-y)$ and $\cos(x+y)=\cos x\cos y-\sin x\sin y$ (see, e.g., [11, pp. 96-97]) yields

$$\cos\frac{h\pi}{n+1}\sin\frac{jh\pi}{n+1}\sin\frac{kh\pi}{n+1}$$
$$= \cos\frac{h\pi}{n+1}\left[-\frac{1}{2}\left(\cos\frac{(j+k)h\pi}{n+1}-\cos\frac{(j-k)h\pi}{n+1}\right)\right] = -\frac{1}{2}\left[\cos\frac{(j+k)h\pi}{n+1}\cos\frac{h\pi}{n+1}-\cos\frac{(j-k)h\pi}{n+1}\cos\frac{h\pi}{n+1}\right]$$
$$= -\frac{1}{2}\left[\frac{1}{2}\left(\cos\frac{(j+k+1)h\pi}{n+1}+\cos\frac{(j+k-1)h\pi}{n+1}\right)-\frac{1}{2}\left(\cos\frac{(j-k+1)h\pi}{n+1}+\cos\frac{(j-k-1)h\pi}{n+1}\right)\right]$$
$$= -\frac{1}{4}\left[\cos\frac{(j+k+1)h\pi}{n+1}+\cos\frac{(j+k-1)h\pi}{n+1}-\cos\frac{(j-k+1)h\pi}{n+1}-\cos\frac{(j-k-1)h\pi}{n+1}\right] \tag{8}$$

for all $h,j,k\in\{1,\dots,n\}$, and hence,

$$\left[U_nD_nU_n^{\top}\right]_{j,k} = \sum_{h=1}^{n}[U_n]_{j,h}\left[D_nU_n^{\top}\right]_{h,k} = \sum_{h=1}^{n}[U_n]_{j,h}\sum_{l=1}^{n}[D_n]_{h,l}\left[U_n^{\top}\right]_{l,k} = \sum_{h=1}^{n}[U_n]_{j,h}\sum_{l=1}^{n}[D_n]_{h,l}[U_n]_{k,l}$$
$$= \sum_{h=1}^{n}[U_n]_{j,h}[D_n]_{h,h}[U_n]_{k,h} = \frac{-4a}{n+1}(-1)^{\lceil\frac{j}{2}\rceil+\lceil\frac{k}{2}\rceil}\sum_{h=1}^{n}\left((-1)^{\frac{1+n+2h}{2}}\right)^{j+k}\cos\frac{h\pi}{n+1}\sin\frac{jh\pi}{n+1}\sin\frac{kh\pi}{n+1}$$
$$= \frac{-4at_{j,k}}{n+1}\sum_{h=1}^{n}\left((-1)^{j+k}\right)^{h}\cos\frac{h\pi}{n+1}\sin\frac{jh\pi}{n+1}\sin\frac{kh\pi}{n+1}$$
$$= \frac{at_{j,k}}{n+1}\sum_{h=1}^{n}\left((-1)^{j+k}\right)^{h}\left[\cos\frac{(j+k+1)h\pi}{n+1}+\cos\frac{(j+k-1)h\pi}{n+1}-\cos\frac{(j-k+1)h\pi}{n+1}-\cos\frac{(j-k-1)h\pi}{n+1}\right],$$

where $t_{j,k} = (-1)^{\lceil \frac{j}{2} \rceil + \lceil \frac{k}{2} \rceil} \left((-1)^{\frac{1+n}{2}} \right)^{j+k}$ and $j,k \in \{1,\dots,n\}$. Therefore, using (6) and (7) we conclude that

$$\left[U_n D_n U_n^\top \right]_{j,k} = \begin{cases} \frac{at_{j,k}}{n+1} \left[n + (-1) - (-1) - (-1) \right] = at_{j,k} & \text{if } j+k=n, \\ \frac{at_{j,k}}{n+1} \left[(-1) + n - (-1) - (-1) \right] = at_{j,k} & \text{if } j+k=n+2, \\ \frac{at_{j,k}}{n+1} \left[0+0-0-0 \right] = 0 & \text{if } j+k \text{ is even}, \\ \frac{at_{j,k}}{n+1} \left[(-1)+(-1)-(-1)-(-1) \right] = 0 & \text{otherwise}, \end{cases}$$

$$= \begin{cases} a(-1)^{\frac{j}{2}+\frac{k+1}{2}} (-1)^{\frac{1+n}{2}} = a(-1)^{n+1} = a & \text{if } j+k=n \text{ and } j \text{ is even}, \\ a(-1)^{\frac{j+1}{2}+\frac{k}{2}} (-1)^{\frac{1+n}{2}} = a(-1)^{n+1} = a & \text{if } j+k=n \text{ and } j \text{ is odd}, \\ a(-1)^{\frac{j}{2}+\frac{k+1}{2}} (-1)^{\frac{1+n}{2}} = a(-1)^{n+2} = -a & \text{if } j+k=n+2 \text{ and } j \text{ is even}, \\ a(-1)^{\frac{j+1}{2}+\frac{k}{2}} (-1)^{\frac{1+n}{2}} = a(-1)^{n+2} = -a & \text{if } j+k=n+2 \text{ and } j \text{ is odd}, \\ 0 & \text{otherwise}, \end{cases}$$

$$= [\text{antitridiag}_n(a, 0, -a)]_{j,k}, \qquad j,k \in \{1,\dots,n\}.$$

□

B Proof of Theorem 3

We divide the proof into two steps. In the first step we prove that U_n is an orthogonal matrix. For convenience we rewrite (2) as

$$s_{j,k} = \frac{(-1)^{\lceil \frac{j}{2} \rceil} (-1)^j \left(1 + (-1)^{\frac{n+2j+2k}{2}} \right)}{2}, \qquad j,k \in \{1,\dots,n\}.$$

Applying (6) and (7) yields

$$\begin{aligned} \left[U_n^\top U_n \right]_{j,k} &= \sum_{h=1}^{n} [U_n]_{h,j} [U_n]_{h,k} \\ &= \sum_{h=1}^{n} \sqrt{2} \frac{(-1)^{\lceil \frac{h}{2} \rceil} (-1)^h \left(1 + (-1)^{\frac{n+2h+2j}{2}} \right)}{2} [Y_n]_{h,j} \sqrt{2} \frac{(-1)^{\lceil \frac{h}{2} \rceil} (-1)^h \left(1 + (-1)^{\frac{n+2h+2k}{2}} \right)}{2} [Y_n]_{h,k} \\ &= \frac{1}{2} \sum_{h=1}^{n} \left(1 + (-1)^{\frac{n+2h+2j}{2}} \right) \left(1 + (-1)^{\frac{n+2h+2k}{2}} \right) [Y_n]_{h,j} [Y_n]_{h,k} \\ &= \frac{1}{n+1} \sum_{h=1}^{n} \left(1 + (-1)^{\frac{n+2h+2j}{2}} \right) \left(1 + (-1)^{\frac{n+2h+2k}{2}} \right) \sin \frac{hj\pi}{n+1} \sin \frac{hk\pi}{n+1} \\ &= \frac{1}{n+1} \sum_{h=1}^{n} \left[\left(1 + (-1)^{j+k} \right) + (-1)^h (-1)^{\frac{n}{2}} \left((-1)^j + (-1)^k \right) \right] \left[-\frac{1}{2} \left(\cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right) \right] \\ &= -\frac{1+(-1)^{j+k}}{2(n+1)} \sum_{h=1}^{n} \left[\cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right] \\ &\quad - (-1)^{\frac{n}{2}} \frac{(-1)^j + (-1)^k}{2(n+1)} \sum_{h=1}^{n} (-1)^h \left[\cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right] \\ &= \begin{cases} -\frac{1}{n+1}[-1-n] - (-1)^{\frac{n}{2}} \frac{(-1)^j}{n+1}[0-0] = 1 & \text{if } j=k, \\ -\frac{1}{n+1}[-1-(-1)] - (-1)^{\frac{n}{2}} \frac{(-1)^j+(-1)^k}{2(n+1)}[0-0] = 0 & \text{if } j \neq k \text{ and } j+k \text{ is even}, \\ 0-0=0 & \text{if } j \neq k \text{ and } j+k \text{ is odd}, \end{cases} \\ &= [I_n]_{j,k}, \qquad j,k \in \{1,\dots,n\}. \end{aligned}$$

In the second step we prove that $\text{antitridiag}_n(a, 0, -a) = U_n D_n U_n^{-1}$, or equivalently, $\text{antitridiag}_n(a, 0, -a) = U_n D_n U_n^\top$, where $D_n = \text{diag}(\lambda_1, \ldots, \lambda_n)$. We have

$$\begin{aligned}
&\left[U_n D_n U_n^\top\right]_{j,k} \\
&= \sum_{h=1}^{n} [U_n]_{j,h} [D_n]_{h,h} [U_n]_{k,h} = \frac{(-1)^{\lceil \frac{j}{2} \rceil + \lceil \frac{k}{2} \rceil}(-1)^{j+k}}{2} \sum_{h=1}^{n} \left(1 + (-1)^{\frac{n+2j+2h}{2}}\right)\left(1 + (-1)^{\frac{n+2k+2h}{2}}\right) \lambda_h [Y_n]_{j,h} [Y_n]_{k,h} \\
&= \frac{(-1)^{\lceil \frac{j}{2} \rceil + \lceil \frac{k}{2} \rceil}(-1)^{j+k}}{2} \sum_{h=1}^{n} \left(1 + (-1)^{j+k} + (-1)^h (-1)^{\frac{n}{2}} \left((-1)^j + (-1)^k\right)\right) \lambda_h [Y_n]_{j,h} [Y_n]_{k,h} \\
&= \frac{(-1)^{\lceil \frac{j}{2} \rceil + \lceil \frac{k}{2} \rceil}(-1)^{j+k}}{2} \left(\left(1 + (-1)^{j+k}\right) \sum_{h=1}^{n} \lambda_h [Y_n]_{j,h} [Y_n]_{k,h} + (-1)^{\frac{n}{2}} \left((-1)^j + (-1)^k\right) \sum_{h=1}^{n} (-1)^h \lambda_h [Y_n]_{j,h} [Y_n]_{k,h}\right) \\
&= -\frac{2at_{j,k}}{n+1} \sum_{h=1}^{n} \cos\frac{h\pi}{n+1} \sin\frac{jh\pi}{n+1} \sin\frac{kh\pi}{n+1} - \frac{2ar_{j,k}}{n+1} \sum_{h=1}^{n} (-1)^h \cos\frac{h\pi}{n+1} \sin\frac{jh\pi}{n+1} \sin\frac{kh\pi}{n+1},
\end{aligned}$$

where $t_{j,k} = (-1)^{\lceil \frac{j}{2} \rceil + \lceil \frac{k}{2} \rceil}(1 + (-1)^{j+k})$, $r_{j,k} = (-1)^{\lceil \frac{j}{2} \rceil + \lceil \frac{k}{2} \rceil}(-1)^{\frac{n}{2}} \left((-1)^j + (-1)^k\right)$, and $j, k \in \{1, \ldots, n\}$. Using (6), (7), and (8) we conclude that

$$\begin{aligned}
\left[U_n D_n U_n^\top\right]_{j,k} &= \frac{at_{j,k}}{2(n+1)} \sum_{h=1}^{n} \left[\cos\frac{(j+k+1)h\pi}{n+1} + \cos\frac{(j+k-1)h\pi}{n+1} - \cos\frac{(j-k+1)h\pi}{n+1} - \cos\frac{(j-k-1)h\pi}{n+1}\right] \\
&\quad + \frac{ar_{j,k}}{2(n+1)} \sum_{h=1}^{n} (-1)^h \left[\cos\frac{(j+k+1)h\pi}{n+1} + \cos\frac{(j+k-1)h\pi}{n+1} - \cos\frac{(j-k+1)h\pi}{n+1} - \cos\frac{(j-k-1)h\pi}{n+1}\right] \\
&= \begin{cases}
\frac{at_{j,k}}{2(n+1)} [0 + 0 - 0 - 0] + \frac{ar_{j,k}}{2(n+1)} \left[n + (-1) - (-1) - (-1)\right] = \frac{ar_{j,k}}{2} & \text{if } j + k = n, \\
\frac{at_{j,k}}{2(n+1)} [0 + 0 - 0 - 0] + \frac{ar_{j,k}}{2(n+1)} \left[-1 + n - (-1) - (-1)\right] = \frac{ar_{j,k}}{2} & \text{if } j + k = n + 2, \\
0 & \text{if } j + k \text{ is odd}, \\
\frac{at_{j,k}}{2(n+1)} [0 + 0 - 0 - 0] + \frac{ar_{j,k}}{2(n+1)} \left[-1 + (-1) - (-1) - (-1)\right] = 0 & \text{otherwise},
\end{cases} \\
&= \begin{cases}
\frac{a}{2}(-1)^{\frac{j}{2} + \frac{k}{2}}(-1)^{\frac{n}{2}} 2 = a(-1)^n = a & \text{if } j + k = n \text{ and } j \text{ is even}, \\
\frac{a}{2}(-1)^{\frac{j+1}{2} + \frac{k+1}{2}}(-1)^{\frac{n}{2}} (-2) = -a(-1)^{n+1} = a & \text{if } j + k = n \text{ and } j \text{ is odd}, \\
\frac{a}{2}(-1)^{\frac{j}{2} + \frac{k}{2}}(-1)^{\frac{n}{2}} 2 = a(-1)^{n+1} = -a & \text{if } j + k = n + 2 \text{ and } j \text{ is even}, \\
\frac{a}{2}(-1)^{\frac{j+1}{2} + \frac{k+1}{2}}(-1)^{\frac{n}{2}} (-2) = -a(-1)^{n+2} = -a & \text{if } j + k = n + 2 \text{ and } j \text{ is odd}, \\
0 & \text{otherwise},
\end{cases} \\
&= [\text{antitridiag}_n(a, 0, -a)]_{j,k}, \qquad j, k \in \{1, \ldots, n\}.
\end{aligned}$$

□

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