Spec. Matrices 2016; 4:80–100 DE GRUYTER OPEN

Research Article Open Access

Ajda Fošner* and Bojan Kuzma

Preserving zeros of Lie product on alternate matrices

DOI 10.1515/spma-2016-0009
Received July 8, 2015; accepted December 13, 2015

Abstract: We study continuous maps on alternate matrices over complex field which preserve zeros of Lie product.

Keywords: Alternate matrix, Lie product, general preserver

MSC: 15A86, 15A99, 15B99

1 Introduction and the main theorem

Linear preserver problem is the problem which concerns the characterization of linear operators on matrix algebras as well as on more general rings and operator algebras that leave certain functions, subsets, relations, etc., invariant. It is one of the most active subjects in the last few decades (for surveys of the topic we refer the reader to the papers [13, 14, 24]). This kind of problems arise in most parts of mathematics because in many cases the corresponding results provide important information on the automorphisms of the underlying structures. In the last few decades a lot of results on linear preservers and also general (non-linear) preservers have been obtained (see [16]).

One among the important preserver problems is classifying maps that preserve commutativity (see, for example, [2, 3, 5, 18–22, 25–27] and references therein). Part of the rationale behind studying this kind of problems is the fact that in associative algebras quite a few elements do commute (for example, every polynomial in x commutes with x). Even more importantly, the assumption of preserving commutativity can be considered as the assumption of preserving zero Lie products. Because of applications in quantum mechanics it is also interesting to study the problem of characterizing general commutativity preserving maps. These are the maps that preserve zeros of Lie product and are not assumed to satisfy any additional algebraic assumption like additivity (see e.g. [17]). Here, let us also mention two papers [4] and [6] dealing with maps preserving zeros of a usual product.

In the present paper we will be interested in describing commutativity preserving maps on the algebra of all $n \times n$ alternate complex matrices. We will only assume that the map is injective and continuous but will not presume it is linear. Let us remark that our results are in the spirit of Šemrl [26] who studied injective continuous maps which preserve zeros of Lie product on complex matrices. Moreover, we emphasize that without imposing some additional regularity conditions on the map, like continuity, we cannot hope for a nice structural result (see also [25, 26]).

Let us list some mostly standard notation. Throughout, $n \geq 3$ will be an integer and M_n will be the algebra of all $n \times n$ matrices over the field of complex numbers. Let E_{ij} be the standard basis of M_n. By $\mathrm{Alt}_n \subset M_n$ we will denote the subspace of all alternate matrices (i.e., matrices with the property $A^t = -A$). Here A^t denotes

***Corresponding Author: Ajda Fošner:** Ajda Fošner, University of Primorska, Faculty of Management, Cankarjeva 5, SI-6000 Koper, Slovenia, E-mail: ajda.fosner@fm-kp.si
Bojan Kuzma: Bojan Kuzma, University of Primorska, FAMNIT, Glagoljaška 8, SI-6000 Koper
and Institute of Mathematics, Physics and Mechanics, Department of Mathematics, Jadranska 19, SI-1000 Ljubljana, Slovenia, E-mail: bojan.kuzma@upr.si

the transpose of a matrix $A \in M_n$. For $A = (a_{ij}) \in M_n$ we will denote by $\overline{A} = \overline{(a_{ij})} = (\overline{a_{ij}})$ the conjugate of A. As usual, we use the notation $\mathrm{diag}(a_1, \dots, a_k)$ to denote the $k \times k$ diagonal matrix with diagonal entries $a_1, \dots, a_k$.

We will study injective continuous maps ϕ with the property

$$AB = BA \implies \phi(A)\phi(B) = \phi(B)\phi(A).$$

This condition can be written as

$$[A, B] = 0 \implies [\phi(A), \phi(B)] = 0,$$

where $[A, B] = AB - BA$ is the Lie product. Note that alternate matrices are closed under the Lie product.

We will represent vectors $x \in \mathbb{C}^n$ as $n \times 1$ complex matrices. Note that the standard basis $e_1, e_2, \dots, e_n$ of $\mathbb{C}^n$ is the set of all $n \times 1$ matrices having all entries equal to zero but one that is equal to one. If $x, y \in \mathbb{C}^n$ are two linearly independent nonzero vectors, then $xy^t - yx^t = x \wedge y$ is a rank two alternate matrix. Every rank two alternate matrix can be written in this form.

A matrix A is said to be nonderogatory if every eigenvalue of A has geometric multiplicity one. Let us point out that a matrix $B \in M_n$ commutes with a nonderogatory matrix $A \in M_n$ if and only if there is a complex polynomial p such that $B = p(A)$ (see [12, p. 135]).

The main idea behind our proof is to utilize the Brouwer's invariance of domain theorem [10, p. 344] stating that if U is an open subset of $\mathbb{R}^m$ and $F\colon U \to \mathbb{R}^m$ is a continuous injective map, then $F(U)$ is open. In particular, there is no injective continuous map from $\mathbb{R}^k$ into $\mathbb{R}^m$ whenever $m < k$. We acknowledge that the same idea was already used before, say in Petek and Šemrl's characterization of continuous maps which preserve adjacency of matrices in one direction (see [23]). Later, the same idea was used in [26] and [9].

The following is a basis for our arguments in Section 2 and Section 3. If $x^t A x = 0$ for every column vector $x \in \mathbb{C}^n$, a complex matrix A is alternate. Each alternate matrix is congruent to a block-diagonal matrix. More precisely, there exists an invertible matrix S such that

$$SAS^t = \bigoplus_{i=1}^{m} J \oplus 0_{n-2m},$$

where

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consequently, the rank of each alternate matrix is an even integer and a minimal possible nonzero rank is 2. Moreover, since $SAS^t = \sum_{i=1}^m E_{(2i-1)\,(2i)} - E_{(2i)\,(2i-1)} = \sum_{i=1}^m e_{2i-1} \wedge e_{2i}$, we have

$$A = \sum_{i=1}^{m} x_i \wedge y_i$$

for some linearly independent vectors $x_i = S^{-1} e_{2i-1}$ and $y_i = S^{-1} e_{2i}$, $i = 1, \dots, m$.

We will need some further properties of alternate matrices over the complex field. We borrow the following notations and facts from the book by Gantmacher, [11, p. 14–21]. In matrices below, all non-specified entries are zero. Firstly, given an integer $k \geq 2$, let

$$V_k := \sum_{i=1}^{k} E_{(k-i+1)\,i} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in M_k \tag{1}$$

be an anti-diagonal $k \times k$ matrix and let

$$J_k := \sum_{i=1}^{k-1} E_{i\,(i+1)} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

be an elementary $k \times k$ Jordan upper-triangular nilpotent. For odd integers q we also define an alternated version of J_q, that is, a matrix

$$J^{(q)} := \sum_{i=1}^{\frac{q-1}{2}} E_{i\,(i+1)} - \sum_{i=\frac{q+1}{2}}^{q-1} E_{i\,(i+1)} = \begin{pmatrix} 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & 0 & -1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & -1 \\ & & & & & & 0 \end{pmatrix} \in M_q.$$

Note that $J^{(q)}$ is an upper-triangular $q \times q$ nilpotent having precisely $\frac{q-1}{2}$ of $+1$ and the same number of -1 on the first upper-diagonal. Note also that $J^{(q)}$ and J_q are similar. Moreover, let Id_k denotes the $k \times k$ identity matrix and, given an integer p and a scalar λ, we define the $(2p) \times (2p)$ matrix $K_\lambda^{(pp)}$ by

$$K_\lambda^{(pp)} := \tfrac{1}{2}\left(\mathrm{Id}_{2p} - \sqrt{-1}V_{2p}\right) \cdot \begin{pmatrix} \lambda\,\mathrm{Id}_p + J_p & 0 \\ 0 & -\lambda\,\mathrm{Id}_p - J_p \end{pmatrix} \cdot \left(\mathrm{Id}_{2p} + \sqrt{-1}V_{2p}\right). \tag{2}$$

Notice that $K_\lambda^{(pp)}$ is similar to $K_{-\lambda}^{(pp)}$, hence by [11, Theorem 4, p. 9] they are also orthogonally similar, that is, there exists a matrix Q such that

$$K_{-\lambda}^{(pp)} = QK_{-\lambda}^{(pp)}Q^{-1} \quad \text{and} \quad Q^{-1} = Q^t. \tag{3}$$

Secondly, given an odd integer q we define

$$K^{(q)} := \tfrac{1}{2}\left(\mathrm{Id}_q - \sqrt{-1}V_q\right) \cdot J^{(q)} \cdot \left(\mathrm{Id}_q + \sqrt{-1}V_q\right). \tag{4}$$

When $q = 1$ we let $K^{(1)}$ be a zero 1×1 matrix.

Example 1.1. *The exact appearance of matrices defined by formulas* (3)–(4) *will play no role in the present paper. For convenience we only show $K_\lambda^{(44)}$ and $K^{(7)}$, here $i^2 = -1$.*

$$K_\lambda^{(44)} = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 1 & -2i\lambda \\ i & 0 & -i & 0 & 0 & 1 & -2i\lambda & 1 \\ 0 & i & 0 & -i & 1 & -2i\lambda & 1 & 0 \\ 0 & 0 & i & 0 & -2i\lambda & 1 & 0 & 0 \\ 0 & 0 & -1 & 2i\lambda & 0 & i & 0 & 0 \\ 0 & -1 & 2i\lambda & -1 & -i & 0 & i & 0 \\ -1 & 2i\lambda & -1 & 0 & 0 & -i & 0 & i \\ 2i\lambda & -1 & 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix},$$

$$K^{(7)} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & i & 0 \\ -1 & 0 & 1 & 0 & i & 0 & i \\ 0 & -1 & 0 & 1+i & 0 & i & 0 \\ 0 & 0 & -1-i & 0 & -1+i & 0 & 0 \\ 0 & -i & 0 & 1-i & 0 & -1 & 0 \\ -i & 0 & -i & 0 & 1 & 0 & -1 \\ 0 & -i & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $d \geq 1$ be an integer. By induction on d, it is easy to prove the following lemma.

Lemma 1.2. *Let $q = 2k+1$ be an odd integer. Then*

$$(J^{(q)})^d = \begin{pmatrix} 0_{(q-d)\times d} & D \\ 0_{d\times d} & 0_{d\times(q-d)} \end{pmatrix},$$

where

$$D = \begin{cases} \mathrm{Id}_{k-d+1} \oplus \mathrm{diag}(-1, (-1)^2, \dots, (-1)^{d-1}) \oplus (-1)^d \, \mathrm{Id}_{k-d+1}, & d \le k \\ (-1)^{d-k+1} \mathrm{diag}(-1, (-1)^2, \dots, (-1)^{2k-d+1}), & k < d \le 2k \\ 0, & d \ge 2k+1 \end{cases}.$$

One easily verifies that $(\mathrm{Id}_k - \sqrt{-1}V_k)^t = (\mathrm{Id}_k - \sqrt{-1}V_k)$ is symmetric with $(\mathrm{Id}_k - \sqrt{-1}V_k)^2 = -2\sqrt{-1}V_k$ and $\frac{1}{2}(\mathrm{Id}_k - \sqrt{-1}V_k) \cdot (\mathrm{Id}_k + \sqrt{-1}V_k) = \mathrm{Id}_k$. Moreover, $J^{(q)}$ is similar to J_q because they are both nilpotents with maximal nilindex. Hence, it is easy to see that $K_\lambda^{(pp)}$ and $K^{(q)}$ are alternate matrices which are similar to $(J_p \oplus J_p)$ and to J_q, respectively. In a sense, these are the only possible cases. Namely, by [11, Corollary on p. 21], any alternate matrix is orthogonally similar to a block-diagonal alternate matrix of the form $\bigoplus_{i=1}^{n_1} K_{\lambda_i}^{(p_i p_i)} \oplus \bigoplus_{j=1}^{n_2} K^{(q_j)}$. More precisely, if $A \in M_n$ is alternate, then there exists a matrix $Q \in M_n$ with $Q^t Q = \mathrm{Id}_n$ and there exist integers $n_1 \ge 0$, $n_2 \ge 0$, integers $p_1, \dots, p_{n_1}$ and $q_1, \dots, q_{n_2}$, and scalars $\lambda_1, \dots, \lambda_{n_1}$ such that

$$QAQ^{-1} = QAQ^t = \bigoplus_{i=1}^{n_1} K_{\lambda_i}^{(p_i p_i)} \oplus \bigoplus_{j=1}^{n_2} K^{(q_j)}. \tag{5}$$

If $n_1 = 0$ or $n_2 = 0$, we omit the corresponding summand.

We are now ready to state our main result. Recall that a polynomial p is called odd if $p(-\lambda) = -p(\lambda)$ for each λ.

Main Theorem. *Let $n \ge 15$ and let $\phi : \mathrm{Alt}_n \to \mathrm{Alt}_n$ be an injective continuous map that preserves zeros of Lie product. Then there exists an orthogonal matrix $Q \in M_n$ and for each $A \in \mathrm{Alt}_n$ there is an odd polynomial $p_A(\lambda)$ such that either*

$$\phi(A) = Qp_A(A)Q^t$$

for all $A \in \mathrm{Alt}_n$, or

$$\phi(A) = Qp_A(\overline{A})Q^t$$

for all $A \in \mathrm{Alt}_n$.

Remark 1.3. We believe that the above theorem is true also for matrices of lower dimensions $n = 5, \dots, 14$, but we have not proved this yet. Note also that in the case $n = 4$ we have at least one additional map. Namely, it can be shown that a map $\Phi : \mathrm{Alt}_4 \to \mathrm{Alt}_4$, which swaps positions $(1,4), (2,3)$ and $(4,1), (3,2)$ and leaves the other entries intact, preserves zeros of Lie product.

2 Preliminaries

Let us denote by $A' = \{X \in \mathrm{Alt}_n : AX = XA\}$ the alternate commutant (a.k.a. a centralizer) of a matrix $A \in \mathrm{Alt}_n$ and by $\mathrm{Sp}(A)$ the spectrum of a matrix A. We will start with a well-known Lumer–Rosenblum theorem [15] (see also [1, p. 2] for a short proof valid in general Banach algebras).

Lemma 2.1. *Suppose that $A \in M_n$ and $B \in M_m$. If $0 \notin \mathrm{Sp}(A) + \mathrm{Sp}(B) = \{\alpha + \beta : \alpha \in \mathrm{Sp}(A), \ \beta \in \mathrm{Sp}(B)\}$, then the equation $AX + XB = 0$ has only a zero solution.*

We proceed with a series of lemmas.

Lemma 2.2. *If* $A = K^{(q)}$ *with* q *odd, then* $\dim A' = \frac{(q-1)}{2}$.

Proof. Note that A is nonderogatory. Hence, its ordinary commutant (i.e., $\{X \in M_q : AX - XA = 0\}$) equals to Poly($A$), i.e., the set of all polynomials in A [12, Theorem 3.2.4.2.]. For the alternate commutant, we must pick out those $X \in \text{Poly}(A)$ which are alternate. Now, if $X = p(A) = \alpha_0 \text{Id} + \alpha_1 A + \cdots + \alpha_{q-1}A^{q-1}$ is alternate, then clearly $Y = \alpha_1 A + \alpha_3 A^3 + \alpha_5 A^5 + \cdots + \alpha_{q-2}A^{q-2}$ is also alternate and so is $X - Y = \sum_{i=0}^{\frac{q-1}{2}} \alpha_{2i}A^{2i}$. But note that $(X-Y)^t = \sum_{i=0}^{\frac{q-1}{2}} \alpha_{2i}(A^t)^{2i} = \sum_{i=0}^{\frac{q-1}{2}} \alpha_{2i}(-A)^{2i} = X - Y$. So, $\alpha_0 = \alpha_2 = \cdots = \alpha_{q-1} = 0$. But then $\dim A' = \frac{(q-1)}{2}$. □

Lemma 2.3. *Let* $p \geq 1$ *be any integer. Then the following holds.*
(i) If $\lambda \neq 0$, *then* $\dim \left(K_\lambda^{(pp)}\right)' = p$.
(ii) If $\lambda = 0$ *and* p *is even, then* $\dim \left(K_\lambda^{(pp)}\right)' = 2p$.
(iii) If $\lambda = 0$ *and* p *is odd, then* $\dim \left(K_\lambda^{(pp)}\right)' = 2p - 1$.

Proof. Suppose that $X \in \text{Alt}_{2p}$ satisfies $K_\lambda^{(pp)}X - XK_\lambda^{(pp)} = 0$. Using the definition of $K_\lambda^{(pp)}$, we arrive at

$$\big((\lambda \text{Id}_p + J_p) \oplus (-\lambda \text{Id}_p - J_p)\big) \cdot S^{-1}XS - S^{-1}XS \cdot \big((\lambda \text{Id}_p + J_p) \oplus (-\lambda \text{Id}_p - J_p)\big) = 0 \tag{6}$$

for $S = \left(\text{Id}_{2p} - \sqrt{-1}V_{2p}\right)$. We now decompose $S^{-1}XS = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ into a 2×2 block matrix. Comparing the individual blocks of (6) we derive four equations

$$\begin{aligned}
(\lambda \text{Id}_p + J_p)X_{11} - X_{11}(\lambda \text{Id}_p + J_p) &= 0,\\
(\lambda \text{Id}_p + J_p)X_{12} + X_{12}(\lambda \text{Id}_p + J_p) &= 0,\\
(\lambda \text{Id}_p + J_p)X_{21} + X_{21}(\lambda \text{Id}_p + J_p) &= 0,\\
(\lambda \text{Id}_p + J_p)X_{22} - X_{22}(\lambda \text{Id}_p + J_p) &= 0.
\end{aligned} \tag{7}$$

The first and the last equations simplify into $J_pX_{11} = X_{11}J_p$ and $J_pX_{22} = X_{22}J_p$, respectively. Since J_p is nonderogatory we see that X_{11} and X_{22} have to be polynomials in J_p. This yields that

$$S^{-1}XS = \begin{pmatrix} f_{11}(J_p) & X_{12} \\ X_{21} & f_{22}(J_p) \end{pmatrix},$$

where f_1, f_2 are complex polynomials and X_{12}, X_{21} satisfy (7). Now, recall that X is alternate. Due to $S^t = S$, this gives $(S^{-1}XS)^t = S^t \cdot (-X) \cdot (S^{-1})^t = -SXS^{-1} = -S^2(S^{-1}XS)S^{-2}$. On the other hand, $S^2 = \left(\text{Id}_{2p} - \sqrt{-1}V_{2p}\right)^2 = -2\sqrt{-1}V_{2p}$, and $S^{-2} = \frac{\sqrt{-1}}{2}V_{2p}$. With this in hand we establish

$$\begin{aligned}
\begin{pmatrix} f_{11}(J_p)^t & X_{21}^t \\ X_{12}^t & f_{22}(J_p)^t \end{pmatrix} &= \begin{pmatrix} f_{11}(J_p) & X_{12} \\ X_{21} & f_{22}(J_p) \end{pmatrix}^t \\
&= -V_{2p} \begin{pmatrix} f_{11}(J_p) & X_{12} \\ X_{21} & f_{22}(J_p) \end{pmatrix} V_{2p} \\
&= -\begin{pmatrix} 0 & V_p \\ V_p & 0 \end{pmatrix} \cdot \begin{pmatrix} f_{11}(J_p) & X_{12} \\ X_{21} & f_{22}(J_p) \end{pmatrix} \cdot \begin{pmatrix} 0 & V_p \\ V_p & 0 \end{pmatrix} \\
&= -\begin{pmatrix} V_pf_{22}(J_p)V_p & V_pX_{21}V_p \\ V_pX_{12}V_p & V_pf_{11}(J_p)V_p \end{pmatrix}.
\end{aligned} \tag{8}$$

Comparing the $(1,1)$ block, we get that $f_{11}(J_p) = -V_pf_{22}(J_p)^tV_p$. Since $Z \mapsto V_pf_{22}(Z)^tV_p$ is a flip map (i.e., reflection over anti-diagonal), we easily conclude $V_pf_{22}(J_p)^tV_p = f_{22}(J_p)$. Hence, $f_{11}(J_p) = -f_{22}(J_p)$.

In the case of off-diagonal blocks we are facing two options.

Case 1. Assume $\lambda \neq 0$. Then, by Lemma 2.1, the second equation and the third equation of (7) have only zero solution. So, $X_{12} = 0 = X_{21}$ and, therefore,

$$(K_\lambda^{(pp)})' = \left\{ S^{-1} \begin{pmatrix} f_{11}(J_p) & 0 \\ 0 & -f_{11}(J_p) \end{pmatrix} S : \ f_{11} \in \mathbb{C}[\lambda] \right\} \tag{9}$$

is of dimension p.

Case 2. Assume $\lambda = 0$. Then, the second equation of (7) simplifies into $J_p X_{12} + X_{12} J_p = 0$ with a solution

$$X_{12} = \sum_{j=0}^{n-1} \alpha_j \sum_{i=1}^{n-j} (-1)^i E_{i\,(i+j)}.$$

However, X being alternate further forces $-X_{12} = (V_p X_{12} V_p)^t$, by comparing the $(2,1)$ entries of (8). Note that the right-hand side is a flip map. It is now easy to verify that, with p even, one has $\alpha_1 = 0 = \alpha_3 = \cdots = \alpha_{p-3} = \alpha_{p-1}$, and, with p odd, one has $\alpha_0 = 0 = \alpha_2 = \cdots = \alpha_{p-3} = \alpha_{p-1}$. Therefore, with p even, at block position $(1,2)$ of X we have $\frac{p}{2}$ linearly independent matrices, while with p odd we have $\frac{p-1}{2}$ linearly independent matrices. We can duplicate the calculations also for X_{21}. We already know that the diagonal blocks are of the form described in (9). Combined, we have

$$\dim(K_\lambda^{(pp)})' = \begin{cases} p + 2 \cdot \left(\frac{p}{2}\right) = 2p\,, & p \text{ is even} \\ p + 2 \cdot \left(\frac{p-1}{2}\right) = 2p - 1\,, & p \text{ is odd} \end{cases}. \quad \square$$

Let $n \geq 3$ and $A \in \mathrm{Alt}_n$. We would like to estimate the dimension of A'. Using orthogonal similarity, we may assume that A is alternate and block-diagonal, say

$$A = \bigoplus_{i=1}^{n_1} K^{(q_i)} \oplus \bigoplus_{j=1}^{n_2} K_0^{(p_j p_j)} \oplus \bigoplus_{k=1}^{n_3} K_{\lambda_k}^{(p_k p_k)},$$

with $q_i \geq 1$ odd, $p_j, p_k \geq 1$, and $\lambda_1, \ldots, \lambda_{n_3}$ nonzero. Let us write $X = (X_{ij}) \in A'$ into block form. Then it is easy to see that the diagonal blocks satisfy $X_{ii} \in (K^{(q_i)})'$, $X_{jj} \in (K_0^{(p_j p_j)})'$, and $X_{kk} \in (K_{\lambda_k}^{(p_k p_k)})'$. By the previous lemmas we have

$$\dim(K^{(q_i)})' = \frac{q_i - 1}{2}\,,$$
$$\dim(K_0^{(p_j p_j)})' = d(p_j) = \begin{cases} 2p_j\,, & p_j \text{ even} \\ 2p_j - 1\,, & p_j \text{ odd} \end{cases},$$
$$\dim(K_{\lambda_k}^{(p_k p_k)})' = p_k.$$

Thus, the dimension of diagonal blocks of matrices from A' equals

$$\sum_{i=1}^{n_1} \frac{q_i - 1}{2} + \sum_{j=1}^{n_2} d(p_j) + \sum_{k=1}^{n_3} p_k.$$

It remains to calculate the dimension of off-diagonal blocks of matrices from A'.

First, consider $X_{ii'}$, $1 \leq i < i' \leq n_1$. This block must satisfy the equation $K^{(q_i)} X_{ii'} = X_{ii'} K^{(q_{i'})}$. Note that $K^{(q_i)}$ is similar to an elementary Jordan upper-triangular matrix J_{q_i}. It is easy to see that the dimension of the solutions of the above equation equals the dimension of the solutions of a similar equation $J_{q_i} Y = Y J_{q_{i'}}$. Also, one easily computes that this dimension is $\min\{q_i, q_{i'}\}$.

Next, consider X_{ij}, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. This block must satisfy the equation $K^{(q_i)} X_{ij} = X_{ij} K_0^{(p_j p_j)}$. Note that $K_0^{(p_j p_j)}$ is similar to $J_{p_j} \oplus J_{p_j}$, so the dimension of the solutions of the above equation equals the dimension of the solutions of the equation $J_{q_i} Y = Y(J_{p_j} \oplus J_{p_j})$. Writing $Y = (Y_1 | Y_2)$ we easily reduce this case to the previous case and find that the dimension of the solutions is $2\min\{q_i, p_j\}$.

Further, consider $X_{jj'}$, $1 \le j < j' \le n_2$. This block must satisfy the equation $K_0^{(p_jp_j)} X_{ij} = X_{ij} K_0^{(p_{j'}p_{j'})}$. Arguing as above, we reduce this equation to the equation $(J_{p_j} \oplus J_{p_j})Y = Y(J_{p_{j'}} \oplus J_{p_{j'}})$, which easily establishes that the dimension of the solutions equals $4\min\{p_j, p_{j'}\}$.

Now, consider X_{ik} or X_{jk}, $1 \le i \le n_1$, $1 \le j \le n_2$, $1 \le k \le n_3$. These blocks must be zero because in the equation they are multiplied with a nilpotent matrix on the left and with an invertible matrix on the right side. By the theory of elementary operators, this yields that $X_{ik} = 0 = X_{jk}$.

At the end, consider $X_{kk'}$, $1 \le k < k' \le n_3$. If $\lambda_k = \lambda_{k'}$, then we easily reduce this case to the previous cases to find that the dimension of these blocks equals $4\min\{k, k'\}$. On the other hand, if $\lambda \ne \lambda'$, then $X_{kk'} = 0$.

Consequently,

$$\begin{aligned} \dim A' \le \sum_{i=1}^{n_1} \frac{q_i - 1}{2} + \sum_{j=1}^{n_2} d(p_j) + \sum_{k=1}^{n_3} p_k + \sum_{1 \le i < i' \le n_1} \min\{q_i, q_{i'}\} + \sum_{\substack{1 \le i \le n_1 \\ 1 \le j \le n_2}} 2\min\{q_i, p_j\} \\ + \sum_{1 \le j < j' \le n_2} 4\min\{p_j, p_{j'}\} + \sum_{1 \le k < k' \le n_3} 4\min\{p_k, p_{k'}\}. \end{aligned} \tag{10}$$

Lemma 2.4. *Let $n \ge 3$ and $A \in \mathrm{Alt}_n$. Then $A' = \mathrm{Alt}_n$ if and only if $A = 0$.*

Proof. If $A = 0$, then, of course, $A' = \mathrm{Alt}_n$. Now, suppose that $A \ne 0$. According to (10), we can easily see that in this case $\dim A' < \frac{n(n-1)}{2}$ and, therefore, $A' \ne \mathrm{Alt}_n$. □

Lemma 2.5. *Let $n \ge 9$ and $A \in \mathrm{Alt}_n$ be a nonzero matrix. Then A is of minimal rank if and only if* $\dim A' \ge \frac{(n-2)(n-3)}{2} + 1$.

Proof. Let $A \in \mathrm{Alt}_n$ be a nonzero alternate matrix. Using orthogonal similarity, we may write A in a block-diagonal form as in the previous investigations.

Now, if $\operatorname{rk} A = 2$ (here, $\operatorname{rk} A$ denotes the rank of the matrix A), then either $A = K^{(3)}$ or $A = K_\lambda^{(11)}$, $\lambda \ne 0$, or $A = K_0^{(22)}$. In the first case we have $n_1 = n - 2$, $q_1 = 3$, $q_2 = \cdots = q_{n_1} = 1$, while $n_2 = 0 = n_3$. By the remarks above, it is easy to see that in this case $\dim A' = \frac{(n-2)(n-3)}{2} + 1$. Likewise we argue in the second case, where $\dim A' = \frac{(n-2)(n-3)}{2} + 1$, and in the last case, where $\dim A' = \frac{(n-4)(n-5)}{2} + 2(n-4) + 4 = \frac{(n-4)(n-1)}{2} + 4$.

It remains to show that if $\operatorname{rk} A > 2$, then the dimension of A' is strictly smaller than $\frac{(n-2)(n-3)}{2} + 1$.

Let m be a dimension of all nilpotent blocks in A. Then we can write $A = A_1 \oplus A_2 \in \mathrm{Alt}_m \oplus \mathrm{Alt}_{n-m}$, where A_1 is nilpotent and A_2 is invertible. It is easy to show that $A' = A_1' \oplus A_2' \subseteq \mathrm{Alt}_m \oplus \mathrm{Alt}_{n-m}$ which gives

$$\dim A' \le \dim \mathrm{Alt}_m + \dim \mathrm{Alt}_{n-m} = \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}.$$

This is strictly below $\frac{(n-2)(n-3)}{2} + 1$ whenever $3 \le m \le n-3$ because $n \ge 9$. Recall that if $m = 2$, then $\dim A' \le \dim \mathrm{Alt}_2 + (\dim \mathrm{Alt}_{n-2} - 1) < \frac{(n-2)(n-3)}{2} + 1$ since $A_2' = \mathrm{Alt}_{n-2}$ is possible only when $A_2 = 0$. Note also that we cannot have $m = n-1$ since A is alternate. Hence, the equality holds only if $m \in \{0, 1, n-2, n\}$.

Case m = 0. Note that in this case A is invertible. Let us fix an eigenvalue $\lambda \in \mathrm{Sp}(A)$ and write A block-diagonally as $A = A_1 \oplus A_2$, where $A_1 = \bigoplus K_\lambda^{p_ip_i}$ is the sum of alternate blocks with eigenvalues $-\lambda, \lambda$, while $\mathrm{Sp}(A_2) \cap \{-\lambda, \lambda\} = \emptyset$. Let m_1 be the dimension of A_1. Observe that $m_1 \notin \{0, 1, n-1\}$ because A is invertible. If $3 \le m_1 \le n-3$, then, arguing as above, we get that $\dim A' < \frac{(n-2)(n-3)}{2} + 1$, as desired. The same is true if either $m_1 = 2$ (by Lemma 2.4, $0 \ne A_2$ is of dimension $n - 2 > 2$) or $m_1 = n-2$ (in this case we just change the roles of A_1 and A_2). At the end, if $m_1 = n$, then we can write $A = \bigoplus_{k=1}^s K_\lambda^{(p_kp_k)}$ with $\sum_{k=1}^s 2p_k = n$ and $1 \le p_1 \le \cdots \le p_s$. We already know (according to previous observations) that the k-th diagonal block of A' has dimension p_k and the (k, h), $k < h$, off-diagonal block of A' has dimension $\min\{2p_k, 2p_h\}$. Therefore,

$$\dim A' = \sum_{k=1}^s p_k + \sum_{1 \le k < h \le s} \min\{2p_k, 2p_h\} = 2sp_1 + 2(s-1)p_2 + \cdots + 2p_s - \frac{n}{2}.$$

Note that $sp_1 \leq p_1 + \cdots + p_s = \frac{n}{2}$ and $(s-1)p_2 \leq p_2 + \cdots + p_s = \frac{n}{2} - p_1 \leq \frac{n}{2} - 1$ and, inductively, $(s-k+1)p_k \leq p_k + \cdots + p_s = \frac{n}{2} - p_1 - \cdots - p_{k-1} \leq \frac{n}{2} - (k-1)$ for every $1 \leq k \leq s$. Therefore,

$$\begin{aligned}\dim A' &= 2\big(sp_1 + (s-1)p_2 + \cdots + p_s\big) - \frac{n}{2}\\ &\leq 2\big(\tfrac{n}{2} + (\tfrac{n}{2} - 1) + \cdots + (\tfrac{n}{2} - s + 1)\big) - \frac{n}{2}\\ &\leq 2\Big(\sum_{k=1}^{\frac{n}{2}} k\Big) - \frac{n}{2} = \frac{n^2}{4}.\end{aligned}$$

But this is strictly below $\frac{(n-2)(n-3)}{2} + 1$ whenever $n \geq 9$.

Case m = 1. In this case we can write $A = 0 \oplus A_2 \in 0 \oplus \mathrm{Alt}_{n-1}$, where A_2 is invertible. As above, we easily see that $\dim A' = \dim A_2' < \frac{((n-1)-2)((n-1)-3)}{2} + 1 \leq \frac{(n-2)(n-3)}{2} + 1$, as desired.

Case m = n − 2. In this case we can write $A = A_1 \oplus K_\lambda^{(11)}$, where A_1 is nonzero. Observe that $A' \subset \mathrm{Alt}_{n-2} \oplus \mathrm{Alt}_2$. Moreover, the inclusion is strict because $A_1 \neq 0$. Hence, $\dim A' < \dim \mathrm{Alt}_{n-2} + 1 = \frac{(n-2)(n-3)}{2} + 1$, as claimed.

Case m = n. In this case A is nilpotent and we can decompose it into $A = \bigoplus_{i=1}^{n_1} K^{(q_i)} \oplus \bigoplus_{j=1}^{n_2} K_0^{(p_j p_j)}$ with $q_1 \geq q_2 \geq \cdots \geq q_{n_1}$ odd and $p_1 \geq p_2 \geq \cdots \geq p_{n_2}$.

If $q_1 \geq 5$, then we have $A = K^{(q_1)} \oplus A_2$, where A_2 contains all other nilpotent blocks of A. If $X = \begin{pmatrix} X_{11} & X_{12} \\ -X_{12}^t & X_{22} \end{pmatrix} \in A'$, then $X_{11} \in (K^{(q_1)})'$, $X_{22} \in A_2' \subseteq \mathrm{Alt}_{n-q_1}$, and $K^{(q_1)} X_{12} = X_{12} A_2$. To compute the dimension of $(1,2)$ block in A', we decompose X_{12} into blocks according to the block structure of $A_2 = \bigoplus_{i=2}^{n_1} K^{(q_i)} \oplus \bigoplus_{j=1}^{n_2} K_0^{(p_j p_j)}$. By previous observations, we see that the dimension of $(1,2)$ block equals $\sum_{i=2}^{n_1} \min\{q_1, q_i\} + \sum_{j=1}^{n_2} 2\min\{q_1, p_j\} \leq \sum_{i=2}^{n_1} q_i + \sum_{j=1}^{n_2} 2p_j = n - q_1$. Hence, we may estimate the dimension of A' as follows

$$\dim A' \leq \dim(K^{(q_1)})' + \dim \mathrm{Alt}_{n-q_1} + (n - q_1) = \frac{q_1 - 1}{2} + \frac{(n-q_1)(n-q_1-1)}{2} + (n - q_1).$$

This is a strictly decreasing function of $q_1 \leq n$ and it is strictly smaller than $\frac{(n-2)(n-3)}{2} + 1$ whenever $q_1 > 3$. So, we are done in this case.

If $p_1 \geq 3$, then we can write $A = K_0^{(p_1 p_1)} \oplus A_2$, where A_2 contains all other nilpotent blocks of A. Proceeding as above, we see that

$$\begin{aligned}\dim A' &\leq \dim(K_0^{(p_1 p_1)})' + \dim \mathrm{Alt}_{n-2p_1} + \Big(\sum_{i=1}^{n_1} 2\min\{p_1, q_i\} + \sum_{j=2}^{n_2} 4\min\{p_1, p_j\}\Big)\\ &\leq 2p_1 + \frac{(n-2p_1-1)(n-2p_1)}{2} + \Big(\sum_{i=1}^{n_1} 2q_i + \sum_{j=2}^{n_2} 4p_j\Big)\\ &= \Big(2p_1 + \sum_{j=2}^{n_2} 2p_j + \sum_{i=1}^{n_1} q_i\Big) + \Big(\sum_{j=2}^{n_2} 2p_j + \sum_{i=1}^{n_1} q_i\Big) + \frac{(n-2p_1-1)(n-2p_1)}{2}\\ &= n + (n - 2p_1) + \frac{(n-2p_1-1)(n-2p_1)}{2}.\end{aligned}$$

Note that this is a strictly decreasing function of $p_1 \in [3, \frac{n}{2}]$ and it takes a maximum at $p_1 = 3$. Since $n \geq 9$ it follows that the maximum is strictly smaller than $\frac{(n-2)(n-3)}{2} + 1$. Hence, $\dim A' < \frac{(n-2)(n-3)}{2} + 1$.

Now, suppose that there are at least two nonzero blocks in the decomposition of A. By the above, we may assume that all blocks in A are of dimension at most 4. First, let A be orthogonal similar to $A = K_0^{(22)} \oplus K_0^{(22)} \oplus A_2$. Then

$$\begin{aligned}\dim A' &\leq 4 + 4 + \frac{(n-8)(n-9)}{2} + \Big(\sum_{i=1}^{n_1} 2\min\{2, q_i\} + \sum_{j=2}^{n_2} 4\min\{2, p_j\}\Big) + \Big(\sum_{i=1}^{n_1} 2\min\{2, q_i\} + \sum_{j=3}^{n_2} 4\min\{2, p_j\}\Big)\\ &\leq \Big(4 + 4 + \sum_{i=1}^{n_1} q_i + \sum_{j=2}^{n_2} 2p_j\Big) + \Big(\sum_{i=1}^{n_1} q_i + \sum_{j=3}^{n_2} 2p_j\Big) + \Big(\sum_{i=1}^{n_1} 2q_i + \sum_{j=2}^{n_2} 4p_j\Big) + \frac{(n-8)(n-9)}{2}\end{aligned}$$

$$=n+(n-8)+2(n-4)+\frac{(n-8)(n-9)}{2}.$$

It is easy to see that this is strictly smaller than $\frac{(n-2)(n-3)}{2}+1$ for $n \geq 9$.

Next, let A be orthogonally similar to $K^{(3)} \oplus K_0^{(22)} \oplus A_2$. In this case

$$\begin{aligned}\dim A' &\leq \dim K^{(3)} + \dim K_0^{(22)} + \dim \mathrm{Alt}_{n-7} + \Big(2\min\{3,2\} + \sum_{i=7}^{n}\min\{3,1\}\Big) + 2(n-7)\\ &= 1+4+\frac{(n-7)(n-8)}{2}+4+3(n-7)\\ &= \frac{(n-7)(n-2)}{2}+9\end{aligned}$$

and it is easy to see that this is strictly smaller than $\frac{(n-2)(n-3)}{2}+1$ for $n \geq 9$.

Assume lastly that A is orthogonally similar to $K^{(3)} \oplus K^{(3)} \oplus A_2$. In this case

$$\begin{aligned}\dim A' &\leq \dim K^{(3)} + \dim K^{(3)} + \dim \mathrm{Alt}_{n-6} + \Big(\min\{3,3\} + \sum_{i=6}^{n}\min\{3,1\}\Big) + \sum_{i=6}^{n}\min\{3,1\}\\ &= 1+1+\frac{(n-6)(n-7)}{2}+3+2(n-6)\\ &= \frac{(n-6)(n-3)}{2}+5\end{aligned}$$

and this is strictly smaller than $\frac{(n-2)(n-3)}{2}+1$ for $n \geq 9$. The proof is completed. □

We will also require an upper bound for dimension of the alternate commutant of rank-four alternate matrices.

Lemma 2.6. *Let $n \geq 15$ and let $A \in \mathrm{Alt}_n$ be of rank-four. Then,* $\dim A' \lneq \frac{(n-3)(n-4)}{2}$

Proof. If $A \in \mathrm{Alt}_n$, then, from its Jordan decomposition and the fact that each block of a block-diagonal alternate matrix is itself alternate and hence of even rank, we see that $\mathrm{rk}\, A = 4$ if and only if, up to orthogonal similarity, (i) $A = K_\lambda^{(22)} \oplus 0_{n-4}$ for $\lambda \neq 0$, (ii) $A = K_0^{(33)} \oplus 0_{n-6}$, (iii) $A = K^{(5)} \oplus 0_{n-5}$, (iv) $A = K_0^{(22)} \oplus K_0^{(22)} \oplus 0_{n-8}$, (v) $A = K_\lambda^{(11)} \oplus K_0^{(22)} \oplus 0_{n-6}$ with $\lambda \neq 0$, (vi) $A = K_\lambda^{(11)} \oplus K_\mu^{(11)} \oplus 0_{n-4}$ for $\lambda, \mu \neq 0$, (vii) $A = K^{(3)} \oplus K^{(3)} \oplus 0_{n-6}$, (viii) $A = K^{(3)} \oplus K_0^{(2,2)} \oplus 0_{n-7}$, or (ix) $A = K^{(3)} \oplus K_\lambda^{(11)} \oplus 0_{n-4}$ with $\lambda \neq 0$.

Now, it is easy to see that in each of the cases (i)–(ix), the dimension of an alternate commutant is a quadratic polynomial in n. Namely, it is an elementary that, with an alternate block matrix X kept fixed, $(X \oplus 0_{n-i})' = \begin{pmatrix} X' & \star \\ -\star^t & \mathrm{Alt}_{n-i}\end{pmatrix}$, where the dimension of a rectangular space of matrices $\star$ is a linear function of n. So, after some calculations we see that under (i), $\dim A' = \frac{(n-9)n+24}{2}$, under (ii), $\dim A' = \frac{(n-9)n+28}{2}$, under (iii), $\dim A' = \frac{(n-9)n+24}{2}$, under (iv), $\dim A' = \frac{(n-9)n+40}{2}$, under (v), $\dim A' = \frac{(n-9)n+28}{2}$, under (vi), $\dim A' \leq \dim(K_\lambda^{(11)} \oplus K_\lambda^{(11)})' = \frac{(n-9)n+28}{2}$, under (vii), $\dim A' = \frac{(n-9)n+28}{2}$, under (viii), $\dim A' = \frac{(n-9)n+32}{2}$, and under (ix), $\dim A' = \frac{(n-9)n+24}{2}$. From here, one easily finds that the matrix from the case (iv) has the maximal dimension of its alternate commutant among all matrices in (i)–(ix). This dimension equals $\frac{(n-9)n+40}{2}$ and, since $n \geq 15$, this is smaller than $\frac{(n-3)(n-4)}{2}$, as claimed.

□

Lemma 2.7. *The set of all diagonalizable alternate matrices is dense in* Alt_n.

Proof. Given an odd integer q, the matrix

$$L^{(q)} = \frac{1}{2}(\mathrm{Id}_q - \sqrt{-1}V_q)(-J^{(q)})^t(\mathrm{Id}_q + \sqrt{-1}V_q) \in M_q$$

is alternate. Namely, if we multiply alternate matrix

$$K^{(q)} = \frac{1}{2}(\mathrm{Id}_q - \sqrt{-1}V_q)J^{(q)}(\mathrm{Id}_q + \sqrt{-1}V_q)$$

with symmetric matrix V_q on the left and on the right, and use $V_qJ^{(q)}V_q = -(J^{(q)})^t$ we find $L^{(q)} = V_qK^{(q)}V_q$. Moreover, also $X = (L^{(q)})^{q-2}$ is alternate because alternate matrices are closed under forming odd powers. Hence, given $\varepsilon > 0$, note that a matrix $K^{(q)} + \varepsilon X$ is alternate and it is similar to $J^{(q)} + \varepsilon(-(J^{(q)})^t)^{q-2} = J^{(q)} + (-1)^{\frac{q+1}{2}}\varepsilon(-E_{(q-1)\,1} + E_{q2})$. By expanding the determinant on the first column, we see that the characteristic polynomial of this matrix is equal to $-2\varepsilon\lambda - \lambda^q$. In particular, for a nonzero ε, the alternate matrix $K^{(q)} + \varepsilon X$ is diagonalizable because it has pairwise distinct eigenvalues and approaches $K^{(q)}$ as $\varepsilon \to 0$.

Similar, given any integer p, the matrix

$$L^{(pp)} := \frac{1}{2}(\mathrm{Id}_{2p} - \sqrt{-1}V_{2p})\,(J_p \oplus (-J_p))^t\,(\mathrm{Id}_{2p} + \sqrt{-1}V_{2p}) \tag{11}$$

is alternate. If p is even, then the matrix $X = (L^{(pp)})^{p-1}$ is alternate as well. For any scalar λ_0 it is easy to see that the characteristic polynomial of $K^{(pp)}_{\lambda_0} + \varepsilon X$ is the same as the characteristic polynomial of $(\lambda_0\,\mathrm{Id}_p + J_p + \varepsilon(J_p^t)^{p-1}) \oplus (-\lambda_0\,\mathrm{Id}_p - J_p - \varepsilon(J_p^t)^{p-1})$, which is equal to $((\lambda_0-\lambda)^p - \varepsilon)\cdot((\lambda_0+\lambda)^p - \varepsilon)$. Since the eigenvalues of the first block are equal to the zeros of the polynomial $(\lambda_0 - \lambda)^p - \varepsilon$, which are pairwise distinct if $\varepsilon \neq 0$, we see that the first block is diagonalizable for every $\varepsilon > 0$. The same is true for the second block. Hence, $K^{(pp)}_{\lambda_0} + \varepsilon X$ is diagonalizable alternate matrix whenever $\varepsilon > 0$.

If p is odd, then the matrix $X = (L^{(pp)})^{p-2}$ is alternate because it is an odd power of an alternate matrix $L^{(pp)}$. As above, we see that the characteristic polynomial of $K^{(pp)}_{\lambda_0} + \varepsilon X$ is equal to $(\lambda_0 - \lambda)\big((\lambda_0 - \lambda)^{p-1} - 2\varepsilon\big)\cdot(-1)(\lambda_0+\lambda)\big((\lambda_0+\lambda)^{p-1} - 2\varepsilon\big)$ (decompose the determinant of each block in the matrix $(\lambda_0\,\mathrm{Id}_p + J_p + \varepsilon(J_p^t)^{p-2}) \oplus (-\lambda_0\,\mathrm{Id}_p - J_p - \varepsilon(J_p^t)^{p-2})$, which is similar to $K^{(pp)}_{\lambda_0} + \varepsilon X$, by the first column). Hence, we deduce that the matrix $K^{(pp)}_{\lambda_0} + \varepsilon X$ is diagonalizable and it approaches $K^{(pp)}_{\lambda_0}$ as $\varepsilon \to 0$. Consequently, the set of all diagonalizable alternate matrices is dense in Alt_n. □

For a nonzero vector $x \in \mathbb{C}^n$ we denote

$$L_x := \{x \wedge v \;:\; v \in \mathbb{C}^n\}.$$

It is easy to see that $L_x = L_y$ precisely when x, y are linearly dependent. Namely, otherwise we could assume $x = e_1$ and $y = e_2$. But then $e_1 \wedge e_3 \in L_{e_1}$ is not in L_{e_2}. It is also easy to show that $L_x \cap L_y = \mathbb{C}(x \wedge y)$ whenever $L_x \neq L_y$.

In the proof of the main theorem we will also need the following lemmas.

Lemma 2.8. *If $A, B \in L_x$ are linearly independent, then* $\dim(A' \cap B') \geq \frac{(n-3)(n-4)}{2}$.

Proof. Write $A = x \wedge y$ and $B = x \wedge z$. Since A, B are linearly independent, so are the vectors x, y, z. Thus, there exists invertible P such that $Px = e_1$, $Py = e_2$, and $Pz = e_3$. Now, pick any alternate $X \in A'$. Then, $(Xx)y^t - (Xy)x^t = XA = AX = x(X^ty)^t - y(X^tx)^t = -x(Xy)^t + y(Xx)^t$. If we multiply this equation on the right by P and on the left by P^t, we get $(PXx)e_2^t - (PXy)e_1^t = -e_1(PXy)^t + e_2(PXx)^t$. Comparing both sides, we easily obtain $PXx = \alpha e_1 + \beta e_2$ and $PXy = \gamma e_1 - \alpha e_2$ for some scalars α, β, γ. Therefore,

$$Xx = \alpha x + \beta y, \qquad Xy = \gamma x - \alpha y.$$

Likewise we derive

$$Xx = \alpha' x + \beta' z, \qquad Xz = \gamma' x - \alpha' z$$

for any alternate $X \in B'$. Since x, y, z are linearly independent, $X \in A' \cap B'$ implies that

$$Xx = \alpha x, \qquad Xy = \gamma x - \alpha y, \qquad Xz = \gamma' x - \alpha z.$$

Inversely, given X with the above property, X clearly belongs to $A' \cap B'$, provided it is alternate. To check the alternateness, we let $Q = P^{-1}$. Then $x = Qe_1$. We can claim similar for y and z. Thus, the above equations reduce into

$$Q^tXQe_1 = \alpha Q^tQe_1, \quad Q^tXQe_2 = \gamma Q^tQe_1 - \alpha Q^tQe_2,$$

$$Q^tXQe_3 = \gamma' Q^tQe_1 - \alpha Q^tQe_3.$$

Note that X is alternate precisely when Q^tXQ is. So, writing the symmetric matrix Q^tQ as $Q^tQ = (s_{ij})$, the above three conditions give

$$Q^tXQ = \begin{pmatrix} \alpha s_{11} & \gamma s_{11} - \alpha s_{12} & \gamma' s_{11} - \alpha s_{13} & \star_1 \\ \alpha s_{12} & \gamma s_{12} - \alpha s_{22} & \gamma' s_{12} - \alpha s_{23} & \star_2 \\ \alpha s_{13} & \gamma s_{13} - \alpha s_{23} & \gamma' s_{13} - \alpha s_{33} & \star_3 \\ \vdots & \vdots & \vdots & \bigstar \end{pmatrix},$$

where $\star_i$ represents the remainder of the i-th row. Clearly, this is an alternate matrix precisely when $\alpha s_{11} = 0 = \gamma s_{12} - \alpha s_{22} = \gamma' s_{13} - \alpha s_{33}$ and $\gamma s_{11} - \alpha s_{12} = -\alpha s_{12}$, etc. So, $\alpha = 0 = \gamma = \gamma'$, unless the upper-left 3×3 block of Q^tQ is zero. However, we are free to choose the alternate $(n-3)\times(n-3)$ lower-right block, represented by $\bigstar$. There are precisely $\frac{(n-3)(n-4)}{2}$ linearly independent alternate matrices in this block, which gives our lower estimate. □

Remark 2.9. The upper bound for $\dim A' \cap B'$ equals $3 + \frac{(n-3)(n-4)}{2}$. This is possible precisely when the upper-left 3×3 block of Q^tQ is zero. Then, $X \in A' \cap B'$ is equivalent to

$$Q^tXQ = \begin{pmatrix} 0 & 0 & 0 & -\alpha s_{14} & \dots \\ 0 & 0 & 0 & -(\gamma s_{14} - \alpha s_{24}) & \dots \\ 0 & 0 & 0 & -(\gamma' s_{14} - \alpha s_{34}) & \dots \\ \alpha s_{14} & \gamma s_{14} - \alpha s_{24} & \gamma' s_{14} - \alpha s_{34} & & \\ \vdots & \vdots & \vdots & \bigstar & \end{pmatrix}.$$

This can happen, for example, if $A = (e_1 + ie_2) \wedge (e_3 + ie_4)$ and $B = (e_1 + ie_2) \wedge (e_5 + ie_6)$ (i.e., $A^2 = 0 = B^2 = AB = BA$).

Lemma 2.10. *Let $\Omega \subset \mathrm{Alt}_n$ be a subset of alternate matrices such that* $\operatorname{rk} A, \operatorname{rk} B, \operatorname{rk}(A+B) \le 2$ *for every $A, B \in \Omega$. Then either $\Omega \subseteq L_{\hat{x}}$ for some nonzero vector $\hat{x}$ or there exists invertible $P \in M_n$ such that $\Omega \subseteq P(\mathrm{Alt}_3 \oplus 0_{n-3})P^t$ lies in at most three dimensional subspace.*

Proof. Let us choose two linearly independent matrices $A_1, A_2 \in \Omega$ and write them as $A_i = x_i \wedge y_i$, $i = 1, 2$. Since $\operatorname{rk}(A_1 + A_2) \le 2$ we may assume that $x_1 = x_2 := \hat{x}$. Namely, at least one among the triples x_1, y_1, x_2 or x_1, y_1, y_2 must be linearly dependent. Without loss of generality, assume the first one is. Write $x_2 = \alpha x_1 + \beta y_1$ and note that $x_1 \wedge y_1 = (\alpha x_1 + \beta y_1) \wedge (\frac{1}{\alpha})y_1 = x_2 \wedge (\frac{1}{\alpha})y_1$, if $\alpha \ne 0$. However, if $\alpha = 0$, then we may as well assume that $\beta = 1$ and we have $x_1 \wedge y_1 = y_1 \wedge (-x_1) = x_2 \wedge (-x_1)$. In any case, $A_1, A_2 \in L_{x_2}$. Moreover, with the help of some invertible matrix P we may further assume that $\hat{x} = e_1$, $y_1 = e_2$, and $y_2 = e_3$, that is, $A_1 = e_1 \wedge e_2 \in L_{e_1}$ and $A_2 = e_1 \wedge e_3 \in L_{e_1}$. Suppose that there exists a matrix $A_3 = x_3 \wedge y_3 \in \Omega \setminus L_{e_1}$. Due to $\operatorname{rk}(A_1 + A_3) \le 2$, either x_3 or y_3 is a linear combination of e_1, e_2. Without loss of generality we may assume the former and write $x_3 = \alpha e_1 + \beta e_2$. Clearly, $\beta \ne 0$, otherwise $A_3 \in L_{e_1}$. Similarly, $\operatorname{rk}(A_3 + A_2) \le 2$ implies that at least one among x_3 and y_3 is a linear combination of e_1, e_3. But for x_3 this is already impossible, so $y_3 = \gamma e_1 + \delta e_3$. Consequently, $A_3 \in \mathrm{Alt}_3 \oplus 0_{n-3}$ whenever $A_3 \in \Omega \setminus L_{e_1}$.

It remains to show that $\Omega \subseteq \mathrm{Alt}_3 \oplus 0_{n-3}$. In fact, pick any matrix $A_4 \in \Omega$ and assume erroneously that it is linearly independent of A_1, A_2, A_3. Then, as before, due to $\operatorname{rk}(A_2 + A_4), \operatorname{rk}(A_1 + A_4) \le 2$ we must have $A_4 \in L_{e_1}$. So, $A_4 = e_1 \wedge y_4$, where y_4 is linearly independent of e_1, e_2, e_3. But then, $A_3 + A_4 = (\frac{\alpha}{\beta} e_1 + e_2) \wedge (\beta\gamma e_1 + \beta\delta e_3) + e_1 \wedge y_4$ is of rank four, a contradiction. □

For nonzero vectors $x, y \in \mathbb{C}^n$, we denote $[x] = \{\alpha x \ :\ \alpha \in \mathbb{C}\}$ and $[x] + [y] = \{\alpha x + \beta y \ :\ \alpha, \beta \in \mathbb{C}\}$.

Lemma 2.11. *If x, y are linearly independent vectors, then $[z] \subseteq [x] + [y]$ precisely when the intersection $(L_x \cap L_z) \cap (L_y \cap L_z)$ is nonzero.*

Proof. Suppose $[z] \subseteq [x] + [y]$. Then $z = \alpha x + \beta y$ for some scalars α, β. If $\beta = 0$ then $L_z = L_x$ and $(L_x \cap L_z) \cap (L_y \cap L_z) = L_x \cap L_y = \mathbb{C}(x \wedge y) \neq 0$. The same is true when $\alpha = 0$. Now, if α and β are both nonzero, then $(L_x \cap L_z) = \mathbb{C}(x \wedge z) = \mathbb{C}x \wedge (\alpha x + \beta y) = \mathbb{C}(x \wedge y)$ and $(L_y \cap L_z) = \mathbb{C}y \wedge (\alpha x + \beta y) = \mathbb{C}(y \wedge x) = \mathbb{C}(x \wedge y) \neq 0$.

Now, assume that $(L_x \cap L_z) \cap (L_y \cap L_z) \neq 0$. Then $\lambda(x \wedge z) = \mu(y \wedge z)$ for some nonzero scalars λ, μ. If x, y, z were linearly independent, there would exist an invertible P such that $Px = e_1, Py = e_2$, and $Pz = e_3$. Multiplying the above identity with P on the left and P^t on the right side yields that $\lambda(e_1 \wedge e_3) = \mu(e_2 \wedge e_3)$, a contradiction. □

3 Proof of the main theorem

Let $\phi : \mathrm{Alt}_n \to \mathrm{Alt}_n$ be an injective continuous map that preserves zeros of Lie product. Then for every $A \in \mathrm{Alt}_n$, we have

$$\phi(A') \subseteq \left(\phi(A)\right)'.$$

In particular,

$$\phi(\mathrm{Alt}_n) = \phi(0') \subseteq \phi(0)' \subseteq \mathrm{Alt}_n.$$

Since ϕ is injective and continuous, it follows by the invariance of domain theorem that $\phi(0)'$ cannot be contained inside a proper linear subspace of Alt_n. This yields that $\phi(0)' = \mathrm{Alt}_n$ and, consequently, $\phi(0) = 0$. The remainder of the proof is divided into several steps.

Step 1. The set $\Xi_2 \subseteq \mathrm{Alt}_n$ of rank two alternate matrices is invariant under ϕ. Moreover, $\phi(\mathbb{C}A) \subseteq \mathbb{C}\phi(A)$ for every $A \in \Xi_2$.

Let $0 \neq A \in \mathrm{Alt}_n$. By Lemma 2.5 $\dim A' \geq \frac{(n-2)(n-3)}{2} + 1$ if and only if $\operatorname{rk} A = 2$. This yields that every $A \in \mathrm{Alt}_n$ of rank two is mapped by ϕ into an alternate matrix of rank two. Namely otherwise, ϕ would map A', which is a linear subspace of dimension at least $\frac{(n-2)(n-3)}{2} + 1$, continuously and injectively into $\phi(A)'$, whose dimension is strictly smaller than $\frac{(n-2)(n-3)}{2} + 1$. But this is impossible by the invariance of domain theorem. Therefore, for every $A \in \Xi_2$ of rank two there exists $B \in \Xi_2$ such that $\phi(A) = B$.

It remains to prove $\phi(\mathbb{C}A) \subseteq \mathbb{C}B$. If there was a nonzero complex number λ such that $\phi(\lambda A) \notin \mathbb{C}B$, then ϕ would map $A' = (\lambda A)'$ injectively and continuously into $B' \cap \phi(\lambda A)'$. Now, any $X \in B' \cap \phi(\lambda A)'$ also commutes with $B + \phi(\lambda A)$. Hence, if $\operatorname{rk}(B + \phi(\lambda A)) \geq 4$, then, by Lemma 2.5, $\dim B' \cap \phi(\lambda A)' \leq \dim(B + \phi(\lambda A))' \lneq \frac{(n-2)(n-3)}{2} + 1 \leq \dim A'$. However, if $\operatorname{rk}(B + \phi(\lambda A)) = 2$, then Remark 2.9 implies $\dim(B' \cap \phi(\lambda A)') \lneq \dim A'$. Each of these two cases contradicts Brouwer's invariance of domain theorem.

Step 2. Let x be a nonzero vector. Then, $\operatorname{rank}(\phi(x \wedge y_1) + \phi(x \wedge y_2)) \leq 2$ for each $x \wedge y_1, x \wedge y_2 \in L_x$.

Indeed, write $\phi(x \wedge y_1) = \tilde{x}_1 \wedge \tilde{y}_1$ and $\phi(x \wedge y_2) = \tilde{x}_2 \wedge \tilde{y}_2$. Note that

$$\phi\left((x \wedge y_1)' \cap (x \wedge y_2)'\right) \subseteq (\tilde{x}_1 \wedge \tilde{y}_1)' \cap (\tilde{x}_2 \wedge \tilde{y}_2)'.$$

Now, suppose erroneously that $\operatorname{rk}(\tilde{x}_1 \wedge \tilde{y}_1 + \tilde{x}_2 \wedge \tilde{y}_2) > 2$. Then, by Lemma 2.8, the space $(x \wedge y_1)' \cap (x \wedge y_2)'$ of dimension at least $\frac{(n-3)(n-4)}{2}$ would be mapped injectively and continuously into a subspace $(\tilde{x}_1 \wedge \tilde{y}_1)' \cap (\tilde{x}_2 \wedge \tilde{y}_2)' \subseteq (\tilde{x}_1 \wedge \tilde{y}_1 + \tilde{x}_2 \wedge \tilde{y}_2)'$ which by Lemma 2.6 has a strictly smaller dimension, a contradiction to the invariance of domain theorem. Therefore, $\operatorname{rk}(\tilde{x}_1 \wedge \tilde{y}_1 + \tilde{x}_2 \wedge \tilde{y}_2) \leq 2$.

Step 3. For each L_x there exists some $L_{\tilde{x}}$ such that $\phi(L_x) \subseteq L_{\tilde{x}}$.

In fact, if this would not be the case for some x, then the previous step together with Lemma 2.10 would imply that ϕ maps the $n-1$ dimensional space L_x continuously and injectively into at most three dimensional subspace, a contradiction.

Step 4. Let x be a nonzero vector such that $\phi(L_x) \subseteq L_{\tilde{x}} \cap L_{\tilde{x}'}$ for some vectors $\tilde{x}$ and $\tilde{x}'$. Then $\tilde{x}$ and $\tilde{x}'$ must be linearly dependent.

To see this, note that $\phi(L_x) \subseteq L_{\tilde{x}}$ implies $\dim(\operatorname{Lin} \phi(L_x)) = \dim L_{\tilde{x}} = n - 1$ by the invariance of domain theorem (here, $\operatorname{Lin} \phi(L_x)$ denotes the linear span of the set $\phi(L_x)$). But then $\phi(L_x) \subseteq L_{\tilde{x}} \cap L_{\tilde{x}'}$ implies that $L_{\tilde{x}} = \operatorname{Lin} \phi(L_x) = L_{\tilde{x}'}$ which is possible precisely when $\tilde{x}, \tilde{x}'$ are linearly dependent.

We can hence introduce a well-defined map φ on a projective space $\mathbb{PC}^n$ by letting

$$\varphi : [x] \mapsto [\tilde{x}],$$

where $\tilde{x}$ satisfies $\phi(L_x) \subseteq L_{\tilde{x}}$. To prove that this is a projectivity, assume that $[z] \subseteq [x] + [y]$. We can clearly suppose that x, y are linearly independent, otherwise $[x] = [y] = [z]$ and, hence, there is nothing to prove. By Lemma 2.11, we must have $(L_x \cap L_z) \cap (L_y \cap L_z) \neq \{0\}$. Since ϕ is injective and $\phi(0) = 0$, this further implies

$$\{0\} \neq \phi\big((L_x \cap L_z) \cap (L_y \cap L_z)\big) \subseteq (\phi(L_x) \cap \phi(L_z)) \cap (\phi(L_y) \cap \phi(L_z)).$$

Thus, by Lemma 2.11, $\varphi([z]) \subseteq \varphi([x]) + \varphi([y])$.

Step 5. The map φ is injective.

Let x_1, x_2 be two linearly independent vectors and suppose that $\varphi([x_1]) = \varphi([x_2]) = [\tilde{x}]$ for some $\tilde{x}$. Then, by the definition of φ and the invariance of domain theorem, $\phi(L_{x_1})$ and $\phi(L_{x_2})$ are open subsets of $L_{\tilde{x}}$ containing $\phi(0) = 0$. Let $x_1 \wedge y \in L_{x_1} \backslash L_{x_2}$. As $\lim_{\lambda\to 0} \phi(\lambda x_1 \wedge y) = \phi(0) = 0$, we can find a small enough nonzero complex number λ such that $\phi(\lambda x_1 \wedge y) \in \phi(L_{x_2})$, contradicting the injectivity of ϕ. Thus, φ is injective.

Step 6. The span of lines in $\operatorname{Im}\varphi$ equals $\mathbb{PC}^n$. Hence, $\operatorname{Im}\varphi$ is not contained in a projective line.

To prove this, denote $[\tilde{e}_1] := \varphi([e_1])$. Note that L_{e_1} and $L_{\tilde{e}_1}$ are both $n-1$ dimensional linear subspaces and ϕ maps continuously and injectively the first one into the second one. So, by the invariance of domains, $\phi(L_{e_1})$ is an open subset inside $L_{\tilde{e}_1}$ that contains a zero matrix. Therefore, given any vector z, linearly independent of $\tilde{e}_1$, there exists some y such that $\phi(e_1 \wedge y) = \lambda\tilde{e}_1 \wedge z$ for small enough nonzero scalar λ. Clearly, $e_1 \wedge y \neq 0$, because $\phi(0) = 0$. Therefore, e_1, y are linearly independent. Now, $e_1 \wedge y \in L_{e_1} \cap L_y$. Thus, $\lambda\tilde{e}_1 \wedge z = \phi(e_1 \wedge y) \in \phi(L_{e_1} \cap L_y) \subseteq \phi(L_{e_1}) \cap \phi(L_y) \subseteq L_{\tilde{e}_1} \cap L_{\tilde{y}}$. But this is possible only when $\tilde{y} = \alpha\tilde{e}_1 + \beta z$ for some scalars α, β. Moreover, $\beta \neq 0$ because linear independence of e_1, y and injectivity of φ forces $L_{\tilde{y}} \neq L_{\tilde{e}_1}$. We may now assume that $\beta = 1$. So, given any z, linearly independent from $\tilde{e}_1$, there is some $y \neq 0$ and a scalar α such that $\varphi([y]) = [\alpha\tilde{e}_1 + z]$. But then the image of φ spans $\mathbb{PC}^n$: it clearly contains $[\tilde{e}_1]$ and it also contains $[\alpha_i\tilde{e}_1 + e_i]$ for every i (except for one possible exception, when $\tilde{e}_1, e_i$ are linearly dependent). But then this n lines (or $n + 1$ lines, if $\tilde{e}_1, e_i$ are always independent) span the whole projective space.

Step 7. There exists an invertible matrix $\hat{Q}$ and a field homomorphism $\sigma : \mathbb{C} \to \mathbb{C}$ such that for each nonzero alternate $x \wedge y$ we have $\phi(x \wedge y) = \lambda_{x\wedge y}\hat{Q}(x^\sigma \wedge y^\sigma)\hat{Q}^t$ for appropriate scalar $\lambda_{x\wedge y} \in \mathbb{C}$.

By a nonsurjective version of the fundamental theorem of projective geometry (see e.g. [7, Theorem 3.1]) there exists a field homomorphism $\sigma : \mathbb{C} \to \mathbb{C}$ and a σ-linear map $T : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi([x]) = [Tx]$ for every line $[x] \in \mathbb{PC}^n$. Consequently, $\phi(L_x) \subseteq L_{Tx}$. Thus, from $x \wedge y \in L_x \cap L_y$ we derive that $\phi(x \wedge y) \in \phi(L_x \cap L_y) = \phi(L_x) \cap \phi(L_y) \subseteq \mathbb{C}\,(Tx \wedge Ty)$. It follows that $\phi(x \wedge y) = \frac{1}{\beta}(Tx) \wedge Ty$, where a scalar β depends on the operator $x \wedge y$. It is also easy to see that $Tx = \hat{Q}x^\sigma$ for some matrix $\hat{Q} \in M_n$. Since $\operatorname{Lin}(\operatorname{Im}\varphi) = \mathbb{C}^n$, the matrix $\hat{Q}$ is invertible. Moreover, $\phi(x \wedge y) = \frac{1}{\beta}\hat{Q}x^\sigma \wedge \hat{Q}y^\sigma = \frac{1}{\beta}\hat{Q}(x^\sigma \wedge y^\sigma)\hat{Q}^t$.

Step 8. Homomorphism σ is either a complex conjugation or the identity.

To see this, consider $\phi(e_1 \wedge (\alpha e_2 + e_3)) = \mu_\alpha\hat{Q}(e_1 \wedge (\sigma(\alpha)e_2 + e_3))\hat{Q}^t$ for appropriate nonzero scalars μ_α. The continuity of ϕ forces the continuity of the functions $\mu_\alpha\sigma(\alpha)$ and μ_α. This gives that σ is a continuous homomorphism of $\mathbb{C}$. Thus, it is either a complex conjugation or the identity. Replacing ϕ by the map $X \mapsto \phi(\overline{X})$, $X \in \operatorname{Alt}_n$, if necessary, we may assume in the sequel that σ is the identity.

Step 9. The matrix $\hat{Q}^t\hat{Q}$ is a scalar multiple of the identity matrix.

To see this, choose any orthogonal matrix W. Given any distinct i, j, k, l, observe that $W(e_i \wedge e_j)W^t$ commutes with $W(e_k \wedge e_l)W^t$. Hence,

$$\begin{aligned}0 &= \phi(W(e_i \wedge e_j)W^t)\phi(W(e_k \wedge e_l)W^t) - \phi(W(e_k \wedge e_l)W^t)\phi(W(e_i \wedge e_j)W^t)\\ &= \mu\big(\hat{Q}W(e_i \wedge e_j)W^t\hat{Q}^t\hat{Q}W(e_k \wedge e_l)W^t\hat{Q}^t - \hat{Q}W(e_k \wedge e_l)W^t\hat{Q}^t\hat{Q}W(e_i \wedge e_j)W^t\hat{Q}^t\big),\end{aligned}$$

where $\mu = \lambda_{W(e_i\wedge e_j)W^t}\lambda_{W(e_k\wedge e_l)W^t} \neq 0$. In particular,

$$(e_i \wedge e_j)W^t\hat{Q}^t\hat{Q}W(e_k \wedge e_l) = (e_k \wedge e_l)W^t\hat{Q}^t\hat{Q}W(e_i \wedge e_j).$$

Comparing the (i,k), (i,l), (j,k), and (j,l) positions, we derive that the entries of $W^t\hat{Q}^t\hat{Q}W$ corresponding to those positions vanish. Due to arbitrariness of i,j,k,l, we see that $W^t\hat{Q}^t\hat{Q}W$ is diagonal for every orthogonal W. Choosing $W = \mathrm{Id}_n$, we conclude that $\hat{Q}^t\hat{Q}$ is diagonal. Furthermore, choosing $W = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\right) \oplus \mathrm{Id}_{n-2}$, we see that the entries at the positions $(1,1)$ and $(2,2)$ of $\hat{Q}^t\hat{Q}$ are equal. Choosing in this succession $W = 0_i \oplus \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\right) \oplus \mathrm{Id}_{n-2-i}$ for $i = 1,\dots,n-2$, we derive that all diagonal entries of $\hat{Q}^t\hat{Q}$ are equal. This implies that $\hat{Q}^t\hat{Q} = \lambda\,\mathrm{Id}_n$ for some nonzero scalar λ, as desired. Therefore, $Q := \frac{1}{\sqrt{\lambda}}\hat{Q}$ is an orthogonal matrix with $\phi(X) = \mu_X\hat{Q}X\hat{Q}^t = \frac{\mu_X}{\lambda}QXQ^t$ for every rank two alternate matrix X.

Step 10. According to above observations, we may replace ϕ by the map $X \mapsto Q^t\phi(X)Q$. The new map is still continuous, preserves zeros of Lie product, and fixes every alternate matrix of minimal rank modulo scalar multiples.

Step 11. If D is a diagonalizable alternate matrix then there exists an odd polynomial $p_D(\lambda)$ (i.e., $p_D(\lambda) = -p_D(-\lambda)$) such that $\phi(D) = p_D(D)$ (i.e., ϕ acts locally polynomially on diagonalizable alternate matrices).

To see this, let D be a diagonalizable alternate matrix. It follows from (5) that there exists an orthogonal matrix W such that

$$D = \begin{cases} W(\alpha_1 J) \oplus (\alpha_2 J) \oplus \cdots \oplus (\alpha_k J)W^t; & n \text{ even}\\ W\big((\alpha_1 J) \oplus (\alpha_2 J) \oplus \cdots \oplus (\alpha_k J) \oplus 0\big)W^t; & n \text{ odd}\end{cases}$$

for some scalars α_i, where J is a 2×2 matrix of the form $J = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Clearly, D commutes with a minimal rank matrix $A_i = W(0_{2i} \oplus J \oplus 0_{n-2i-2})W^t$ for every $i = 1,\dots,\lfloor n/2\rfloor$. Consequently, also $\phi(D)$ commutes with $\phi(A_i) = \gamma_i A_i \neq 0$, which forces $\phi(D) = W((\beta_1 J) \oplus (\beta_2 J) \oplus \cdots \oplus (\beta_k J))W^t$, respectively, $\phi(D) = W((\beta_1 J) \oplus (\beta_2 J) \oplus \cdots \oplus (\beta_k J) \oplus 0)W^t$ for some scalars β_i. Take a block alternate matrix $X = \begin{pmatrix}J & J\\ J & J\end{pmatrix} \oplus 0$ and note that this matrix is of minimal rank. If, say $\alpha_1 = \alpha_2$, then X commutes with D. Hence, also its image, $\phi(X) = \gamma_X X \neq 0$, commutes with $\phi(D)$ which is possible only when $\beta_1 = \beta_2$. Likewise we show that $\alpha_i = \alpha_j$ forces $\beta_i = \beta_j$, $i,j = 1,\dots,k$.

It is known that there exists a polynomial $p_D(\lambda)$ with $p_D(\alpha_i) = \beta_i$ for $i = 1,\dots,n$. Hence, $\phi(D) = p_D(D)$. In other words, ϕ acts locally as polynomial on diagonalizable alternate matrices. Moreover, we will show that these polynomials can be chosen to be odd (recall that a polynomial $f(\lambda)$ is odd if $f(-\lambda) = -f(\lambda)$, or equivalently, if $f(\lambda) = \sum_i a_{2i+1}\lambda^{2i+1}$ for some scalars $a_{2i+1} \in \mathbb{C}$). In fact, suppose that A and $p(A)$ are both alternate for some polynomial $p(\lambda)$. Decompose $p(\lambda) = p_+(\lambda) + p_-(\lambda)$ into a sum of even and odd parts. Clearly, A^{2i} are symmetric matrices for each nonnegative integer i. So, $p_+(A)$ is a linear combination of symmetric matrices and, hence, it is symmetric. Likewise one can show that $p_-(A)$ is alternate. So, the symmetric matrix $p_+(A) = p(A) - p_-(A)$ is a difference of two alternate matrices $p(A)$ and $p_-(A)$, whence $p_+(A) = 0 = p(A) - p_-(A)$. Thus, we can assume that $p(\lambda) = p_-(\lambda)$ is an odd polynomial, i.e., $p(-\lambda) = -p(\lambda)$.

Step 12. $\phi(X) \in X'$ for every alternate X.

In fact, by Step 11 we know that ϕ is a locally polynomial map on the set of all diagonalizable alternate matrices. In particular, every diagonalizable alternate matrix A commutes with $\phi(A)$. According to Lemma 2.7, the set of all diagonalizable alternate matrices is dense in Alt_n. Thus, by the continuity of ϕ, every $X \in \mathrm{Alt}_n$ commutes with $\phi(X)$. In particular $\phi(X) \in X'$, and, if alternate X is nonderogatory, then $\phi(X) \in X' = \mathrm{Poly}(X) \cap \mathrm{Alt}_n$ [12, Theorem 3.2.4.2.].

Step 13. Action of ϕ on a general alternate A.

As explained in (5), there exists an orthogonal Q such that

$$A = Q^t\Big(\bigoplus_{i=1}^{n_1} K_{\lambda_i}^{(p_i p_i)} \oplus \bigoplus_{j=1}^{n_2} K^{(q_j)}\Big)Q = Q^t\big(A_1 \oplus A_2\big)Q. \tag{12}$$

To ease the proof, we will continue in substeps. Also, we will temporarily modify ϕ into a map

$$\phi_Q(X) := Q\phi(Q^t XQ)Q^t,$$

which clearly shares the same properties as ϕ. In particular, all steps until the present one hold for ϕ_Q as well. The benefit of introducing ϕ_Q is that $\phi(A)$ is a polynomial in A if and only if $\phi_Q(A_1 \oplus A_2)$ is a polynomial in block-diagonal $A_1 \oplus A_2$.

Substep 13.a. There exist odd polynomials $f_1(\lambda), \dots, f_{n_1}(\lambda)$ and odd polynomials $h_1(\lambda), \dots, h_{n_2}(\lambda)$ such that

$$\phi_Q(A_1 \oplus A_2) = \Big(\bigoplus_{i=1}^{n_1} f_i(K_{\lambda_i}^{(p_ip_i)}) \oplus \bigoplus_{j=1}^{n_2} h_j(K^{(q_j)})\Big) \tag{13}$$

To see this, choose continuous functions $\lambda_1(\varepsilon), \dots, \lambda_{n_1}(\varepsilon)$, nonzero and in absolute value pairwise distinct at each fixed $\varepsilon > 0$, such that $\lim_{\varepsilon \searrow 0} \lambda_i(\varepsilon) = \lambda_i$ for $i = 1, \dots, n_1$. Define

$$A_1(\varepsilon) := \bigoplus_{i=1}^{n_1} K_{\lambda_i(\varepsilon)}^{(p_ip_i)}, \qquad A(\varepsilon) := \big(A_1(\varepsilon) \oplus A_2\big).$$

By its definition (2) we see that, at each fixed $\varepsilon > 0$, every diagonal block of $A_1(\varepsilon)$ is a nonderogatory matrix with eigenvalues $\{-\lambda_i(\varepsilon), \lambda_i(\varepsilon) : \ i = 1, \dots, n_1\}$. Hence, due to $0 \neq |\lambda_i(\varepsilon)| \neq |\lambda_j(\varepsilon)|$ for $i \neq j$, we derive that $A(\varepsilon)$ is nonderogatory at each fixed $\varepsilon > 0$ and so by Step 12 and the fact that $(X_1 \oplus X_2)' = X_1' \oplus X_2'$ whenever block matrices X_1, X_2 have no eigenvalue in common (see Lemma 2.1 or [8, Theorem 2 and Remark on p. 127]), we have:

$$\phi_Q(A(\varepsilon)) \in A(\varepsilon)' = \big(A_1(\varepsilon)' \oplus A_2'\big) = \Big(\bigoplus_{i=1}^{n_1} (K_{\lambda_i(\varepsilon)}^{(p_ip_i)})' \oplus A_2'\Big). \tag{14}$$

Since $\lambda_i(\varepsilon) \neq 0$ for $\varepsilon > 0$, the equation (9) gives $(K_{\lambda_i(\varepsilon)}^{(p_ip_i)})' = (K_{\lambda=1}^{(p_ip_i)})'$. Also, the matrices $K_{\lambda=1}^{(p_ip_i)}$ are nonderogatory, and therefore

$$(K_{\lambda=1}^{(p_ip_i)})' = \mathrm{Poly}(K_{\lambda=1}^{(p_ip_i)}) \cap \mathrm{Alt}_n = \mathrm{Poly}(K_{\lambda_i}^{(p_ip_i)}) \cap \mathrm{Alt}_n\,.$$

Combined with (14) we derive

$$\phi_Q(A(\varepsilon)) \in \Big(\bigoplus_{i=1}^{n_1} \mathrm{Poly}(K_{\lambda_i}^{(p_ip_i)}) \oplus A_2'\Big); \qquad \varepsilon > 0.$$

Hence, by continuity and the fact that the set on the right side of the above equation is closed,

$$\phi_Q(A_1 \oplus A_2) = \lim_{\varepsilon \searrow 0} \phi_Q(A(\varepsilon)) \in \Big(\bigoplus_{i=1}^{n_1} \mathrm{Poly}(K_{\lambda_i}^{(p_ip_i)}) \oplus A_2'\Big). \tag{15}$$

Since $\phi_Q(A_1 \oplus A_2)$ is alternate, there are hence odd polynomials (see last paragraph of Step 11) $f_1(\lambda), \dots, f_{n_1}(\lambda)$ such that

$$\phi_Q(A_1 \oplus A_2) \in \Big(\bigoplus_{i=1}^{n_1} f_i(K_{\lambda_i}^{(p_ip_i)}) \oplus A_2'\Big). \tag{16}$$

Likewise, it follows from (4) that the block constituents of $A_2 = \bigoplus_{j=1}^{n_2} K^{(q_j)}$, i.e., matrices $K^{(q_j)}$, are nonderogatory. Hence, if $X \in A_2' = \big(\bigoplus K^{(q_j)}\big)'$ is partition into blocks conformally to A_2 then its j-th diagonal block satisfies

$$X_{jj} \in (K^{(q_j)})' = \mathrm{Poly}(K^{(q_j)}) \cap \mathrm{Alt}_{q_j}\,. \tag{17}$$

Moreover, recall from the proof of Lemma 2.7 that the matrices

$$A_\varepsilon := \Big(\bigoplus_{i=1}^{n_1} X_\varepsilon^{(p_ip_i)} \oplus \bigoplus_{j=1}^{n_2} X_\varepsilon^{(q_j)}\Big), \tag{18}$$

where

$$\begin{cases} X_\varepsilon^{(1)} := 0, \\ X_\varepsilon^{(q_j)} := K^{(q_j)} + \frac{\varepsilon}{2}(\mathrm{Id}_{q_j} - \sqrt{-1}V)\big(E_{(q_j-1)\,1} - E_{q_j 2})(\mathrm{Id}_{q_j} + \sqrt{-1}V), & q_j \geq 3 \\ X_\varepsilon^{(p_ip_i)} := K_{\lambda_i+\varepsilon}^{(p_ip_i)} + \varepsilon(L^{(pp)})^{p-1}, & \text{for } p \text{ even} \\ X_\varepsilon^{(p_ip_i)} := K_{\lambda_i+\varepsilon}^{(p_ip_i)} + \varepsilon(L^{(pp)})^{p-2}, & \text{for } p \text{ odd} \end{cases},$$

($L^{(pp)}$ is defined by equation (11)) are alternate, diagonalizable, and converge to $A_1 \oplus A_2$ as $\varepsilon \to 0$. By Step 11, $\phi_Q(A_\varepsilon)$ are odd polynomials (i.e., $p(\lambda) = \sum a_{2i+1}\lambda^{2i+1}$) in A_ε and, by the continuity of ϕ_Q, they converge to $\phi_Q(A_1 \oplus A_2)$ as $\varepsilon \to 0$. Note that a polynomial in a block-diagonal matrix A_ε is itself a block-diagonal matrix. Therefore, $\phi_Q(A_1 \oplus A_2)$ is also a block-diagonal alternate matrix. It then follows from (16)–(17) that there exist odd polynomials $h_1(\lambda), \dots, h_{n_2}(\lambda)$ which satisfied the desired equation (13).

Substep 13.b. To finish the proof that $\phi_Q(A_1 \oplus A_2)$ is a polynomial in $A_1 \oplus A_2$ we have to show that whenever in equation (13) we have (a) $\lambda_i = \pm\lambda_{i'}$ or (b) $\lambda_i = 0$ then there exists a common polynomial $f(\lambda)$ such that, under (a),

$$f_i(K_{\lambda_i}^{(p_ip_i)}) \oplus f_{i'}(K_{\lambda_{i'}}^{(p_{i'}p_{i'})}) = f(K_{\lambda_i}^{(p_ip_i)}) \oplus f(K_{\lambda_{i'}}^{(p_{i'}p_{i'})})$$

respectively, under (b)

$$f_i(K_{\lambda_i}^{(p_ip_i)}) \oplus h_j(K^{(q_j)}) = f(K_{\lambda_i}^{(p_ip_i)}) \oplus f(K^{(q_j)}).$$

Also, we have to show that there exists a common polynomial $h(\lambda)$ such that $\bigoplus h_j(K^{(q_j)}) = \bigoplus h(K^{(q_j)})$. Once this is shown the proof ends because it is well-known that

$$\mathrm{Poly}(X_1 \oplus X_2) = \mathrm{Poly}(X_1) \oplus \mathrm{Poly}(X_2)$$

whenever matrices X_1, X_2 have no eigenvalue in common.

We thus have to observe three cases.

Substep 13.c. Assume first that $K_{\lambda_i}^{(p_ip_i)}$ and $K_{\lambda_{i'}}^{(p_{i'}p_{i'})}$ share the same eigenvalue, that is, $\lambda_i = \pm\lambda_{i'}$. By (3) the block $K_{-\lambda_i}^{(p_ip_i)}$ is orthogonally similar to $K_{\lambda_i}^{(p_ip_i)}$. Hence we may assume that in (12),

$$\text{whenever} \quad \lambda_i = \pm\lambda_{i'} \qquad \text{then actually} \qquad \lambda_i = \lambda_{i'}.$$

For the sake of simplicity, we will also assume that $p_i \le p_{i'}$ and that $i = 1$ and $i' = 2$, that is, the first two diagonal blocks in (13) share the same eigenvalue. Then, by the definition of $K_{\lambda_1}^{(p_1p_1)}$ (see (2)), we see that $A_1 \oplus A_2$ commutes with alternate block matrices

$$T_k = \begin{pmatrix} 0 & X_k \\ -X_k^t & 0 \end{pmatrix} \oplus 0_{n-2p_1-2p_2}, \qquad k = 1, 2,$$

where

$$X_1 = \tfrac{1}{2}\left(\mathrm{Id}_{2p_1} - \sqrt{-1}V_{2p_1}\right) \cdot \begin{pmatrix} [0|I] & 0_{p_1\times p_2} \\ 0_{p_1\times p_2} & [0|0] \end{pmatrix} \cdot \left(\mathrm{Id}_{2p_2} + \sqrt{-1}V_{2p_2}\right),$$

$$X_2 = \tfrac{1}{2}\left(\mathrm{Id}_{2p_1} - \sqrt{-1}V_{2p_1}\right) \cdot \begin{pmatrix} [0|0] & 0_{p_1\times p_2} \\ 0_{p_1\times p_2} & [0|I] \end{pmatrix} \cdot \left(\mathrm{Id}_{2p_2} + \sqrt{-1}V_{2p_2}\right),$$

and I denotes the identity matrix of the size $p_1 \times p_1$. Assume that $p_1 = p_2$. In this case, let $X = X_1 + X_2 = \frac{1}{2}\left(\mathrm{Id}_{2p_1} - \sqrt{-1}V_{2p_1}\right)\mathrm{Id}_{2p_1}\left(\mathrm{Id}_{2p_1} + \sqrt{-1}V_{2p_1}\right) = \mathrm{Id}_{2p_1}$ and $T = T_1 + T_2 = \begin{pmatrix} 0 & \mathrm{Id}_{2p_1} \\ -\mathrm{Id}_{2p_1} & 0 \end{pmatrix} \oplus 0_{n-4p_1}$. Then $T^3 = -T$. In particular, T is diagonalizable. Thus, ϕ_Q maps T into an odd polynomial in T. But due to $T^3 = -T$ it is fixed by ϕ_Q, modulo multiplication by a nonzero scalar. We infer that $\phi_Q(A_1 \oplus A_2)$ must also commute with T. Hence, the first two diagonal blocks of $\phi_Q(A_1 \oplus A_2)$ are intertwined with a matrix X. Since $X = \mathrm{Id}_{2p_1}$ we get that, in identity (13), $f_1(K_{\lambda_1}^{(p_1p_1)}) = f_2(K_{\lambda_2}^{(p_2p_2)})$. Since $p_2 = p_1$ and $\lambda_1 = \lambda_2$ we can hence take $f(\lambda) = f_1(\lambda)$ in this case.

On the other hand, if $p_1 \lneq p_2$, then we have $T_1^3 = -T_1$ and $T_2^3 = -T_2$ so, by the above, T_1 and T_2 are both fixed by ϕ_Q, modulo scalars. Since ϕ_Q preserves commutativity, $\phi_Q(A_1 \oplus A_2)$ commutes with every linear combination of $\phi_Q(T_1) \in \mathbb{C}T_1$ and $\phi_Q(T_2) \in \mathbb{C}T_2$. In particular, $\phi_Q(A_1 \oplus A_2)$ also commutes with

$T = (T_1 + T_2)$. By taking the first two blocks from (13) and using definition (2) we see that $f_1\big((\lambda_1 \operatorname{Id}_{p_1} + J_{p_1}) \oplus (-\lambda_1 \operatorname{Id}_{p_1} - J_{p_1})\big) \oplus f_2\big((\lambda_2 \operatorname{Id}_{p_2} + J_{p_2}) \oplus (-\lambda_2 \operatorname{Id}_{p_2} - J_{p_2})\big)$ commutes with

$$T_1 + T_2 = \begin{pmatrix} 0 & \begin{pmatrix} [0|I] & 0 \\ 0 & [0|I] \end{pmatrix} \\ -\begin{pmatrix} [0|I] & 0 \\ 0 & [0|I] \end{pmatrix}^t & 0 \end{pmatrix}$$

Again, $\begin{pmatrix} [0|I] & 0 \\ 0 & [0|I] \end{pmatrix}$ intertwines blocks $f_1\big((\lambda_1 \operatorname{Id}_{p_1} + J_{p_1}) \oplus (-\lambda_1 \operatorname{Id}_{p_1} - J_{p_1})\big)$ and $f_2\big((\lambda_2 \operatorname{Id}_{p_2} + J_{p_2}) \oplus (-\lambda_2 \operatorname{Id}_{p_2} - J_{p_2})\big)$. It is straightforward that

$$f_1(\lambda_1 \operatorname{Id}_{p_1} + J_{p_1})\,[0|I] = [0|I]\,f_2(\lambda_2 \operatorname{Id}_{p_2} + J_{p_2}); \qquad \lambda_2 = \lambda_1.$$

Since polynomials $f_1(\lambda), f_2(\lambda)$ are odd and since we can assume that the degree of $f_i(\lambda)$ is smaller than the degree of minimal polynomial of corresponding block, i.e., smaller than p_i, we can write

$$f_1(\lambda_1 \operatorname{Id}_{p_1} + J_{p_1}) = \begin{pmatrix} f_1(\lambda_1) & 0 & \alpha_1 & 0 & \alpha_2 & \cdots\cdots \\ & f_1(\lambda_1) & 0 & \alpha_1 & 0 & \alpha_2 \quad \cdots \\ & & \ddots & & \ddots & \quad \ddots \end{pmatrix}$$

for suitable scalar α_i and likewise for $f_2(\lambda_1 \operatorname{Id}_{p_2} + J_{p_2})$. It then easily follows that $f_1(\lambda)$ and $f_2(\lambda)$ coincide in the first $\min\{p_1, p_2\} = p_1$ monomials. Hence, the present substep is done by taking $f(\lambda) = f_2(\lambda)$.

Substep 13.d. From Substep 13.c it follows that ϕ fixes every square-zero alternate matrix, modulo scalars, because such matrices take the form $O\big(\bigoplus K^{(22)}_{\lambda=0} \oplus 0\big)O^t$ for some orthogonal O.

Substep 13.e. Next, assume that $A_1 \oplus A_2$ contains two blocks $K^{(q_j)}$ and $K^{(q_{j'})}$. As above, we may assume that $q_j \le q_{j'}$ and that $j = 1$ and $j' = 2$. Moreover, for simplicity we will temporarily permute the diagonal blocks so that, in the present substep only,

$$A_1 \oplus A_2 = K^{(q_1)} \oplus K^{(q_2)} \oplus \ldots.$$

If $q_1 = q_2$, then $A_1 \oplus A_2$ commutes with an alternate block matrix

$$T = \begin{pmatrix} 0 & \operatorname{Id}_{q_1} \\ -\operatorname{Id}_{q_1}^t & 0 \end{pmatrix} \oplus 0_{n-q_1-q_2}$$

which satisfies $T^3 = -T$. Hence, T is fixed by ϕ_Q, modulo multiplication by a nonzero scalar. As in the first case, it is easy to see that the first two blocks $B_1 = h_1(K^{(q_1)})$ and $B_2 = h_2(K^{(q_2)})$ of $\phi_Q(A_1 \oplus A_2)$ coincide. Hence, $B_1 \oplus B_2 = h(K^{(q_1)}) \oplus h(K^{(q_2)})$ where $h(\lambda) = h_1(\lambda)$.

On the other hand, assume $q_2 = q_1 + 2d$ for some integer $d \ge 1$. Then, by Lemma 1.2, $(J^{(q_1)})^d = \begin{pmatrix} 0 & D_1 \\ 0 & 0 \end{pmatrix}$, where D_1 is a diagonal $(q_1 - d) \times (q_1 - d)$ matrix with ± 1 on the diagonal. Clearly, if $d \ge q_1$, then $(J^{(q_1)})^d = 0$. Similarly,

$$(J^{(q_2)})^d = \begin{pmatrix} 0 & D_2 \\ 0 & 0 \end{pmatrix}, \tag{19}$$

where D_2 is a diagonal $(q_2 - d) \times (q_2 - d)$ matrix with ± 1 on the diagonal. Obviously, D_2 is a nonzero matrix since $q_2 > d$. Moreover, if we write $q_1 = 2k+1$, then $q_2 = q_1 + 2d = 2(k+d)+1$, and, by formula in Lemma 1.2,

$$D_2 = \operatorname{Id}_{k+1} \oplus \operatorname{diag}\big(-1, (-1)^2, \ldots, (-1)^{d-1}\big) \oplus (-1)^d \operatorname{Id}_{k+1}. \tag{20}$$

Now it easily follows that, if we remove from D_2 the first d and the last d rows and columns, we obtain D_1. More precisely, D_1 is a compression of the matrix D_2 onto the subspace generated by $\{e_{d+1}, \ldots, e_{q_1}\}$ and $D_1 = 0$ if $d \ge q_1$.

If d is odd, then we define the matrix

$$T = S^{-1}\left(\begin{pmatrix} -\sqrt{-1}\left(J^{(q_1)}\right)^d & \hat{X} \\ -V_{q_2}\hat{X}^t V_{q_1} & \sqrt{-1}\left(J^{(q_2)}\right)^d \end{pmatrix} \oplus 0_{n-q_1-q_2}\right) S,$$

where $\hat{X} = \left(0_{q_1\times 2d}|\hat{D}\right)$, $\hat{D}$ is the compression of $-D_2$ onto the last q_1 basis vectors, and $S = (\mathrm{Id}_{q_1} + \sqrt{-1}V_{q_1}) \oplus (\mathrm{Id}_{q_2} + \sqrt{-1}V_{q_2}) \oplus \mathrm{Id}_{n-q_1-q_2}$. Observe that (19) implies $\hat{X} = -P(J^{(q_2)})^d$, where $P\colon \mathbb{C}^{q_2} \to \mathbb{C}^{q_1}$ is a projection onto coordinates $(d+1)$ through (q_2-d). By Lemma 1.2, $\hat{X}J^{(q_2)} = P(J^{(q_2)})^{d+1}$ maps basis vector e_i into 0 for $i = 1, \dots, (2d+1)$ and into $-(\pm)_i\, e_{i-2d-1} \in \mathbb{C}^{q_1}$ for $i = (2d+2), \dots, (2d+q_1) = q_2$, where

$$(\pm)_i = \begin{cases} (-1)^{d+1}; & i > q_2 - k \\ (-1)^{d+k+1-i}; & q_2-(d+k) < i \le q_2 - k\,. \\ 1; & \text{otherwise} \end{cases}$$

On the other hand, $\hat{X}e_i = 0$ for $1 \le i \le 2d$ and $\hat{X}e_i = -\widehat{(\pm)}_i e_{i-2d}$ for $i \ge 2d+1$, where

$$\widehat{(\pm)}_i = \begin{cases} (-1)^{d}; & i > q_2 - k - 1 \\ (-1)^{d+k-i-1}; & q_2-(d+k)-1 < i \le q_2 - k - 1\,. \\ 1; & \text{otherwise} \end{cases}$$

Apply $J^{(q_1)}$ and use the fact that $J^{(q_1)}e_{i-2d} = e_{i-2d-1}$ if $i-2d \le \frac{q_1-1}{2}$ and $J^{(q_1)}e_{i-2d} = -e_{i-2d-1}$ if $i-2d > \frac{q_1-1}{2}$. It is then a straightforward calculation that

$$J^{(q_1)}\hat{X} = \hat{X}J^{(q_2)}. \tag{21}$$

Next, it is easy to see that T is alternate. Assume first $d < q_1$. Then the block structure of STS^{-1} equals

$$\begin{array}{c} \\ q_1-d \\ d \\ d \\ q_1-d \\ d \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} d & q_1-d & d & d & q_1-d & d \end{array} \\ \left(\begin{array}{ccc|cccc} 0 & -\sqrt{-1}\hat{D}_2 & 0 & 0 & -\hat{D}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -D_2'' \\ \hline (D_2'')^f & 0 & 0 & \sqrt{-1}D_2' & 0 & 0 \\ 0 & (\hat{D}_2)^f & 0 & 0 & \sqrt{-1}\hat{D}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{-1}D_2'' \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \oplus 0_{n-q_1-q_2}.$$

Here, $X^f = VX^tV$, $V = \sum_i E_{i\,(n-i+1)}$, is a flip map, i.e., reflection over anti-diagonal, and $D_2', D_2'' \in M_d$, $\hat{D}_2 \in M_{q_1-d}$ are appropriate diagonal matrices such that $D_2 = D_2' \oplus \hat{D}_2 \oplus D_2''$. It follows from (20) that

$$(\hat{D}_2)^f = -\hat{D}_2.$$

We claim that $T^2 = 0$. In fact, STS^{-1} maps the basis vectors $e_{q_1+1}, \dots, e_{q_1+d}$ already into zero. It further maps the vectors e_k, $k = 1, \dots, d$, into $\pm e_{q_1+k}$ which are annihilated by another application of STS^{-1}. Moreover, it maps e_{q_1+d+k} into $\pm\sqrt{-1}e_{q_1+k}$ for $k = 1, \dots, d$, which are annihilated by the second application of STS^{-1}. Next, STS^{-1} maps the vectors e_{d+k}, $k = 1, \dots, q_1-d$ into $\pm(\sqrt{-1}e_k + e_{q_1+d+k})$, which are further mapped by STS^{-1} into

$$\pm(\sqrt{-1}STS^{-1}e_k + STS^{-1}e_{q_1+d+k}) = \pm(-\sqrt{-1}e_{k+q_1} + \sqrt{-1}e_{q_1+k}) = 0.$$

Finally, the vectors e_{q_1+2d+k}, $k = 1, \dots, q_1$ are mapped to $STS^{-1}e_{q_1+2d+k} = \pm(e_k - \sqrt{-1}e_{q_1+d+k}) = \pm\sqrt{-1}(\sqrt{-1}e_k + e_{q_1+d+k})$, which, as we showed above, are annihilated by the second application of STS^{-1}. This shows $T^2 = 0$ if $d < q_1$. By the similar analysis we can show that $T^2 = 0$ also when $d \ge q_1$ (the block structure of STS^{-1} is same as before except that now, $\hat{D}_2 = 0$). Consequently, by Substep 13.d,

$$\phi_Q(T) \in \mathbb{C}T.$$

We next prove that $A_1 \oplus A_2$ commutes with T. It suffices to show that $J^{(q_1)} \oplus J^{(q_2)}$ commutes with $\begin{pmatrix} 0 & \hat{X} \\ -V_{q_2}\hat{X}^t V_{q_1} & 0 \end{pmatrix}$. This is equivalent to the fact that $J^{(q_1)}\hat{X} = \hat{X}J^{(q_2)}$ and $J^{(q_2)}V_{q_2}\hat{X}^t V_{q_1} = V_{q_2}\hat{X}^t V_{q_1} J^{(q_1)}$. The first equality follows from (21) while the second one is equivalent to the first one which is seen by applying the bijective operation $Z \mapsto (V_{q_1} Z V_{q_2})^t$ to the first equality, and in the resulting

$$(V_{q_1}(J^{(q_1)}\hat{X})V_{q_2})^t = (V_{q_1}(\hat{X}J^{(q_2)})V_{q_2})^t$$

rewrite the left-hand side into $(V_{q_1}J^{(q_1)}\hat{X}V_{q_2})^t = (V_{q_1}J^{(q_1)}V_{q_1} \cdot V_{q_1}\hat{X}V_{q_2})^t = (V_{q_1}\hat{X}V_{q_2})^t \cdot (V_{q_1}J^{(q_1)}V_{q_1})^t = (V_{q_2}\hat{X}^t V_{q_1})^t \cdot (-J^{(q_1)})$ and likewise rewrite the right-hand side.

It follows that $\phi_Q(A_1 \oplus A_2)$ commutes with $\phi_Q(T) \in \mathrm{Poly}(T) = \mathbb{C}T$ as well. This is equivalent to the fact that $h_1(J^{(q_1)}) \oplus h_2(J^{(q_2)})$ commutes with $\begin{pmatrix} 0 & \hat{X} \\ -V_{q_2}\hat{X}^t V_{q_1} & 0 \end{pmatrix}$ for appropriate odd polynomials $h_1(\lambda)$ and $h_2(\lambda)$ of degrees smaller than q_1 or q_2 respectively. Actually, this is further equivalent to the fact that $h_1(J^{(q_1)})\hat{X} = \hat{X}h_2(J^{(q_2)})$. Recall that $J^{(q_1)}\hat{X} = \hat{X}J^{(q_2)}$, hence it easily follows that for each $s = 0, 1, \dots, q_1$, we have

$$(J^{(q_1)})^s\hat{X} = \left(0_{q_1\times 2d} | W_s\right) = \hat{X}(J^{(q_2)})^s,$$

where $W_s \in M_{q_1}$ is strictly upper-triangular matrix with ±1 on s-th super diagonal and zeros elsewhere. Thus, writing

$$h_1(x) = \alpha_1 x + \alpha_3 x^3 + \cdots + \alpha_{q_1-2}x^{q_1-2}, \quad h_2(x) = \beta_1 x + \beta_3 x^3 + \cdots + \beta_{q_2-2}x^{q_2-2}$$

then $h_1(J^{(q_1)})\hat{X} = \hat{X}h_2(J^{(q_2)})$ implies $\sum \alpha_i W_i = \sum \beta_j W_j$. Since matrices $W_1, W_3, \dots, W_{q_1-2}$ are linearly independent while $W_{q_1} = W_{q_1+2} = \cdots = 0$ we deduce $\alpha_i = \beta_i$ for $i = 1, \dots, q_1 - 2$. This shows that $h_1(J^{(q_1)}) = h_2(J^{(q_1)})$ and we are done as before.

The case when d is even can be treated in a similar way. We just have to take the block matrix

$$T = S^{-1}\left(\begin{pmatrix} -\sqrt{-1}\left(J^{(q_1)}\right)^{d+1} & \hat{X} \\ -V_{q_2}\hat{X}^t V_{q_1} & \sqrt{-1}\left(J^{(q_2)}\right)^{d+1} \end{pmatrix} \oplus 0_{n-q_1-q_2}\right) S,$$

where S is similar as above except that now $\hat{X} = \left(0_{q_1\times 2d} \middle| \begin{pmatrix} 0_{(q_1-1)\times 1} & \hat{D}' \\ 0_{1\times 1} & 0_{1\times(q_1-1)} \end{pmatrix}\right)$ with diagonal matrix $\hat{D}'$ being a compression of $-D_2'$ onto the last $(q_1 - 1)$ basis vectors, where diagonal matrix D_2' comes from $(J^{(q_2)})^{d+1} = \begin{pmatrix} 0 & D_2' \\ 0 & 0 \end{pmatrix}$. Similarly as in the previous case, we derive that $T^2 = 0$ and that T commutes with $A_1 \oplus A_2$. So, we are done as above.

Substep 13.f. Before continuing with the last case we will show that every cube-zero alternate matrix is fixed by ϕ_Q, modulo scalars. So, let T be an alternate matrix with $T^3 = 0$. Since every cube-zero alternate matrix consists of square-zero blocks and cube-zero blocks we may write without loss of generality that $T = (K^{(2,2)} \oplus \dots \oplus K^{(2,2)}) \oplus (K^{(3)} \oplus \dots \oplus K^{(3)}) \oplus 0$. Now, by subssteps 13.c and 13.e, we know that $\phi_Q(T)$ has a similar block-diagonal structure with each block being an odd polynomial of the corresponding block. That is, $\phi_Q(T) = (\alpha_1 K^{(2,2)} \oplus \dots \oplus \alpha_r K^{(2,2)}) \oplus (\beta_1 K^{(3)} \oplus \dots \oplus \beta_s K^{(3)}) \oplus 0$. Furthermore, from substep 13.c we know that $\alpha_1 = \dots = \alpha_r$ and from subsetep 13.e we learn that $\beta_1 = \dots = \beta_s$. Hence, $\phi_Q(T) = \alpha(K^{(2,2)} \oplus \dots \oplus K^{(2,2)}) \oplus \beta(K^{(3)} \oplus \dots \oplus K^{(3)}) \oplus 0$. It only remains to show that $\alpha = \beta$. To this end, we may assume that $T = K^{(2,2)} \oplus K^{(3)} \oplus 0$ and $\phi_Q(T) = \alpha K^{(2,2)} \oplus \beta K^{(3)} \oplus 0$. Using $X = \begin{pmatrix} 0 & -1 & 0 & i \\ 0 & 0 & \frac{1}{2} + \frac{i}{2} & 0 \\ 0 & 0 & -\frac{1}{2} + \frac{i}{2} & 0 \end{pmatrix}$, define alternate $Y = \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix}$. Since $(\operatorname{rank} Y, \operatorname{rank} Y^2, \operatorname{rank} Y^3) = (4, 2, 0)$, it is easy to see that Y is similar to $K^{(3)} \oplus K^{(3)} \oplus 0$ and, hence, fixed by ϕ_Q, modulo scalars. Moreover, Y commutes with T. Therefore, Y has to commute with $\phi_Q(T)$ as well. But then scalars α and β have to be equal. Hence, every cube-zero alternate matrix is fixed by ϕ_Q, modulo scalars.

Substep 13.g. Suppose lastly that $K_{\lambda_i}^{(p_i p_i)}$ and $K^{(q_j)}$ share the same eigenvalue, i.e, $\lambda_i = 0$. Again for simplicity we temporarily permute the diagonal blocks so that, in the present substep only,

$$A_1 \oplus A_2 = K^{(q_j)} \oplus K_0^{(p_i p_i)} \oplus \dots .$$

Define an alternate block matrix

$$T = \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \oplus 0_{n-q_j-2p_i},$$

where

$$X = \tfrac{1}{2}\left(\mathrm{Id}_{q_j} - \sqrt{-1}V_{q_j}\right) \cdot \hat{X} \cdot \left(\mathrm{Id}_{2p_i} + \sqrt{-1}V_{2p_i}\right)$$

and $\hat{X}$ is a $q_j \times 2p_i$ matrix defined as follows. If $q_j \le p_i$, then we take $\hat{X} = \begin{pmatrix} 0_{q_j\times(p_i-q_j)} & D & 0_{q_j\times p_i} \end{pmatrix}$, where

$$D = \mathrm{Id}_{k_0} \oplus \mathrm{diag}\left((-1), (-1)^2, \dots, (-1)^{k_0-1}\right), \qquad k_0 = \tfrac{q_j+1}{2}.$$

If $q_j > p_i$, then we take $\hat{X} = \begin{pmatrix} [D|0_{p_i\times p_i}] \\ 0_{(q_j-p_i)\times 2p_i} \end{pmatrix}$, where

$$D = \begin{cases} \mathrm{Id}_{p_i}; & p_i \le k_0 := \frac{q_j+1}{2} \\ \mathrm{Id}_{k_0} \oplus \mathrm{diag}\left((-1), (-1)^2, \dots, (-1)^{p_i-k_0}\right); & p_i > k_0 := \frac{q_j+1}{2} \end{cases}.$$

An easy exercise gives that $J^{(q_j)}\hat{X} = \hat{X}\left(J_{p_i} \oplus (-J_{p_i})\right)$, so $A_1 \oplus A_2$ commutes with T. Let us show that

$$T^3 = 0.$$

Firstly, recall that $T^2 = -(XX^t \oplus X^tX)$ is block-diagonal. We claim that $XX^t = 0$. To see this, note that $(\mathrm{Id}_{2p_i} - \sqrt{-1}V_{2p_i})^2 = -2\sqrt{-1}V_{2p_i}$ and that $V_{q_j}^2 = \mathrm{Id}_{q_j}$. This yields that

$$XX^t = \sqrt{-1}(\mathrm{Id}_{q_j} - \sqrt{-1}V_{q_j}) \cdot \hat{X}V_{2p_i}\hat{X}^t V_{q_j} \cdot V_{q_j}(\mathrm{Id}_{q_j} - \sqrt{-1}V_{q_j}).$$

Now, it is easy to see that $\hat{X} \cdot V_{2p_i}\hat{X}^t V_{q_j} = 0$ which gives the claimed $XX^t = 0$. Then also $X^tXX^t = 0$, so $T^3 = 0$.

Consequently, by substep 13.f, T is fixed by ϕ_Q, modulo a scalar. Since $A_1 \oplus A_2$ commutes with T, then $\phi_Q(A_1 \oplus A_2)$ commutes with $\phi_Q(T) \in \mathbb{C}T$ as well. Recall that

$$\phi_Q(A_1 \oplus A_2) = h_j\left(K^{(q_j)}\right) \oplus f_i\left(K_0^{(p_i p_i)}\right) \oplus \dots$$

where h_j and f_i are odd polynomials of degrees less that q_j and p_i, respectively. It follows easily from $\phi_Q(A_1 \oplus A_2)T = T\phi_Q(A_1 \oplus A_2)$ that h_j and f_i coincide in the first $\min\{q_j - 1, p_i - 1\}$ monomials, as desired (here, we use that $(J^{(q_j)})^k\hat{X} = \hat{X}(J_{p_i} \oplus (-J_{p_i}))^k \neq 0$ for $k = 1, \dots, \min\{q_j - 1, p_i - 1\}$). The proof is completed.

References

[1] R. Bhatia, P. Rosenthal, How and why to solve the operator equation $AX - XB = Y$, *Bull. London Math. Soc.* **29** (1997), 1–21.

[2] M. Brešar, Commuting traces of biadditive mappings, commutativity preserving mappings, and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993), 525–546.

[3] M. Brešar, C. R. Miers, Commutativity preserving mappings of von Neumann algebras, *Canad. J. Math.* **45** (1993), 695–708.

[4] J.-T. Chan, C.-K. Li, N.-S. Sze, Mappings on matrices: invariance of functional values of matrix products, *J. Aust. Math. Soc.* **81** (2006), 165–184.

[5] M. D. Choi, A. Jafarian, H. Radjavi, Linear maps preserving commutativity, *Linear Algebra Appl.* **87** (1987), 227–241.

[6] J. Cui, J. Hou, Maps leaving functional values of operator products invariant, *Linear Algebra Appl.* **428** (2008), 1649–1663.

[7] C.-A. Faure, An elementary proof of the fundamental theorem of projective geometry, *Geom. Dedicata* **90** (2002), 145–151.

[8] P.A. Fillmore, D.A. Herrero, W.E. Longstaff, The hyperinvariant subspace lattice of a linear transformation, *Linear Algebra Appl.* **17** (1977), 125–132.

[9] A. Fošner, B. Kuzma, T. Kuzma, N.-S. Sze, Maps preserving matrix pairs with zero Jordan product, *Linear Multilinear Algebra* **59** (2011), 507–529.
[10] W. Fulton, *Algebraic topology: a first course*, Springer, Graduate Texts in Mathematics 153, New York (1995).
[11] F. R. Gantmacher, *Applications of the theory of matrices*, Interscience Publishers, Inc., New York (1959).
[12] R. A. Horn, C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge (1985).
[13] C.-K. Li, S. Pierce, Linear preserver problems, *Amer. Math. Monthly* **108** (2001), 591–605.
[14] C.-K. Li, N.-K. Tsing, Linear preserver problems: A brief introduction and some special techniques, *Linear Algebra Appl.* **162-164** (1992), 217–235.
[15] G. Lumer, M. Rosenblum, Linear operator equations, *Proc. Amer. Math. Soc.* **10** (1959), 32–41.
[16] L. Molnár, *Selected preserver problems on algebraic structures of linear operators and on function spaces*, Springer, Lecture Notes in Mathematics 1895, Berlin (2007).
[17] L. Molnár, P. Šemrl, Nonlinear commutativity preserving maps on self-adjoint operators, *Q. J. Math.* **56** (2005), 589–595.
[18] M. Omladič, H. Radjavi, P. Šemrl, Preserving commutativity, *J. Pure Appl. Algebra* **156** (2001), 309–328.
[19] T. Petek, Additive mappings preserving commutativity, *Linear Multilinear Algebra* **42** (1997), 205–211.
[20] T. Petek, Mappings preserving spectrum and commutativity on Hermitian matrices, *Linear Algebra Appl.* **290** (1999), 167–191.
[21] T. Petek, A note on additive commutativity-preserving mappings, *Publ. Math. (Debr.)* **56** (2000), 53–61.
[22] T. Petek, H. Sarria, Spectrum and commutativity preserving mappings on H_2, *Linear Algebra Appl.* **364** (2003), 317–319.
[23] T. Petek, P. Šemrl, Adjacency preserving maps on matrices and operators, *Proc. R. Soc. Edinb.* **132** (2002), 661–684.
[24] S. Pierce et.al., A survey of linear preserver problems, *Linear Multilinear Algebra* **33** (1992), 1–192.
[25] P. Šemrl, Non-linear commutativity preserving maps, *Acta Sci. Math. (Szeged)* **71** (2005), 781–819.
[26] P. Šemrl, Commutativity preserving maps, *Linear Algebra Appl.* **429** (2008), 1051–1070.
[27] W. Watkins, Linear maps that preserve commuting pairs of matrices, *Linear Algebra Appl.* **14** (1976), 29–35.