

## Research Article

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A. K. Bhuniya\* and Sudip Bera

# On some characterizations of strong power graphs of finite groups

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**Abstract:** Let  $G$  be a finite group of order  $n$ . The strong power graph  $\mathcal{P}_s(G)$  of  $G$  is the undirected graph whose vertices are the elements of  $G$  such that two distinct vertices  $a$  and  $b$  are adjacent if  $a^{m_1} = b^{m_2}$  for some positive integers  $m_1, m_2 < n$ . In this article we classify all groups  $G$  for which  $\mathcal{P}_s(G)$  is a line graph. Spectrum and permanent of the Laplacian matrix of the strong power graph  $\mathcal{P}_s(G)$  are found for any finite group  $G$ .

**Keywords:** groups, strong power graphs, line graphs, Laplacian spectrum, Laplacian permanent

**MSC:** 05C25, 05C50

## 1 Introduction

The study of different algebraic structures using graph theory becomes an exciting research topic in the last few decades, leading to many fascinating results and questions [1, 4], - , [7, 15, 16, 20]. Given an algebraic structure  $S$ , there are different formulations to associate a directed or undirected graph to  $S$ , and the algebraic properties of  $S$  are studied in terms of properties of associated graphs.

Directed power graphs associated to semigroups were introduced by Kelarev and Quinn [15]. If  $S$  is a semigroup, then the directed power graph  $\mathcal{P}(S)$  of  $S$  is a directed graph with  $S$  as the set of all vertices and for any two distinct vertices  $u$  and  $v$  of  $S$ , there is an arc from  $u$  to  $v$  if  $v = u^m$  for some positive integer  $m$ . Then Chakrabarty, Ghosh and Sen [6] defined the undirected power graph  $\mathcal{P}(S)$  of a semigroup  $S$  such that two distinct elements  $u$  and  $v$  of  $S$  are edge connected in  $\mathcal{P}(S)$  if  $u = v^m$  or  $v = u^m$  for some positive integer  $m$ . They proved that for a finite group  $G$ , the undirected power graph  $\mathcal{P}(G)$  is complete if and only if  $G$  is a cyclic group of order 1 or  $p^m$  for some prime  $p$  and positive integer  $m$ . In [4], Cameron and Ghosh showed that for two finite abelian groups  $G_1$  and  $G_2$ ,  $\mathcal{P}(G_1) \cong \mathcal{P}(G_2)$  implies that  $G_1 \cong G_2$ . They also conjectured that two finite groups with isomorphic undirected power graphs have the same number of elements of each order.

Singh and Manilal [20] introduced strong power graph as a generalization of the undirected power graph of a finite group. Let  $G$  be a group of order  $n$ . The strong power graph  $\mathcal{P}_s(G)$  of  $G$  is a graph whose vertices are the elements of  $G$  and two distinct vertices  $a$  and  $b$  are adjacent in  $\mathcal{P}_s(G)$  if  $a^{m_1} = b^{m_2}$  for some positive integers  $m_1, m_2 < n$ . Thus a finite group  $G$  is noncyclic if and only if  $\mathcal{P}_s(G)$  is complete. Also  $\mathcal{P}_s(G)$  is connected if and only if  $n$  is composite. By definition, the strong power graph of every group is simple. Henceforth, unless otherwise stated, by a graph we always mean a simple graph.

Here we give several graph theoretic and spectral characterizations of the strong power graph of a finite group. A complete list of the finite groups  $G$  such that  $\mathcal{P}_s(G)$  is a line graph is given in Section 2. In Section 3, the algebraic connectivity and the chromatic number of  $\mathcal{P}_s(G)$  have been found.

For any graph  $\Gamma$ , let  $A(\Gamma)$  be the adjacency matrix and  $D(\Gamma)$  be the diagonal matrix of vertex degrees. Then the Laplacian matrix of  $\Gamma$  is defined as  $L(\Gamma) = D(\Gamma) - A(\Gamma)$ . Clearly  $L(\Gamma)$  is a real symmetric matrix and it is

\*Corresponding Author: A. K. Bhuniya: Department of Mathematics, Visva-Bharati, Santiniketan-731235, India, E-mail: anjankbhuniya@gmail.com

Sudip Bera: Department of Mathematics, Visva-Bharati, Santiniketan-731235, India, E-mail: sudipbera517@gmail.com

well known that  $L(\Gamma)$  is a positive semidefinite matrix with 0 as the smallest eigenvalue. Thus we can assume that the Laplacian eigenvalues are  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n = 0$ . The eigenvalues of the Laplacian matrix  $L(\Gamma)$  is called the Laplacian spectrum of  $\Gamma$ . In the study of connectivity, coloring, energy of graphs Laplacian spectrum plays an important role. According to Mohar [19] the Laplacian eigenvalues are more intuitive and much more important than the eigenvalues of the adjacency matrix. We give a complete characterization of the Laplacian spectrum of strong power graph of any finite group in Section 4.

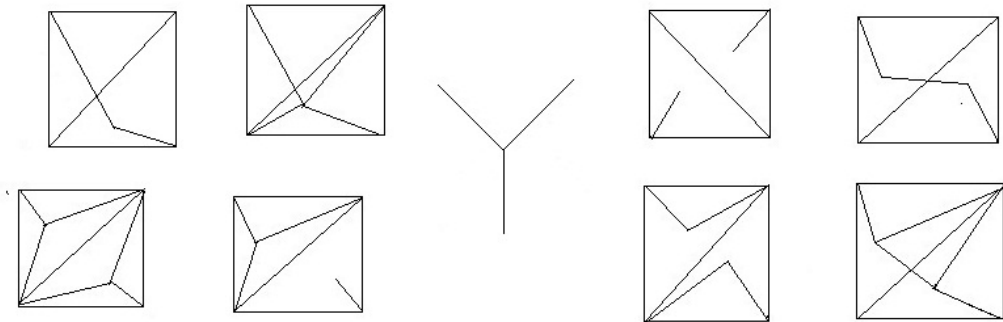
In Section 5, we have derived an explicit formula for the permanent of the Laplacian matrix of strong power graphs of any finite group.

We refer to [9, 21] for the notions of the graph theory, [2, 3] for the matrix theory related to graphs and [14] for the group theoretic background.

## 2 Representing strong power graph as line graph

In this section we characterize the groups  $G$  such that the strong power graph  $\mathcal{P}_s(G)$  of the group  $G$  is a line graph. The line graph of a graph  $\Gamma$  is the graph  $L(\Gamma)$  with the edges of  $\Gamma$  as its vertices, and where two edges of  $\Gamma$  are adjacent in  $L(\Gamma)$  if and only if they are incident in  $\Gamma$ . A subgraph  $T$  of a graph  $\Gamma$  is an induced subgraph if two vertices of  $T$  are adjacent in  $T$  if and only if they are adjacent in  $\Gamma$ . The graph obtained by taking the union of two graphs  $\Gamma_1$  and  $\Gamma_2$  with disjoint vertex set is the disjointed union or sum, denoted by  $\Gamma_1 + \Gamma_2$ . The following result is a fundamental characterization of the line graphs. Proof of this result can be found in [21].

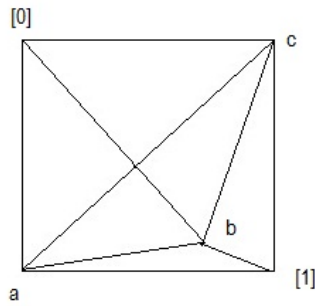
**Lemma 2.1.** *A graph  $G$  is the line graph of some graph if and only if  $G$  does not have any of the following nine graphs as an induced subgraph.*



For any positive integer  $n$ ,  $\mathbb{Z}_n$  denotes the cyclic group of order  $n$  and  $\phi(n)$  denotes the number of positive integers  $r \leq n$  such that  $\gcd(r, n) = 1$ . Then the number of generators of  $\mathbb{Z}_n$  is  $\phi(n)$ . Let  $D'(n) = \{d \in \mathbb{N} \mid d|n, d \neq 1, n\}$ . Thus  $D'(p) = \emptyset$  for every prime  $p$ . Then we have  $\phi(n) > n-4 \Leftrightarrow n-\phi(n) < 4 \Leftrightarrow \sum_{d \in D'(n)} \phi(d) < 3$  or  $n$  is a prime  $\Leftrightarrow n = 4, 9$  or a prime.

**Lemma 2.2.** *If  $\mathcal{P}_s(\mathbb{Z}_n)$  is a line graph then  $n = 4, 9$  or a prime.*

*Proof.* If possible, on the contrary, suppose that  $n \neq 4, 9$  and a prime. Then it follows that  $\phi(n) \leq n-4$  which implies that  $n - \phi(n) - 1 \geq 3$ , and hence  $\mathbb{Z}_n$  has at least three nonzero nongenerators, say  $a, b$  and  $c$ . In this case the following graph is an induced subgraph of  $\mathcal{P}_s(\mathbb{Z}_n)$ .



where  $[0]$  is the zero element and  $[1]$  is the unity in  $\mathbb{Z}_n$ . Therefore, by Lemma 2.1,  $\mathcal{P}_s(\mathbb{Z}_n)$  can not be a line graph. □

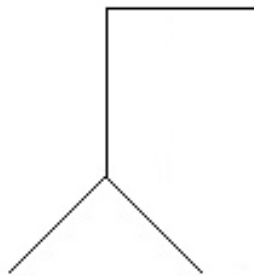
Now we characterize all finite groups  $G$  such that  $\mathcal{P}_s(G)$  is a line graph.

**Theorem 2.3.** *Let  $G$  be a finite group of order  $n$ .*

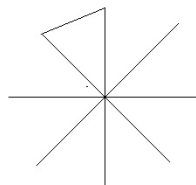
1. *If  $G$  is noncyclic, then  $\mathcal{P}_s(G)$  is a line graph of  $K_{1,n}$ , where  $K_{1,n}$  is the complete bipartite graph whose vertex set  $V = V_1 \cup V_2$ , where  $V_1$  contains only one vertex and  $V_2$  contains  $n$  vertices.*
2. *If  $G$  is cyclic, then  $\mathcal{P}_s(G)$  is a line graph if and only if  $n = 4, 9$  or a prime.*

*Proof.* 1. If  $G$  is noncyclic then  $\mathcal{P}_s(G)$  is a complete graph with  $n$  vertices, and hence  $\mathcal{P}_s(G) = L(K_{1,n})$ .

2. First suppose that  $n = 4, 9$  or a prime. We have  $\mathcal{P}_s(\mathbb{Z}_4)$  is the line graph of the graph



and  $\mathcal{P}_s(\mathbb{Z}_9)$  is the line graph of the graph



If  $n = p$ , a prime number then  $\mathcal{P}_s(\mathbb{Z}_p)$  is the line graph of  $K_{1,p-1} + K_2$ .  
Converse follows from Lemma 2.2. □

### 3 Vertex connectivity and chromatic number of strong power graphs of finite groups

The vertex connectivity or connectivity of a graph  $\Gamma$ , denoted by  $\kappa(\Gamma)$ , is the minimum number of vertices in  $\Gamma$  whose deletion from  $\Gamma$  leaves either a disconnected graph or a graph with only one vertex. Thus if  $\Gamma$  is a complete graph with  $n$  vertices then  $\kappa(\Gamma) = n - 1$ , and if  $\Gamma$  is a disconnected graph then  $\kappa(\Gamma) = 0$ . The strong power graph  $\mathcal{P}_s(G)$  of every finite noncyclic group  $G$  is complete. Thus for any noncyclic finite group  $G$  of order  $n$ , the vertex connectivity  $\kappa(\mathcal{P}_s(G))$  is  $n - 1$ . In the following we give the vertex connectivity of the strong power graphs of cyclic groups.

**Theorem 3.1.** 1. If  $n$  is a prime number then  $\kappa(\mathcal{P}_s(\mathbb{Z}_n)) = 0$ .  
2. If  $n$  is a composite number then  $\kappa(\mathcal{P}_s(\mathbb{Z}_n)) = n - \phi(n) - 1$ .

*Proof.* 1. Suppose that  $n$  is prime. Then order of every nonzero element of  $\mathbb{Z}_n$  is  $n$  which implies that no nonzero element of  $\mathbb{Z}_n$  is adjacent with  $[0]$ . Thus  $\mathcal{P}_s(\mathbb{Z}_n)$  is a disconnected graph with two components  $\{[0]\}$  and  $\mathbb{Z}_n \setminus \{[0]\}$ , which implies that  $\kappa(\mathcal{P}_s(\mathbb{Z}_n)) = 0$ .

2. If  $n$  is composite then  $\mathbb{Z}_n$  has  $n - \phi(n) - 1$  nonzero and nongenerator elements. Each of these nonzero nongenerators is adjacent to every other vertex, which implies that to make the graph  $\mathcal{P}_s(\mathbb{Z}_n)$  disconnected we have to remove at least these  $n - \phi(n) - 1$  nonzero nongenerators. Thus  $\kappa(\mathcal{P}_s(G)) \geq n - \phi(n) - 1$ . Since each generator is adjacent to every nonzero element but not with  $[0]$ , the removal of these  $n - \phi(n) - 1$  nonzero nongenerators makes the remaining graph disconnected with two components, one containing  $[0]$  only and other containing all the generators. Thus  $\kappa(\mathcal{P}_s(\mathbb{Z}_n)) = n - \phi(n) - 1$ .  $\square$

The chromatic number of a graph  $\Gamma$  is the minimum number of colors required to color the vertices so that every pair of adjacent vertices get distinct colors.

**Theorem 3.2.** Let  $G$  be a group of order  $n$ .

1. If  $G$  is cyclic then the chromatic number  $\chi(\mathcal{P}_s(G))$  of the strong power graph  $\mathcal{P}_s(G)$  is  $n - 1$ .
2. If  $G$  is noncyclic then the chromatic number  $\chi(\mathcal{P}_s(G))$  of the strong power graph  $\mathcal{P}_s(G)$  is  $n$ .

*Proof.* 1. The subgraph  $\mathcal{P}_s(\mathbb{Z}_n) \setminus \{[0]\}$  is complete. Since the chromatic number of a complete graph with  $n - 1$  vertices is  $n - 1$ , the chromatic number  $\chi(\mathcal{P}_s(\mathbb{Z}_n)) \geq n - 1$ . Since the generators of  $\mathbb{Z}_n$  are not adjacent with  $[0]$ , either of the colors of the generators can be given to  $[0]$  which implies that  $\chi(\mathcal{P}_s(\mathbb{Z}_n)) \leq n - 1$ . Thus  $\chi(\mathcal{P}_s(\mathbb{Z}_n)) = n - 1$ .

2. The result follows from the fact that the strong power graph  $\mathcal{P}_s(G)$  of the noncyclic group  $G$  is complete.  $\square$

### 4 Laplacian spectrum of the strong power graphs of finite groups

If  $L$  is the Laplacian matrix of a graph  $\Gamma$ , then we denote the characteristic polynomial of  $L$  by  $\Theta(\Gamma, x)$ ; and call it the Laplacian characteristic polynomial of  $\Gamma$ . Thus  $\Theta(\Gamma, x) = |xI_n - L|$ , where  $n$  is the order of  $L$ . For the vertices  $v_1, v_2, \dots, v_i$  in  $\Gamma$ ,  $L_{v_1, v_2, \dots, v_i}(\Gamma)$  is defined as the principal submatrix of  $L(\Gamma)$  formed by deleting the rows and columns corresponding to the vertices  $v_1, v_2, \dots, v_i$ . In particular if  $i = n$ , then for convention it has been taken that  $\Theta(L_{v_1, v_2, \dots, v_n}(\Gamma), x) = 1$ . The distance and adjacency spectrum of strong power graphs is obtained by Ma [16]. In this section, we determine the Laplacian spectrum of strong power graphs of finite groups.

**Theorem 4.1.** For each positive integer  $n \geq 2$ , let  $\bar{s}_i (i = 1, 2, \dots, m)$  be the nonzero nongenerators of  $\mathbb{Z}_n$ . Then  $\Theta(\mathcal{P}_s(\mathbb{Z}_n), x) = x(x - n)^{n - \phi(n) - 1} (x - n + \phi(n) + 1)(x - n + 1)^{\phi(n) - 1}$ , where  $m = n - \phi(n) - 1$ .

*Proof.* The Laplacian matrix  $L(\mathcal{P}_s(\mathbb{Z}_n))$  is the  $n \times n$  matrix whose rows and columns are indexed in order by the nonzero nongenerators  $\bar{s}_i (i = 1, 2, \dots, m)$  and the generators of  $\mathbb{Z}_n$  and  $[0]$  is in last position. Then

$$L(\mathcal{P}_s(\mathbb{Z}_n)) = \begin{pmatrix} n-1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & n-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & n-2 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & \cdots & n-2 & 0 \\ -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & n-\phi(n)-1 \end{pmatrix}.$$

Each row and column sum of the above matrix is zero. Then the characteristic polynomial of  $L(\mathcal{P}_s(\mathbb{Z}_n))$  is  $\Theta(L(\mathcal{P}_s(\mathbb{Z}_n)), x) =$

$$\begin{vmatrix} x-(n-1) & \cdots & 1 & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & 1 & \cdots & \vdots \\ 1 & \cdots & x-(n-1) & 1 & \cdots & 1 & \cdots & 1 \\ 1 & \cdots & 1 & x-(n-2) & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & 1 & \cdots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & x-(n-2) & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & x-(n-\phi(n)-1) \end{vmatrix}$$

Multiply the first row of  $\Theta(L(\mathcal{P}_s(\mathbb{Z}_n)), x)$  by  $(x-1)$  and apply the row operation  $R_1' = R_1 - R_2 - R_3 - \dots - R_{(n-1)} - R_n$ . Then expanding the determinant in terms of the first row we get

$$\Theta(L(\mathcal{P}_s(\mathbb{Z}_n)), x) = \frac{x(x-n)}{(x-1)} \Theta(L_{\bar{s}_1}(\mathcal{P}_s(\mathbb{Z}_n)), x).$$

Again multiplying the first row of  $\Theta(L_{\bar{s}_1}(\mathcal{P}_s(\mathbb{Z}_n)), x)$  by  $(x-2)$  and applying the row operation  $R_1' = R_1 - R_2 - R_3 - \dots - R_{(n-1)}$  we get

$$\Theta(L_{\bar{s}_1}(\mathcal{P}_s(\mathbb{Z}_n)), x) = \frac{(x-1)(x-n)}{(x-2)} \Theta(L_{\bar{s}_1, \bar{s}_2}(\mathcal{P}_s(\mathbb{Z}_n)), x),$$

and so

$$\Theta(L(\mathcal{P}_s(\mathbb{Z}_n)), x) = \frac{x(x-n)^2}{(x-2)} \Theta(L_{\bar{s}_1, \bar{s}_2}(\mathcal{P}_s(\mathbb{Z}_n)), x).$$

Continuing in this way we get

$$\Theta(L(\mathcal{P}_s(\mathbb{Z}_n)), x) = \frac{x(x-n)^{n-\phi(n)-1}}{(x-n+\phi(n)+1)} \Theta(L_{\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m}(\mathcal{P}_s(\mathbb{Z}_n)), x),$$

where

$$\Theta(L_{\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m}(\mathcal{P}_s(\mathbb{Z}_n)), x) = \begin{vmatrix} x-\lambda & 1 & 1 & \cdots & 0 \\ 1 & x-\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & x-m \end{vmatrix}$$

$\lambda = n-2, m = n-\phi(n)-1$  and order of the determinant is  $\phi(n)+1$ . Now expanding  $\Theta(L_{\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m}(\mathcal{P}_s(\mathbb{Z}_n)), x)$  with respect to last row we get

$$\begin{aligned} \Theta(L_{\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m}(\mathcal{P}_s(\mathbb{Z}_n)), x) &= (-1)^{2\phi(n)}(x - m) \begin{vmatrix} x - \lambda & 1 & \cdots & 1 \\ 1 & x - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & x - \lambda \end{vmatrix} \\ &= (x - n + \phi(n) + 1)(x - \lambda + \phi(n) + 1 - 2)(x - \lambda - 1)^{\phi(n)-1} \\ &= (x - n + \phi(n) + 1)^2(x - n + 2 - 1)^{\phi(n)-1} \\ &= (x - n + \phi(n) + 1)^2(x - n + 1)^{\phi(n)-1}. \end{aligned}$$

Thus  $\Theta(L(\mathcal{P}_s(\mathbb{Z}_n)), x) = x(x - n)^{n-\phi(n)-1}(x - n + \phi(n) + 1)(x - n + 1)^{\phi(n)-1}$ .

□

**Proposition 4.2.** *Let  $G$  be a group of order  $n$ .*

1. *If  $G$  is cyclic, then the laplacian spectrum of  $\mathcal{P}_s(G)$  is*

$$\left( \begin{array}{cccc} 0 & n & n - \phi(n) - 1 & n - 1 \\ 1 & n - \phi(n) - 1 & 1 & \phi(n) - 1 \end{array} \right).$$

2. *If  $G$  is noncyclic, then the Laplacian spectrum of  $\mathcal{P}_s(G)$  is*

$$\left( \begin{array}{cc} 0 & n \\ 1 & n - 1 \end{array} \right).$$

The algebraic connectivity of a graph  $\Gamma$ , denoted by  $a(\Gamma)$ , is the second smallest Laplacian eigenvalue of  $\Gamma$  [8]. Now the algebraic connectivity has received special attention due to its huge applications on connectivity problems, isoperimetric numbers, genus, combinatorial optimizations and many other problems. It follows immediately from Proposition 4.2 that:

**Corollary 4.3.** *Let  $G$  be a group of order  $n$ .*

- 1 *If  $G$  is a cyclic group then  $a(\mathcal{P}_s(G)) = n - \phi(n) - 1$ .*
- 2 *If  $G$  is a noncyclic group then  $a(\mathcal{P}_s(G)) = n$ .*

Another important application of Laplacian spectrum is on the number of spanning trees of a graph. A spanning tree  $T$  of a graph  $\Gamma$  is a subgraph which is a tree having same vertex set as  $\Gamma$ . If  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n = 0$  are the Laplacian eigenvalues of a graph  $\Gamma$  of  $n$ -vertices, then the number of spanning trees of  $\Gamma$ , denoted by  $\tau(\Gamma)$ , is  $\frac{\lambda_1 \lambda_2 \dots \lambda_{n-1}}{n}$  [Theorem 4.11; [2]]. Thus from Proposition 4.2 we have:

**Corollary 4.4.** *Let  $G$  be a group of order  $n$ .*

- 1 *If  $G$  is a cyclic group then  $\tau(\mathcal{P}_s(G)) = n^{n-\phi(n)-2}(n - \phi(n) - 1)(n - 1)^{\phi(n)-1}$ .*
- 2 *If  $G$  is a noncyclic group then  $\tau(\mathcal{P}_s(G)) = n^{n-2}$ .*

The graph energy is defined in terms of the spectrum of the adjacency matrix. Depending on the well-developed spectral theory of the Laplacian matrix, recently Gutman et. al [10] have defined the Laplacian energy of a graph  $\Gamma$  with  $n$  vertices and  $m$  edges as:  $LE(\Gamma) = \sum_{i=1}^n |\lambda_i - \frac{2m}{n}|$ , where  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n = 0$  are the Laplacian eigenvalues of the graph  $\Gamma$ . This definition has been adjusted so that the Laplacian energy becomes equal to the energy for any regular graph. For various properties of Laplacian energy we refer [11–13]. From Proposition 4.2 we have

**Corollary 4.5.** *Let  $G$  be a finite group of order  $n$ .*

- 1 *If  $G$  is cyclic then  $LE(\mathcal{P}_s(G)) = 2(n - 1) - \frac{4\phi(n)}{n}$ .*
- 2 *If  $G$  is noncyclic then  $LE(\mathcal{P}_s(G)) = 2(n - 1)$ .*

## 5 Permanent of the Laplacian of strong power graph

If  $A = (a_{ij})$  be a square matrix of order  $n$ , then the permanent of  $A$ , denoted by  $per(A)$ , is  $per(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ , where  $S_n$  is the set of all permutations of  $1, 2, \dots, n$ . The permanent function was introduced by Binet and independently by Cauchy in 1812. It is quite difficult to determine the permanent of a square matrix. We refer to [17] and [18] for more on permanent. In this section we have determined the permanent of the Laplacian matrix of strong power graph of any finite group explicitly. Our method is based on the following observation. Let  $A = (a_{ij})$  be a matrix of order  $n$ . Then  $per(A)$  is equal to the coefficient of  $x_1 x_2 \cdots x_n$  in the expression  $(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) \cdots (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n)$ . Throughout the rest of this section we make the following convention: for any  $n$ -functions  $f_1(x), f_2(x), \dots, f_n(x)$  we denote  $f_1(x)f_2(x) \cdots f_n(x) = f_1(x)f_2(x) \cdots f_{i-1}(x)f_{i+1}(x) \cdots f_n(x)$ , and for any  $n$  variables  $x_1, x_2, \dots, x_n$ , the coefficient of  $x_{n_1}x_{n_2} \cdots x_{n_k}$  in a polynomial  $F(x_1, x_2, \dots, x_n)$  will be denoted by  $C_{x_{n_1}x_{n_2} \cdots x_{n_k}}(F(x_1, x_2, \dots, x_n))$ . Here we first find the permanent of adjacency matrix of any graph with  $m+n+1$  vertices such that  $m+n$  vertices form a clique and the rest vertex is adjacent with  $n$  vertices. We have the following Lemma,

**Lemma 5.1.** *The permanent of the adjacency matrix of any graph with  $m+n+1$  vertices such that  $m+n$  vertices form a clique and the rest vertex is adjacent with  $n$  vertices is*

$$n! \left[ \sum_{r=1}^{m+n} (-1)^{r-1} (m+n-r)! \left\{ \binom{m+n-1}{r-1} + (n-1) \binom{m+n-2}{r-1} \right\} \right].$$

*Proof.* The required permanent is the coefficient of  $x_1 x_2 \cdots x_{m+n+1}$  in  $F(x_1, x_2, \dots, x_{m+n+1}) = (X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+1} + x_{m+n+1})(X - x_{m+2} + x_{m+n+1}) \cdots (X - x_{m+n} + x_{m+n+1})(x_{m+1} + x_{m+2} + \cdots + x_{m+n})$ , where  $X = x_1 + x_2 + \cdots + x_{m+n}$ .

Now we have  $F(x_1, x_2, \dots, x_{m+n+1}) = \prod_{i=1}^m (X - x_i) [x_{m+n+1}^{m+n} + \cdots + x_{m+n+1} \sum_{i=1}^n (X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+i}) \cdots (X - x_{m+n}) + \prod_{i=1}^n (X - x_{m+i}) (\sum_{i=1}^n x_{m+i})]$  which shows that

$$\begin{aligned} & C_{x_1 x_2 \cdots x_{m+n+1}}(F(x_1, x_2, \dots, x_{m+n+1})) \\ &= C_{x_1 x_2 \cdots x_{m+n}} \prod_{i=1}^m (X - x_i) \left[ \sum_{i=1}^n (X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+i}) \cdots (X - x_{m+n}) \right] \left( \sum_{i=1}^n x_{m+i} \right) \\ &= n C_{x_1 x_2 \cdots x_{m+n}} (x_{m+1} \prod_{i=1}^m (X - x_i) \left[ \sum_{i=1}^n (X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+i}) \cdots (X - x_{m+n}) \right]) \\ &= n C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} \left( \prod_{i=1}^m (X - x_i) \left[ \sum_{i=1}^n (X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+i}) \cdots (X - x_{m+n}) \right] \right) \\ &= n [ C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} ((X - x_1) \cdots (X - x_m)(X - x_{m+2}) \cdots (X - x_{m+n})) + \\ & \quad C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} ((X - x_1) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+3}) \cdots (X - x_{m+n})) + \cdots \\ & \quad \cdots + C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} ((X - x_1) \cdots (X - x_m)(X - x_{m+1}) \cdots (X - x_{m+n-1})) ] \end{aligned}$$

Now,  $(X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+2}) \cdots (X - x_{m+n}) = X^{m+n-1} - X^{m+n-2} \sum_{i \neq (m+1)} x_i + \cdots + (-1)^{m+n-1} x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}$ , shows that  $C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} ((X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+2}) \cdots (X - x_{m+n})) = \sum_{r=1}^{m+n} (-1)^{r-1} (m+n-r)! \binom{m+n-1}{r-1}$ , and  $(X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+3}) \cdots (X - x_{m+n}) = X^{m+n-1} - X^{m+n-2} \sum_{i \neq (m+2)} x_i + \cdots + (-1)^{m+n-1} x_1 x_2 \cdots x_m x_{m+1} x_{m+3} \cdots x_{m+n}$  shows that  $C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} ((X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+3}) \cdots (X - x_{m+n})) = \sum_{r=1}^{m+n} (-1)^{r-1} (m+n-r)! \binom{m+n-2}{r-1}$ .

Also  $C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} ((X - x_1)(X - x_2) \cdots (X - x_{m+2})(X - x_{m+4}) \cdots (X - x_{m+n})) = \cdots = C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}} ((X -$

$$x_1)(X - x_2) \cdots (X - x_m) \cdots (X - x_{m+n-1}) \\ = \sum_{r=1}^{m+n} (-1)^{r-1} (m+n-r)! \binom{m+n-2}{r-1}.$$

Hence the permanent of the adjacency matrix of the stated graph is

$$n \left[ \sum_{r=1}^{m+n} (-1)^{r-1} (m+n-r)! \left\{ \binom{m+n-1}{r-1} + (n-1) \binom{m+n-2}{r-1} \right\} \right].$$

□

Now from the above lemma we get the permanent of the adjacency matrix of strong power graph of any finite cyclic group. The cyclic group  $\mathbb{Z}_n$  has  $m = \phi(n)$  generators none of which is adjacent to  $[0]$ . Thus the set  $\mathbb{Z}_n \setminus \{[0]\}$  of all  $n - 1$  nonzero vertices forms a clique and  $[0]$  is adjacent to each of the  $n - (\phi(n) + 1)$  nonzero nongenerators. Hence from Lemma 5.1 it follows immediately that:

**Theorem 5.2.** *The permanent of the adjacency matrix of the strong power graph  $\mathcal{P}_s(\mathbb{Z}_n)$  of  $\mathbb{Z}_n$  is*

$$(n - \phi(n) - 1) \left[ \sum_{r=1}^{n-1} (-1)^{r-1} (n-1-r)! \left\{ \binom{n-2}{r-1} + (n-2 - \phi(n)) \binom{n-3}{r-1} \right\} \right].$$

Now we compute the permanent of the Laplacian matrix of the strong power graph of any finite group. For this, first we prove the following lemma.

**Lemma 5.3.** *The permanent of the Laplacian matrix of a graph  $\Gamma$  with  $m + n + 1$  vertices such that  $m + n$  vertices form a clique and the rest vertex is edge connected with  $n$  vertices is  $\sum_{r=1}^{m+n} (m+n-r)! F_r(d)$ , where  $F_r(d) = \sum_{i+j=r-1} \binom{m}{i} (d+2)^j (d+1)^i \left[ n \binom{n-1}{j} + n(n-1) \binom{n-2}{j} + (-1)^{m+n-r+1} (d-m+1)(m+n-r+1) \binom{n}{j} \right]$ , and  $d = m + n - 1$ .*

*Proof.* Consider  $F(x_1, x_2, \dots, x_{m+n+1}) = \prod_{i=1}^m (X + (d+1)x_i) \prod_{i=1}^n (X + (d+2)x_{m+i} - x_{m+n+1}) ((d-m+1)x_{m+n+1} + \sum_{i=1}^n -x_{m+i})$ , where  $X = -(x_1 + x_2 + \dots + x_{m+n})$ . Then the permanent of the Laplacian matrix of  $\Gamma$  is  $C_{x_1 x_2 \dots x_{m+n+1}}(F(x_1, x_2, \dots, x_{m+n+1}))$  which is equal to  $C_{x_1 x_2 \dots x_{m+n+1}}(\prod_{i=1}^m (X + (d+1)x_i) \prod_{i=1}^n (X + (d+2)x_{m+i} - x_{m+n+1}) (\sum_{i=1}^n -x_{m+i})) + C_{x_1 x_2 \dots x_{m+n}}((d-m+1) \prod_{i=1}^m X + (d+1)x_i) \prod_{i=1}^n (X + (d+2)x_{m+i}))$ . Now  $(d-m+1) \prod_{i=1}^m (X + (d+1)x_i) \prod_{i=1}^n (X + (d+2)x_{m+i}) = (d-m+1)[X^{m+n} + X^{m+n-1} \sum f_1(d)x_i + X^{m+n-2} \sum f_{12}(d)x_1 x_2 + \dots + f_{123\dots m+n}(d)x_1 x_2 \dots x_{m+n}]$ , where  $f_{123\dots j}(d)$  is a product of some  $(d+1)$  and  $(d+2)$  which is clear from the context. So  $C_{x_1 x_2 \dots x_{m+n}}((d-m+1) \prod_{i=1}^m (X + (d+1)x_i) \prod_{i=1}^n (X + (d+2)x_{m+i})) = (d-m+1) \sum_{r=1}^{m+n+1} (-1)^{m+n-r+1} (m+n-r+1)! f_r(d)$ , where  $f_r(d) = \sum_{i+j=r-1} \binom{m}{i} \binom{n}{j} (d+2)^j (d+1)^i$ . Now proceeding as the proof of Lemma 5.1 we get  $C_{x_1 x_2 \dots x_{m+n+1}}(\prod_{i=1}^m (X + (d+1)x_i) \prod_{i=1}^n (X + (d+2)x_{m+i} - x_{m+n+1}) (\sum_{i=1}^n -x_{m+i})) = n \sum_{r=1}^{m+n} (-1)^{m+n-r} (m+n-r)! \left[ \sum_{i+j=r-1} \binom{m}{i} \left( \binom{n-1}{j} + (n-1) \binom{n-2}{j} \right) (d+2)^j (d+1)^i \right]$ .

Hence the permanent of the Laplacian matrix of  $\Gamma$  is

$$\sum_{r=1}^{m+n} \{ (-1)^{m+n-r} (m+n-r)! F_r(d) \} + (d-m+1) \sum_{i+j=m+n} \binom{m}{i} \binom{n}{j} (d+2)^j (d+1)^i,$$

where

$$F_r(d) = \sum_{i+j=r-1} \binom{m}{i} (d+2)^j (d+1)^i \left\{ n \binom{n-1}{j} + n(n-1) \binom{n-2}{j} - (d-m+1)(m+n-r+1) \binom{n}{j} \right\}.$$

□

**Theorem 5.4.** *Let  $G$  be a group of order  $n$ .*



1. If  $G$  is cyclic then the permanent of the Laplacian matrix of strong power graph of  $G$  is  $\sum_{r=1}^{n-1} \{(-1)^{n-r-1}(n-r-1)!F_r(d)\} + (n-\phi(n)-1) \sum_{i+j=n-1} \binom{\phi(n)}{i} \binom{n-\phi(n)-1}{j} n^i(n-1)^j$ , where
- $$F_r(d) = \sum_{i+j=r-1} \binom{\phi(n)}{i} n^i(n-1)^j \{ (n-\phi(n)-1) \binom{n-\phi(n)-2}{j} + (n-\phi(n)-1)(n-\phi(n)-2) \binom{n-\phi(n)-3}{j} - (n-\phi(n)-1)(n-r) \binom{n-\phi(n)-1}{j} \}.$$
2. If  $G$  is noncyclic then the permanent of the Laplacian matrix of the strong power graph  $\mathcal{P}_s(G)$  is  $(-1)^n n!(1 - \frac{n}{1!} + \frac{n^2}{2!} - \frac{n^3}{3!} + \dots + (-1)^n \frac{n^n}{n!})$ .

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