

## Research Article

## Open Access

Special Issue: Proceedings of the 24th International Workshop on Matrices and Statistics

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# On a criterion of $D$ -stability for $P$ -matrices

DOI 10.1515/spma-2016-0017

Received October 23, 2015; accepted March 22, 2016

**Abstract:** In this paper, we study positive stability and  $D$ -stability of  $P$ -matrices. We introduce the property of  $D_\theta$ -stability, i.e., the stability with respect to a given order  $\theta$ . For an  $n \times n$   $P$ -matrix  $\mathbf{A}$ , we prove a new criterion of  $D$ -stability and  $D_\theta$ -stability, based on the properties of matrix scalings.

**Keywords:**  $P$ -matrices,  $Q^2$ -matrices,  $D$ -stability, matrix scalings,  $P$ -matrix powers, stabilization

**MSC:** 15A12, 15A18, 15A75

## 1 Introduction

An  $n \times n$  real matrix  $\mathbf{A}$  is called *positive stable* or just *stable* if all its eigenvalues have positive real parts (note that if a matrix  $\mathbf{A}$  is positive stable, then  $-\mathbf{A}$  is stable in the traditional Lyapunov sense, since all the eigenvalues of  $-\mathbf{A}$  have negative real parts). Let  $\mathbf{D} = \text{diag}\{d_{11}, \dots, d_{nn}\}$  denote a positive diagonal matrix, i.e., an  $n \times n$  diagonal matrix with positive entries on its principal diagonal (while the rest are zero). An  $n \times n$  real matrix  $\mathbf{A}$  is called  *$D$ -stable* if  $\mathbf{DA}$  is stable for any positive diagonal matrix  $\mathbf{D}$ . The concept of  $D$ -stability was introduced to analyze the stability of systems of differential equations (see [1], [17]). Thus  $D$ -stability has a huge variety of applications in the mathematical modelling (see, for example, [14]). Unfortunately, this property is not easy to verify. For  $n = 3$ , Cain provided a complete description of real  $D$ -stable matrices (see [4]). For  $n = 4$ , a verifiable criterion of  $D$ -stability was proved by Kanovei and Logofet (see [13]). For  $n > 4$ , there are some necessary for  $D$ -stability conditions as well as some classes of structured matrices that are known to be  $D$ -stable (see [15]). However, the problem of characterizing  $D$ -stable matrices of  $n$  dimensions is still open.

In this paper, we mainly focus on the  $D$ -stability of  $P$ -matrices (an  $n \times n$  matrix  $\mathbf{A}$  is called a  *$P$ -matrix* if all its principal minors are positive, i.e., the inequality  $A \begin{pmatrix} i_1 & \cdots & i_k \\ i_1 & \cdots & i_k \end{pmatrix} > 0$  holds for all  $(i_1, \dots, i_k)$ ,  $1 \leq i_1 < \dots < i_k \leq n$ , and all  $k$ ,  $1 \leq k \leq n$ .) Note that  $P$ -matrices are not necessarily stable.

**Example.** Consider

$$\mathbf{A} = \begin{pmatrix} 8.5 & 6 & 45 \\ 5 & 7.5 & 6 \\ 0.9 & 6 & 5 \end{pmatrix}.$$

It is easy to verify that  $\mathbf{A}$  is a  $P$ -matrix with  $A \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 33.75 > 0$ ,  $A \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = 2 > 0$ ,  $A \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = 1.5 > 0$  and  $\det \mathbf{A} = 941.4 > 0$ . However, it is not positive stable, since its eigenvalues are as follows:  $\lambda_1 \approx 21.3235$ ,  $\lambda_{2,3} \approx -0.161757 \pm 6.64246i$ .

There is a number of papers considering the additional conditions upon which  $P$ -matrices are positive stable (see, for example, [5], [11], [12], [18]).

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In this paper, we introduce the following concept which is stronger than stability but weaker than  $D$ -stability.

Given a positive diagonal matrix  $\mathbf{D} = \text{diag}\{d_{11}, \dots, d_{nn}\}$  and a permutation  $\theta = (\theta(1), \dots, \theta(n))$  of the set of indices  $[n] := \{1, \dots, n\}$ , we call the matrix  $\mathbf{D}$  *ordered with respect to  $\theta$* , or  $\theta$ -*ordered*, if it satisfies the inequalities

$$d_{\theta(i)\theta(i)} \geq d_{\theta(i+1)\theta(i+1)}, \quad i = 1, \dots, n - 1.$$

We say that a matrix  $\mathbf{A}$  is  $D$ -stable with respect to the order  $\theta$ , or  $D_\theta$ -stable, if the matrix  $\mathbf{DA}$  is positive stable for every  $\theta$ -ordered positive diagonal matrix  $\mathbf{D}$ .

In this paper, we connect  $D_\theta$ -stability and the properties of matrix scalings to provide the following criterion of  $D_\theta$ -stability for  $P$ -matrices. Here an  $n \times n$  matrix  $\mathbf{A}$  is called a  $Q$ -matrix if the sums of its principal minors of every fixed order are positive, i.e., for any  $k$ ,  $1 \leq k \leq n$ , we have the inequality

$$\sum_{(i_1, \dots, i_k)} A \begin{pmatrix} i_1 & \dots & i_k \\ i_1 & \dots & i_k \end{pmatrix} > 0,$$

where the sum is taken for all the sets  $(i_1, \dots, i_k)$ ,  $1 \leq i_1 < \dots < i_k \leq n$ .

**Theorem 1.1.** *Given a permutation  $\theta = (\theta(1), \dots, \theta(n))$  and an  $n \times n$   $P$ -matrix  $\mathbf{A}$ , let  $(\mathbf{DA})^2$  be a  $Q$ -matrix for any  $\theta$ -ordered positive diagonal matrix  $\mathbf{D}$ . Then  $\mathbf{A}$  is  $D_\theta$ -stable.*

We also provide the following criterion of matrix  $D$ -stability ensuing from the criterion of  $D_\theta$ -stability given above.

**Theorem 1.2.** *Let an  $n \times n$  matrix  $\mathbf{A}$  be a  $P$ -matrix and  $(\mathbf{DA})^2$  be a  $Q$ -matrix for every positive diagonal matrix  $\mathbf{D}$ . Then  $\mathbf{A}$  is  $D$ -stable.*

The paper is organized as follows. In Section 1, we provide the description of the main method used in the paper: the technique of exterior products and compound matrices. In Section 2, we collect definitions and statements concerning  $P$ - and  $Q$ -matrices. In Section 3, ordered diagonal matrices are studied. Section 4 deals with matrix stabilization results. In Section 5, we give the proofs of Theorem 1.1 and Theorem 1.2. Section 6 is due to the applications to some classes of structured matrices.

## 2 Exterior products of operators and their matrices

Let us recall the following basic definitions and facts (see, for example, [8]). Let  $\{e_1, \dots, e_n\}$  be an arbitrary basis in  $\mathbb{R}^n$ . Let  $x_1, \dots, x_j$  ( $2 \leq j \leq n$ ) be arbitrary vectors in  $\mathbb{R}^n$ , defined by their coordinates:  $x_i = (x_1^i, \dots, x_n^i)$  in the basis  $\{e_1, \dots, e_n\}$ . The exterior product  $x_1 \wedge \dots \wedge x_j$  of the vectors  $x_1, \dots, x_j$  is defined as a vector in  $\mathbb{R}^{\binom{n}{j}}$  ( $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ ), with the coordinates  $(\varphi_1, \dots, \varphi_{\binom{n}{j}})$ , where

$$\varphi_\alpha = \begin{vmatrix} x_1^{i_1} & \dots & x_j^{i_1} \\ \dots & \dots & \dots \\ x_1^{i_j} & \dots & x_j^{i_j} \end{vmatrix}.$$

Here the number  $\alpha$  of the coordinate  $\varphi_\alpha$  is the number of the set of indices  $(i_1, \dots, i_j) \subseteq [n]$  in the lexicographic ordering.

We consider the  $j$ th exterior power  $\wedge^j \mathbb{R}^n$  of the space  $\mathbb{R}^n$  as the space  $\mathbb{R}^{\binom{n}{j}}$ . The set of all exterior products of the form  $e_{i_1} \wedge \dots \wedge e_{i_j}$  where  $1 \leq i_1 < \dots < i_j \leq n$  forms a canonical basis in  $\wedge^j \mathbb{R}^n$ .

Now let us recall the following definitions and statements concerning exterior powers of operators and compound matrices. For a linear operator  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , the operator  $\wedge^j A : \mathbb{R}^{\binom{n}{j}} \rightarrow \mathbb{R}^{\binom{n}{j}}$  is defined by the equality

$$(\wedge^j A)(x_1 \wedge \dots \wedge x_j) = Ax_1 \wedge \dots \wedge Ax_j.$$

The operator  $\wedge^j A$  is called the  $j$ th exterior power of the initial operator  $A$ . It is easy to see that  $\wedge^1 A = A$  and  $\wedge^n A$  is one-dimensional and coincides with  $\det A$ .

If  $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$  is the matrix of  $A$  in the basis  $\{e_1, \dots, e_n\}$ , then the matrix of  $\wedge^j A$  in the basis  $\{e_{i_1} \wedge \dots \wedge e_{i_j}\}$ , where  $1 \leq i_1 < \dots < i_j \leq n$ , equals the  $j$ th compound matrix  $\mathbf{A}^{(j)}$  of the initial matrix  $\mathbf{A}$ . (Here the  $j$ th compound matrix  $\mathbf{A}^{(j)}$  consists of all the minors of the  $j$ th order  $A \begin{pmatrix} i_1 & \dots & i_j \\ k_1 & \dots & k_j \end{pmatrix}$ , where  $1 \leq i_1 < \dots < i_j \leq n$ ,  $1 \leq k_1 < \dots < k_j \leq n$ , of the initial  $n \times n$  matrix  $\mathbf{A}$  listed in the lexicographic order (see, for example, [16])).

The following properties of compound matrices are well known.

1. Let  $\mathbf{A}$ ,  $\mathbf{B}$  be  $n \times n$  matrices. Then  $(\mathbf{AB})^{(j)} = \mathbf{A}^{(j)}\mathbf{B}^{(j)}$  for  $j = 1, \dots, n$  (the Cauchy–Binet formula).
2. The  $j$ -th compound matrix  $\mathbf{A}^{(j)}$  of an invertible matrix  $\mathbf{A}$  is also invertible and the following equality holds:  $(\mathbf{A}^{(j)})^{-1} = (\mathbf{A}^{-1})^{(j)}$ ,  $j = 1, \dots, n$  (the Jacobi formula).

### 3 $P$ - and $Q$ -matrices

Here  $\sigma(\mathbf{A})$  denotes the spectrum of the matrix  $\mathbf{A}$ , i.e., the set of all its eigenvalues. Let us recall some basic facts concerning  $P$ -matrices (see [6]) and  $P^2$ -matrices (a matrix  $\mathbf{A}$  is called a  $P^2$ -matrix if both  $\mathbf{A}$  and  $\mathbf{A}^2$  are  $P$ -matrices (see [12])). First, let us list the matrix operations which preserve the class of  $P$ -matrices (see, for example, [19], Theorem 3.1).

**Lemma 3.1.** *Let  $\mathbf{A}$  be a  $P$ -matrix. Then the following matrices are also  $P$ -matrices.*

1.  $\mathbf{A}^T$  (the transpose of  $\mathbf{A}$ );
2.  $\mathbf{A}^{-1}$  (the inverse of  $\mathbf{A}$ );
3.  $\mathbf{DAD}^{-1}$ , where  $\mathbf{D}$  is an invertible diagonal matrix;
4.  $\mathbf{PAP}^{-1}$ , where  $\mathbf{P}$  is a permutation matrix;
5. any principal submatrix of  $\mathbf{A}$ ;
6. the Schur complement of any principal submatrix of  $\mathbf{A}$ .
7. both  $\mathbf{DA}$  and  $\mathbf{AD}$ , where  $\mathbf{D}$  is a positive diagonal matrix.

It was shown by Fiedler and Ptak that every real eigenvalue of a  $P$ -matrix is positive (see [6], p. 385, Theorem 3.3). Hershkowitz established the same spectral properties for a wider class of  $Q$ -matrices (recall that a matrix  $\mathbf{A}$  is called a  $Q$ -matrix if the inequality

$$\sum_{(i_1, \dots, i_k)} A \begin{pmatrix} i_1 & \dots & i_k \\ i_1 & \dots & i_k \end{pmatrix} > 0$$

holds for all  $k$ ,  $1 \leq k \leq n$ .) It was shown in [10], that a set  $\{\lambda_1, \dots, \lambda_n\}$ ,  $\lambda_i \in \mathbb{C}$ , is a spectrum of some  $P$ -matrix if and only if it is a spectrum of some  $Q$ -matrix (see [10], p. 83, Theorem 1). This statement implies

**Theorem 3.1.** *Every real eigenvalue of a  $Q$ -matrix  $\mathbf{A}$  is positive.*

However,  $P$ -matrices are not necessarily positive stable. They may have non-real eigenvalues in the left-hand side of the complex plane. For  $P^2$ -matrices (recall that  $\mathbf{A}$  is called a  $P^2$ -matrix if both  $\mathbf{A}$  and  $\mathbf{A}^2$  are  $P$ -matrices), the question of their positive stability is still open (see [12], p. 122, Question 6.2).

Later we show that, under some additional assumptions, a  $P$ -matrix is  $D_\theta$ -stable with respect to a certain order  $\theta$ . For this, we refer to the spectral properties of  $Q^2$ -matrices (a matrix  $\mathbf{A}$  is called a  $Q^2$ -matrix if  $\mathbf{A}$  and  $\mathbf{A}^2$  are both  $Q$ -matrices) which are known to be unstable (see [11]).

**Theorem 3.2.** *Let  $\mathbf{A}$  be a  $Q^2$ -matrix. Then it has no eigenvalues of the form  $\lambda = \alpha i$ , where  $\alpha \in \mathbb{R}$ .*

*Proof.* It is enough for the proof to observe that  $\lambda \in \sigma(\mathbf{A})$  if and only if  $\lambda^2 \in \sigma(\mathbf{A}^2)$ . Assume that  $\lambda = \alpha_0 i \in \sigma(\mathbf{A})$  for some  $\alpha_0 \in \mathbb{R}$ . Then  $\lambda^2 = -\alpha_0^2 \in \sigma(\mathbf{A}^2)$ . Since  $-\alpha_0^2 \leq 0$  for any  $\alpha_0 \in \mathbb{R}$ , this contradicts Theorem 3.1.  $\square$

## 4 Ordered diagonal matrices

In what follows,  $\Theta_{[n]}$  denotes, as usual, the set of all the permutations of  $[n]$ ,  $\mathcal{D}$  denotes the set of all positive diagonal matrices,  $\mathcal{D}_\theta$  denote the subset of positive diagonal matrices ordered with respect to a given permutation  $\theta \in \Theta_{[n]}$ . Obviously,

$$\mathcal{D} = \bigcup_{\theta \in \Theta_{[n]}} \mathcal{D}_\theta.$$

Let us mention the following fact.

**Lemma 4.1.** *A matrix  $\mathbf{A}$  is  $D$ -stable if and only if it is  $D_\theta$ -stable for any  $\theta \in \Theta_{[n]}$ .*

*Proof.* It is enough to observe that  $\mathcal{D} = \bigcup_{\theta \in \Theta_{[n]}} \mathcal{D}_\theta$ , thus every positive diagonal matrix  $\mathbf{D}$  belongs to one of the classes  $\mathcal{D}_\theta$ .  $\square$

Let us analyze the following properties of the classes  $\mathcal{D}_\theta$ .

**Lemma 4.2.** *Let  $\theta$  be an arbitrary permutation from  $\Theta_{[n]}$ . The following inclusions hold:*

1. *If  $\mathbf{D}_1 \in \mathcal{D}_\theta$ ,  $\mathbf{D}_2 \in \mathcal{D}_\theta$  then  $\mathbf{D}_1 \mathbf{D}_2 \in \mathcal{D}_\theta$ .*
2. *Define  $\mathbf{D}_t := t\mathbf{I} + (1-t)\mathbf{D}$ ,  $t \in [0, 1]$ . If  $\mathbf{D} \in \mathcal{D}_\theta$  then  $\mathbf{D}_t \in \mathcal{D}_\theta$  for every  $t \in [0, 1]$ .*
3. *Denote  $(1, \dots, n)$  the natural order of the set  $[n]$ . If  $\mathbf{D} \in \mathcal{D}_{(1, \dots, n)}$ , then  $\mathbf{P}_\theta \mathbf{D} (\mathbf{P}_\theta)^{-1} \in \mathcal{D}_\theta$ . Here  $\mathbf{P}_\theta$  is an  $n \times n$  permutation matrix which corresponds to the permutation  $\theta$ .*

*Proof.*

1. Let  $\mathbf{D}_1 = \text{diag}\{d_{11}^1, \dots, d_{nn}^1\}$ , satisfying  $d_{\theta(i)\theta(i)}^1 \geq d_{\theta(i+1)\theta(i+1)}^1$  for  $i = 1, \dots, n-1$ . In its turn,  $\mathbf{D}_2 = \text{diag}\{d_{11}^2, \dots, d_{nn}^2\}$ , satisfying  $d_{\theta(i)\theta(i)}^2 \geq d_{\theta(i+1)\theta(i+1)}^2$  for  $i = 1, \dots, n-1$ . Since  $\mathbf{D}_1 \mathbf{D}_2 = \text{diag}\{d_{11}^1 d_{11}^2, \dots, d_{nn}^1 d_{nn}^2\}$ , it obviously satisfies

$$d_{\theta(i)\theta(i)}^1 d_{\theta(i)\theta(i)}^2 \geq d_{\theta(i+1)\theta(i+1)}^1 d_{\theta(i+1)\theta(i+1)}^2$$

for  $i = 1, \dots, n-1$ .

2. Consider  $\mathbf{D}_t = \text{diag}\{d_{11}^t, \dots, d_{nn}^t\}$ . By the construction,  $d_{ii}^t = t + (1-t)d_{ii}$ . Since  $d_{\theta(i)\theta(i)} \geq d_{\theta(i+1)\theta(i+1)}$  for  $i = 1, \dots, n-1$  and both  $t$  and  $1-t$  are nonnegative for all  $t \in [0, 1]$ , we have:

$$(1-t)d_{\theta(i)\theta(i)} \geq (1-t)d_{\theta(i+1)\theta(i+1)}$$

and

$$t + (1-t)d_{\theta(i)\theta(i)} \geq t + (1-t)d_{\theta(i+1)\theta(i+1)}.$$

Thus,  $d_{\theta(i)\theta(i)}^t \geq d_{\theta(i+1)\theta(i+1)}^t$  for  $i = 1, \dots, n-1$ .

3. Since  $\mathbf{D} \in \mathcal{D}_{(1, \dots, n)}$ , we have  $\mathbf{D} = \text{diag}\{d_{11}, \dots, d_{nn}\}$ , where  $d_{ii} \geq d_{i+1, i+1}$ ,  $i = 1, \dots, n-1$ . For  $\tilde{\mathbf{D}} = \mathbf{P}_\theta \mathbf{D} (\mathbf{P}_\theta)^{-1}$ , we obtain that  $\tilde{\mathbf{D}} = \text{diag}\{\tilde{d}_{11}, \dots, \tilde{d}_{nn}\}$ , where  $\tilde{d}_{ii} = d_{\theta^{-1}(i)\theta^{-1}(i)}$ . Thus,  $\tilde{d}_{\theta(i)\theta(i)} = d_{ii} \geq d_{i+1, i+1} = \tilde{d}_{\theta(i+1)\theta(i+1)}$  for  $i = 1, \dots, n-1$ .  $\square$

## 5 Matrix stabilization

Let us recall the following definition (see, for example, [9]). An  $n \times n$  matrix  $\mathbf{A}$  is said to have a *nested sequence of positive principal minors* or simply a *nest*, if there is a permutation  $(i_1, \dots, i_n)$  of the set of indices  $[n]$  such

that

$$A \begin{pmatrix} i_1 & \cdots & i_j \\ i_1 & \cdots & i_j \end{pmatrix} > 0 \quad j = 1, \dots, n.$$

Then we introduce the following matrix property.

An  $n \times n$  positive diagonal matrix  $\mathbf{D}_{(A)}$  is called a *stabilization matrix* for an  $n \times n$  matrix  $\mathbf{A}$  if  $\mathbf{D}_{(A)}\mathbf{A}$  is positive stable (i.e.  $\lambda \in \sigma(\mathbf{D}_{(A)}\mathbf{A})$  implies  $\operatorname{Re}(\lambda) > 0$ ). An  $n \times n$  real matrix  $\mathbf{A}$  is called *stabilizable* if there is at least one stabilization matrix  $\mathbf{D}_{(A)}$ .

The following sufficient condition for the existence of a stabilization matrix were provided by Fisher and Fuller (see [2], p. 728, Theorem 1, also [7]).

**Theorem 5.1** (Fisher, Fuller). *Let  $\mathbf{A}$  be an  $n \times n$  real matrix, all of whose leading principal minors are positive. Then there is an  $n \times n$  positive diagonal matrix  $\mathbf{D}_{(A)}$ , such that all of the eigenvalues of  $\mathbf{D}_{(A)}\mathbf{A}$  are positive and simple.*

**Corollary 5.1.** *An  $n \times n$  real matrix  $\mathbf{A}$  is stabilizable if it has at least one nested sequence of positive principal minors.*

Later we shall use the following statement based on the Fisher–Fuller result.

**Lemma 5.1.** *Let  $\mathbf{A}$  be an  $n \times n$   $P$ -matrix. Then it is stabilizable. Moreover, for any permutation  $\theta$  of the set  $[n]$ , the stabilization matrix  $\mathbf{D}_{(A)}$  can be chosen to belong to the class  $\mathcal{D}_\theta$ .*

*Proof.* Given an arbitrary permutation  $\theta$ , we construct the permutation matrix  $\mathbf{P}_\theta$  and define  $\mathbf{B} := \mathbf{P}_\theta^{-1}\mathbf{A}\mathbf{P}_\theta$ . By Lemma 3.1, Part 4,  $\mathbf{B}$  is also a  $P$ -matrix. Thus its leading principal minors are all positive:

$$b_{11} > 0, \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0, \dots, \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix} = \det(A) > 0.$$

It is stabilizable by the Fisher–Fuller theorem (Theorem 5.1). Show that we can choose a stabilization matrix  $\mathbf{D}_{(B)}$  in the following form:

$$\mathbf{D}_{(B)} = \operatorname{diag}\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}, \tag{1}$$

where  $1 = \epsilon_1^0 > \epsilon_2^0 > \dots > \epsilon_n^0 > 0$ . To do so, we use the induction on  $n$ . For  $n = 1$ , the result is trivial,  $\mathbf{B} = \{b_{11}\}$  and  $\mathbf{D}_{(B)} = \{1\}$ . Let  $n = 2$ . In this case,  $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$  and we define  $\mathbf{D}_{(B)} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$ , where  $\epsilon_2 > 0$

will be chosen later. Then  $\mathbf{D}_{(B)}\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ \epsilon_2 b_{21} & \epsilon_2 b_{22} \end{pmatrix}$ . Let  $M_2(\epsilon_2, \lambda) := \det(\mathbf{D}_{(B)}\mathbf{B} - \lambda I)$ , which is dependent on  $\epsilon_2$ . Then  $M_2(0, \lambda)$  has two simple roots:  $\lambda_1 = b_{11}$  and  $\lambda_2 = 0$ . The roots of  $M_2(\epsilon_2, \lambda)$  are close to  $b_{11}$  and 0, for sufficiently small values of  $\epsilon_2$ . That means both of them are real (since complex eigenvalues must appear in conjugate pairs), and at least one of them is positive. Since  $\det(\mathbf{D}_{(B)}\mathbf{B})$  is positive, the other eigenvalue is also positive. The continuity of eigenvalues implies that there is a positive integer  $\epsilon_2^0$  such that the above spectral properties hold for all  $\epsilon_2, 0 < \epsilon_2 \leq \epsilon_2^0$ . Thus, we can find  $\epsilon_2 < \min(1, \epsilon_2^0)$  and put  $\mathbf{D}_{(B)} := \operatorname{diag}\{1, \epsilon_2\}$ .

Assume the representation (1) is valid for  $n - 1$ . Let us prove it for  $n$ . Let  $\mathbf{B}$  be an  $n \times n$  matrix and let us show that the entry  $\epsilon_n$  of  $\mathbf{D}_{(B)}$  may be chosen to satisfy the inequalities  $0 < \epsilon_n < \epsilon_{n-1}$ . Repeating the proof of the Fisher–Fuller theorem, we apply the induction hypothesis to the  $(n - 1) \times (n - 1)$  leading principal submatrix  $\mathbf{B}_{11}$  obtained from  $\mathbf{B}$  by deleting the last row and the last column. We use the following partitions:

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$$

and

$$\mathbf{D}_{(B)} = \begin{pmatrix} \mathbf{D}_1 & 0 \\ 0 & \epsilon_n \end{pmatrix},$$

where  $\mathbf{D}_1 = \text{diag}\{\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}\}$ , with  $1 = \epsilon_1 > \epsilon_2 > \dots > \epsilon_{n-1} > 0$  and  $\epsilon_n$  to be chosen later.

Repeating the above reasoning, we put  $M_n(\epsilon_n, \lambda) := \det(\mathbf{D}_{(\mathbf{B})}\mathbf{B} - \lambda I)$ . Nonzero roots of  $M_n(0, \lambda)$  are the same that those of  $\det(\mathbf{D}_1\mathbf{B}_{11} - \lambda I)$ , besides, it has a simple root in 0. Considering sufficiently small  $\epsilon_n$  and taking into account that  $\det(\mathbf{D}_{(\mathbf{B})}\mathbf{B}) > 0$ , we obtain that all the roots of  $M_n(\epsilon_n, \lambda)$  are positive and simple. Using the continuity of eigenvalues, we get that there is a positive integer,  $\epsilon_n^0$ , such that the above spectral properties hold for all  $\epsilon_n$  which satisfy  $0 < \epsilon_n \leq \epsilon_n^0$ . Thus, we can choose  $\epsilon_n$  satisfying  $0 < \epsilon_n < \min(\epsilon_n^0, \epsilon_{n-1})$ . As it follows, the stabilization matrix  $\mathbf{D}_{(\mathbf{B})}$  has the form  $\mathbf{D}_{(\mathbf{B})} = \text{diag}\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ , where  $1 = \epsilon_1 > \epsilon_2 > \dots > \epsilon_n > 0$ .

By the construction, all the eigenvalues of  $\mathbf{D}_{\mathbf{B}}\mathbf{B}$  are positive and simple. Let us consider the matrix  $\mathbf{P}_\theta\mathbf{D}_{\mathbf{B}}\mathbf{B}\mathbf{P}_\theta^{-1}$ , which obviously has the same eigenvalues. Since  $\mathbf{D}_{\mathbf{B}}\mathbf{B} = \mathbf{D}_{\mathbf{B}}\mathbf{P}_\theta^{-1}\mathbf{A}\mathbf{P}_\theta$ , we have

$$\mathbf{P}_\theta\mathbf{D}_{\mathbf{B}}\mathbf{B}\mathbf{P}_\theta^{-1} = \mathbf{P}_\theta\mathbf{D}_{\mathbf{B}}\mathbf{P}_\theta^{-1}\mathbf{A}\mathbf{P}_\theta\mathbf{P}_\theta^{-1} = \mathbf{P}_\theta\mathbf{D}_{\mathbf{B}}\mathbf{P}_\theta^{-1}\mathbf{A}.$$

Putting  $\mathbf{D}_{(\mathbf{A})} := \mathbf{P}_\theta\mathbf{D}_{\mathbf{B}}\mathbf{P}_\theta^{-1}$ , we obtain that  $\mathbf{D}_{(\mathbf{A})}$  is a stabilization matrix for  $\mathbf{A}$ . Since  $\mathbf{D}_{\mathbf{B}} \in \mathcal{D}_{(1, \dots, n)}$  by Lemma 4.2, Part 3, we get that  $\mathbf{D}_{(\mathbf{A})} \in \mathcal{D}_\theta$ . □

## 6 Proof of the main theorems

In this section, we prove Theorem 1.1 and Theorem 1.2.

*Proof of Theorem 1.1.* Given a permutation  $\theta = (\theta(1), \dots, \theta(n))$ , we consider the corresponding class  $\mathcal{D}_\theta$  of  $\theta$ -ordered positive diagonal matrices. Prove that  $\mathbf{D}\mathbf{A}$  is positive stable for each  $\mathbf{D} \in \mathcal{D}_\theta$ . Applying Lemma 3.1, Part 7, we get that  $\mathbf{B} := \mathbf{D}\mathbf{A}$  is a  $P$ -matrix for every positive diagonal matrix  $\mathbf{D}$ . Thus,  $\mathbf{B}$  satisfies the conditions of Lemma 5.1. By Lemma 5.1, we obtain that  $\mathbf{B}$  is stabilizable and the stabilization matrix  $\mathbf{D}_{(\mathbf{B})}$  may be chosen from  $\mathcal{D}_\theta$ . Next, we construct the matrices  $\mathbf{D}_t\mathbf{B} := (t\mathbf{I} + (1 - t)\mathbf{D}_{(\mathbf{B})})\mathbf{B}$ . By Lemma 4.2, Parts 1 and 2,  $\mathbf{D}_t\mathbf{D} \in \mathcal{D}_\theta$  for every  $t \in [0, 1]$ . Thus, by the conditions of the theorem, we obtain that  $\mathbf{D}_t\mathbf{B} = \mathbf{D}_t\mathbf{D}\mathbf{A}$  is a  $Q^2$ -matrix for every  $t \in [0, 1]$ . Note that putting  $t = 0$  results in  $\mathbf{B}$  being also a  $Q^2$ -matrix, and it has no eigenvalues on the imaginary axes (see Theorem 3.2). Now we apply to  $\mathbf{B}$  the reasoning of the proof of Carlson’s theorem (see [5]). Since  $\mathbf{D}_{(\mathbf{B})}$  is a stabilization matrix for  $\mathbf{B}$ , we obtain that all the eigenvalues of the matrix  $\mathbf{D}_{(\mathbf{B})}\mathbf{B}$  are in the right-hand side of the complex plane (moreover, they are all positive). Let us consider  $(\mathbf{D}_t\mathbf{B})^2$ , which is a  $Q$ -matrix for any  $t \in [0, 1]$ . Then, applying Theorem 3.2, we obtain that  $\mathbf{D}_t\mathbf{B}$  can not have eigenvalues on the imaginary axis for any  $t \in [0, 1]$ . For  $t = 0$ , we have that  $\mathbf{D}_t\mathbf{B} = \mathbf{D}_{(\mathbf{B})}\mathbf{B}$ , with all eigenvalues being positive. Since the eigenvalues changes continuously with  $t$ , we have that the eigenvalues of  $\mathbf{D}_t\mathbf{B}$  can cross the imaginary axis for none  $t \in [0, 1]$ . So, they must stay in the right-hand side of the complex plane. Thus putting  $t = 1$ , we obtain that  $\mathbf{B}$  is positive stable. Taking into account that  $\mathbf{B} = \mathbf{D}\mathbf{A}$  and  $\mathbf{D}$  is an arbitrary matrix from  $\mathcal{D}_\theta$ , we obtain that  $\mathbf{A}$  is  $\mathcal{D}_\theta$ -stable. □

*Proof of Theorem 1.2.* It is enough to apply Theorem 1.1, then Lemma 4.1. □

## 7 Examples and applications

Here we apply Theorem 1.2 to establish  $D$ -stability of some matrix classes which are known to be stable. One of such classes is strictly row (column) square diagonally dominant for every order of minors matrices introduced by Tang et al. in [18].

An  $n \times n$  matrix  $\mathbf{A}$  is called *strictly row square diagonally dominant for every order of minors* if the following inequalities hold:

$$\left( A \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \right)^2 > \sum_{\alpha, \beta \subset [n], \alpha \neq \beta} \left( A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right)^2$$

for any  $\alpha = (i_1, \dots, i_k)$ ,  $\beta = (j_1, \dots, j_k)$  and all  $k = 1, \dots, n$ . A matrix  $\mathbf{A}$  is called *strictly column square diagonally dominant* if  $\mathbf{A}^T$  is strictly row square diagonally dominant.

The following theorem was proven in [18] (p. 27, Theorem 3).

**Theorem 7.1** (Tang et al.). *Let  $\mathbf{A}$  be a  $P$ -matrix. If  $\mathbf{A}$  is strictly row (column) square diagonally dominant for every order of minors, then  $\mathbf{A}$  is positive stable.*

Now we can establish  $D$ -stability of this matrix class.

**Theorem 7.2.** *Let  $\mathbf{A}$  be a  $P$ -matrix. If  $\mathbf{A}$  is strictly row (column) diagonally dominant for every order of minors, then  $\mathbf{A}$  is  $D$ -stable.*

*Proof.* It is enough to observe that  $(\mathbf{D}\mathbf{A})^2$  is a  $Q^2$ -matrix for each positive diagonal matrix  $\mathbf{D}$  (see [18], p. 27, Proof of Theorem 3).  $\square$

As examples, we mention the following matrix classes which are known to be  $D$ -stable (see [3], [5]).

1. Sign-symmetric matrices (a matrix  $\mathbf{A}$  is called *sign-symmetric* if the inequality

$$A \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} A \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} \geq 0$$

holds for all sets of indices  $(i_1, \dots, i_k)$ ,  $(j_1, \dots, j_k)$ , where  $1 \leq i_1 < \dots < i_k \leq n$ ,  $1 \leq j_1 < \dots < j_k \leq n$ ).

2. Hermitian positive definite matrices.
3. Strictly totally positive matrices and their generalizations.

## 8 Concluding remarks

We define a new property of  $n \times n$  matrices, which is called  $D_\theta$ -stability or  $D$ -stability with respect to the order  $\theta$ , where  $\theta$  is a given permutation of the set of indices  $[n]$ . This property is stronger than stability but weaker than  $D$ -stability. In fact, we show that a matrix is  $D$ -stable if and only if it is stable with respect to any order  $\theta$ . We provide two criteria which ensure  $D_\theta$ -stability and  $D$ -stability of  $P$ -matrices. The conditions are stated in terms of the properties of matrix scallings, which are, as the author hopes, easier to verify.

**Acknowledgement:** The study was partially supported by the National Science Foundation of China (grant number BC0710062).

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