

## Research Article

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# Ranks of permutative matrices

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**Abstract:** A new type of matrix, termed *permutative*, is defined and motivated herein. The focus is upon identifying circumstances under which square permutative matrices are rank deficient. Two distinct ways, along with variants upon them are given. These are a special kind of grouping of rows and a type of partition in which the blocks are again permutative. Other, results are given, along with some questions and conjectures.

**Keywords:**  $h, k$ -partition;  $h, k, g$ -partition; Identically singular; Latin square; Permutative matrix; Polynomial matrix; Row grouping

**MSC:** Primary: 05B20, 15A03; Secondary: 15A15, 15A48

## 1 Introduction

By a (symbolic) *permutative matrix* we mean an  $m$ -by- $n$  matrix whose entries are chosen from among  $n$  independent variables over the nonnegative real numbers in such a way that each row is a different permutation of the  $n$  variables. Associated with each (symbolic) permutative matrix  $A$  is a collection  $\mathcal{P}(A)$  of (numerical) permutative matrices resulting from consistent substitution of distinct positive real numbers for the variables. Two distinct variables are not allowed to take on the same value. Square permutative matrices are of particular interest to us here, but it is convenient to consider ones with  $m < n$  in some situations. When not otherwise indicated, we assume  $m = n$ .

### Example 1.1

The matrix  $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_4 & a_3 & a_1 \\ a_3 & a_2 & a_4 & a_1 \\ a_2 & a_3 & a_1 & a_4 \end{bmatrix}$  is a 4-by-4 (symbolic) permutative matrix and

$$B = \begin{bmatrix} 3 & 2 & 5 & 7 \\ 2 & 7 & 5 & 3 \\ 5 & 2 & 7 & 3 \\ 2 & 5 & 3 & 7 \end{bmatrix} \in \mathcal{P}(A).$$

If every matrix in  $\mathcal{P}(A)$  is nonsingular, we say that the permutative matrix  $A$  is *identically invertible*. Two other possibilities may occur. It may be that some matrices in  $\mathcal{P}(A)$  are singular, while others are invertible. Since the determinant of a (symbolic) permutative matrix is a (homogeneous) polynomial in the  $n$  variables, this means that most matrices in  $\mathcal{P}(A)$  will be nonsingular, and we call such permutative matrices *generically invertible*. The final possibility is that every matrix in  $\mathcal{P}(A)$  is singular, which we term *identically singular*. Each of these possibilities arises, but we are primarily interested in understanding the identically singular

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(symbolic) permutative matrices and their (maximum) ranks. It can happen that more than one rank less than  $n$  can occur in  $\mathcal{P}(A)$ .

Though the formal notion of a permutative matrix is new to this work, particular instances of permutative matrices have arisen a number of times previously. A Latin square [2, 4] is a permutative matrix whose transpose is also permutative and these have been heavily studied in combinatorics and statistical experimental design, etc. Combinatorially, they also occur as multiplication tables of groups. And, a variant has also been studied [1]. Non-Latin-square permutative matrices have also arisen [3]. Finally, the *nonnegative inverse eigenvalue problem* (NIEP) asks which spectra occur among  $n$ -by- $n$  entry-wise nonnegative matrices. Early on [6], it was noted, without proof, that any real spectrum with trace 0 and exactly one positive element does occur. A simple proof [5] may be given with special permutative matrices. For example, if the spectrum  $-a, -b, -c, a + b + c$ , with  $a, b, c > 0$  is desired, then it may be realized by

$$A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix}.$$

Since  $\det(A + xI) = 0$ ,  $x = a, b, c$ , each of  $-a, -b, -c$  is an eigenvalue, and since the row sums are all  $(a + b + c)$ ,  $(a + b + c)$  is also an eigenvalue (and also, since  $\text{Tr}(A) = 0$ ). The same construction, and argument, work generally.

In the next two sections we give a complete accounting of the possibilities for square permutative matrices for  $n = 3$  and 4. Then, we identify the two major ways that we have found for a permutative matrix to be identically singular (rank deficient). The more transparent way is when distinct groups of rows have a common weighted sum, which we call *row grouping*. A more subtle of these is the existence of a special partition: a partition of the rows of a permutative matrix into  $h$  parts and the columns into  $k$  parts such that in each block of the partition there are only as many variables as there are columns is called an  *$h, k$ -partition*. (Each block is, itself, permutative, except, perhaps, for having some repeated rows.) If such a partition exists, with  $h < k$ , then the rank is deficient by at least  $k - h$ . Refinements that involve hybrids of these two ways are also given. We give an algorithm to find an  $h, k$ -partition, mention some open questions, and make some further useful observations along the way.

## 2 The 3-by-3 Case

Here, we give a complete description of what may happen among 3-by-3 (symbolic) permutative matrices. It turns out that they are just of two "kinds", and both are identically invertible. This may be seen by using a natural equivalence relation on permutative matrices that preserves the set of ranks that occur in  $\mathcal{P}(A)$ . We say that  $B$  is equivalent to  $A$  if  $B$  may be obtained from  $A$  via (i) permutation of rows, (ii) permutation of columns, or (iii) permutation of variable names. Of course each of these is reversible.

### Example 2.1

$$\begin{bmatrix} a & b & c \\ a & c & b \\ b & c & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & c \\ c & b & a \\ c & a & b \end{bmatrix}$$

are equivalent 3-by-3 permutative matrices. To see this, interchange the variables  $a$  and  $b$  in the first matrix and then interchange the first 2 columns of the result to arrive at the second matrix.

A priori, there are 120 (symbolic) 3-by-3 permutative matrices:  $3! \times (3! - 1) \times (3! - 2)$ . However, it may be

easily checked that, among these, there are only two equivalence classes. One contains all matrices whose transposes are permutative (i.e. the Latin square), such as

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix},$$

and the other in which each of two of the variables appear twice in a column, such as the matrices in Example 2.1. In the former case the determinant is

$$\pm(a+b+c)(b-a)(b-c)$$

and in the latter case it is (up to change of variables)

$$\pm(a+b+c)(ab+bc+ac-a^2-b^2-c^2)$$

Both are never 0; the second because of the Cauchy-Schwarz inequality, and the first because the variables are distinct and cannot sum to 0 in the definition of permutative matrix.

**Theorem 2.2** Every 3-by-3 (symbolic) permutative matrix is identically invertible.

### 3 The 4-by-4 Permutative Matrices

The  $24 \times 23 \times 22$  4-by-4 permutative matrices fall into 41 equivalence classes. Of these, 5 classes are identically singular; 19 classes are identically invertible, and 17 are generically invertible. We give representative of a few typical equivalence classes.

#### 1. Identically singular 4-by-4 permutative matrices

$$\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ a & c & d & b \end{bmatrix} \text{ determinant} = 0$$

This symbolic permutative matrix always has determinant 0, no matter what the values of  $a, b, c, d$ . The reason will become clear in Section 5.

#### 2. Generically invertible 4-by-4 permutative matrices

$$\begin{bmatrix} a & b & c & d \\ a & c & d & b \\ b & a & d & c \\ b & d & c & a \end{bmatrix} \text{ determinant} = (c-d)(a-b)(a+b-c-d)(a+b+c+d)$$

This permutative matrix is invertible unless  $a+b=c+d$ , which is allowed in a permutative matrix. We call such permutative matrices generically invertible. Other such equivalence classes will be invertible, unless there is some other relation among the entries that cause the determinant to be 0.

#### 3. Identically invertible 4-by-4 permutative matrices

$$\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ a & c & b & d \\ b & a & d & c \end{bmatrix} \text{ determinant} = -(a-b)(b-c)(c-d)(a+b+c+d)$$

$$\begin{bmatrix} a & b & c & d \\ a & b & d & c \\ b & c & a & d \\ c & a & b & d \end{bmatrix} \text{determinant} = -(c-d)(a^2 - ab - ac + b^2 - bc + c^2)(a+b+c+d)$$

In both these cases, the determinant is never 0 given the definition of permutative matrices. In the second case, the second term of the determinant is  $\frac{1}{2}((a-b)^2 + (b-c)^2 + (a-c)^2)$  which also cannot be 0.

## 4 Identical Singularity Resulting from Row Grouping

Since each row of a (symbolic) permutative matrix runs through all positive vectors with no repeated entries, no row can be orthogonal to a fixed numerical vector. Thus, a permutative matrix cannot have a numerical, nonzero, right null vector. It can, however, have a numerical, nonzero left null vector.

**Example 4.1** In the 6-by-6 permutative matrix

$$A = \begin{bmatrix} a & b & c & d & e & f \\ c & e & d & a & f & b \\ c & b & d & a & e & f \\ a & e & c & d & f & b \\ b & f & e & c & d & a \\ d & f & c & b & a & e \end{bmatrix},$$

notice that the sum of the first 2 rows is the same as the sum of the second 2 rows. This means that  $(1, 1, -1, -1, 0, 0)$  is a left null vector for  $A$ , and, in fact a basis for the left null space. Each matrix in  $\mathcal{P}(A)$  has rank 5.

We say that a symbolic matrix  $A$  admits a *g-part* (pure) *row grouping* if a subset of the rows of  $A$  may be partitioned into  $g$  parts so that there is a single row vector that is some nontrivial weighted sum of the rows in each part. We are primarily interested in the concept of row grouping in the context of permutative matrices, but it will occur more generally. Thus, after matrix row operations, the rank deficiency is at least  $g - 1$ . We may identify the particular parts in the partition; note that each part in the partition must contain at least two rows, because the definition of permutative matrices does not allow equal rows. For matrix  $A$  of Example 4.1,  $\{\{1, 2\}, \{3, 4\}\}$  is a 2-part row grouping and the two 1's ( $-1$ 's) in the left null vector correspond to the first (second) part of the row grouping. In the case of a 2-part row grouping, a left null vector may be constructed by placing the weights from the first part in those entries corresponding to the row indices from the first group, and the negatives of the weights from the second group in the positions corresponding to that part of the partition.

Let  $NLN(A)$  denote the *numerical left null space* of a symbolic permutative matrix  $A$ . Let  $e$  denote the column vector of 1's, whose dimension will usually be clear from context.

**Lemma 4.2** If  $A$  is an  $m$ -by- $n$  permutative matrix, then each vector in  $NLN(A)$  is orthogonal to  $e$ , i.e. the sum of its entries is 0.

**Proof:** If  $A$  is based upon the variable  $a_1, a_2, \dots, a_n$ , then  $Ae = (a_1 + \dots + a_n)e$ , so that  $e$  is in the column space of  $A$ . But, if  $x \in NLN(A)$ , then  $x^T A = 0$  and we have  $x^T Ae = 0$  or  $x^T(a_1 + \dots + a_n)e = 0$  which means that  $x^T e = 0$ , as  $a_1 + \dots + a_n \neq 0$ .

□

Now, we have

**Theorem 4.3** Each numerical left null vector of a symbolic permutative matrix corresponds to a 2-part row grouping, and conversely.

**Proof:** First, since a (left) null vector will have rational entries, we may take them to be integers, some positive, some negative (and, perhaps some 0's), by Lemma 4.2. An appropriately weighted sum of the rows corresponding to positive coefficients of the left null vector will then equal the appropriately weighted sum of the rows corresponding to negative entries (there must be some of each), giving a 2-part row grouping. Thus, the 2-parts in such a row grouping may be found with a left numerical null vector. Conversely, a 2-part row grouping may be converted into a left null vector in an obvious way.

□

**Remark 4.4** We note that a) each  $g$ -part row grouping,  $g > 2$ , may be viewed as several 2-part row groupings. And b) there may occur several, independent,  $g$ -part row groupings (with possibly different  $g$ 's) that contribute to  $NLN(A)$ . Theorem 4.3 then means that  $NLN(A)$  encodes all the row groupings that occur in  $A$  and that row groupings entirely explain the rank deficiency that results from  $NLN(A)$ . Of course, we have

**Theorem 4.5** For a symbolic permutative  $m$ -by- $n$  matrix  $A$ , with  $m < n$ , and  $B \in \mathcal{P}(A)$ ,

$$rank(B) \leq m - \dim NLN(A)$$

**Proof:**  $NLN(A) \subseteq$  the left null space of  $B$ . Since  $rank(B) = m$  less the dimension of the left null space of  $B$ , the claim follows. Note that the rank of  $B$  might be smaller, and that  $NLN(A)$  is just intersection of left null space of all  $B$  in  $\mathcal{P}(A)$ .

□

**Example 4.6** We note that, while elements of  $NLN(A)$  may often be  $\pm 1, 0$  vectors, we may have more complicated weights.

$$\begin{bmatrix} a & b & c & d & e & f & g & h & i & j & k & l & m \\ d & e & m & a & b & l & j & k & g & f & c & i & h \\ g & f & h & j & i & m & l & d & k & a & b & e & c \\ j & l & e & c & g & b & i & k & h & m & d & f & a \\ \hline j & f & m & c & i & b & l & d & g & a & k & e & h \\ g & l & c & d & e & m & j & h & k & f & b & i & a \\ d & b & e & a & g & f & i & k & h & j & c & l & m \\ a & l & m & j & e & b & g & k & h & f & d & i & c \\ g & f & h & a & b & m & l & k & i & j & d & e & c \\ d & e & h & c & g & l & j & k & i & m & b & f & a \\ a & e & c & j & b & l & g & d & i & m & k & f & h \\ a & e & h & j & b & f & i & k & g & m & d & l & c \\ j & b & e & d & i & l & g & h & k & a & c & f & m \end{bmatrix}$$

This matrix, having a numerical left null vector  $[3 \ 3 \ 3 \ 3 \ -2 \ -2 \ -2 \ -1 \ -1 \ -1 \ -1 \ -1 \ -1]$ , indicates that the weights in each group of rows within a row grouping do not need to be only 1s (or -1s) and do not need to be the same for all groups.

For an  $m$ -by- $n$  permutative matrix  $A$ , let  $\overline{\mathcal{P}(A)}$  denote the closure of  $\mathcal{P}(A)$ , the set of nonnegative matrices in which equal nonnegative substitutions are allowed for the variables. Then, if we let one variable be 1 and the rest 0, we get a 0,1 matrix in  $\mathcal{P}(A)$  that indicates the positions of that variable. We get such a 0, 1 matrix with  $m$  1's for each variable, and the permutative matrix  $A$  may be written as the linear combination of these

0, 1 matrices whose coefficients are the variables. If the variables are  $a_1, \dots, a_n$ , we may name the 0, 1 matrices  $A_1, A_2, \dots, A_n$ , so that

$$A = a_1A_1 + a_2A_2 + \dots + a_nA_n.$$

**Example 4.7**

If matrix  $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \\ a_3 & a_2 & a_1 \end{bmatrix}$ ,

then  $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ,  $A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ .

The  $NLN(A)$  is simply the intersection of the left null spaces of the matrices  $B \in \mathcal{P}(A)$ . Since  $A_1, \dots, A_n$  span this subspace of matrices,  $NLN(A)$  is the intersection of the left null spaces of  $A_1, \dots, A_n$ , or equivalently the left null space of the the  $m$ -by- $n^2$  matrix:

$$\mathcal{A} = [A_1 \ A_2 \ \dots \ A_n]$$

Now, since  $\mathcal{A}$  is an integer matrix, its left null space has rational entries, and, by clearing denominators, any particular left null vector may be taken to have integer entries (that sum to 0). We then have as a consequence of Theorem 4.3:

**Corollary 4.9** In any  $g$ -part row grouping, the weights within each group of rows may be taken to be integers.

**Example 4.10**

Suppose matrix  $A = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ b & a & c & d \\ a & b & d & c \end{bmatrix}$ . Then,

$$A_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_b = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, A_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and therefore,  $\mathcal{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ .

This means that  $A$ 's left null vector  $[1 \ 1 \ -1 \ -1]^T$  is also a left null vector of  $\mathcal{A}$ .

## 5 Singularity Resulting from $h, k$ -partitions

It may happen that a permutative matrix  $A$  is identically singular, even if  $\dim NLN(A) = 0$ . This results from certain  $h, k$ -partitions. Some trivial  $h, k$ -partitions are of no interest. For example, the matrix itself corresponds to a 1,1-partition; each entry is a block in the  $n, n$ -partition, and each row is a block in the  $n, 1$ -partition. We exclude these from further discussion. The  $h, k$ -partitions, with  $h < k$ , are of most interest; they

always result in rank deficiency. Under additional conditions,  $h, k$ -partitions, with  $h = k$ , may result in rank deficiency.

**Example 5.1** Consider the permutative matrix

$$\left[ \begin{array}{cc|cc} a & b & c & d \\ a & b & d & c \\ \hline c & d & a & b \\ c & d & b & a \end{array} \right]$$

with a 2,3-partition, as indicated. Subtracting the first row from the second and the third from the fourth, and then adding the last column to the third column gives

$$\left[ \begin{array}{cccc} a & b & c+d & d \\ 0 & 0 & 0 & c-d \\ c & d & a+b & b \\ 0 & 0 & 0 & a-b \end{array} \right].$$

Now, interchange of rows 2 and 3 gives

$$\left[ \begin{array}{cccc} a & b & c+d & d \\ c & d & a+b & b \\ 0 & 0 & 0 & c-d \\ 0 & 0 & 0 & a-b \end{array} \right]$$

which has at most 3, and exactly 3, linearly independent columns. Thus,  $A$  is identically singular, and the rank of every matrix in  $\mathcal{P}(A)$  is 3.

To prove our first main result here, we need a well known fact

**Lemma 5.2** If  $A$  is an  $m$ -by- $n$  matrix, over a field, that contains a  $p$ -by- $q$  submatrix of 0's, then

$$\text{rank}(A) \leq (m - p) + (n - q).$$

Now, we may prove

**Theorem 5.3** An  $m$ -by- $n$  permutative matrix  $A$  with an  $h, k$ -partition satisfies

$$\text{rank}(A) \leq n + h - k.$$

Thus, for  $m = n$ , if  $h < k$ ,  $A$  is identically singular.

**Proof:** Because permutation equivalence preserves rank, we may suppose that each part of the row (column) partition consists of consecutive rows (columns), so that the submatrices formed by the partition are contiguous.

Notice, first, that by the definition of an  $h, k$ -partition, the row sums of each submatrix of the partition are constant for that block. Now, for each part of the row partition, subtract the first row from all other rows. This makes all row sums, after the first, 0 within each block. Then, in each part of the column partition, add every other column to the first, so that in our partitioned matrix, each block has only 0's below its upper left entry.

These 0's form a submatrix of  $A$  of size  $(m - h)$ -by- $k$ . Application of Lemma 5.3, then yields

$$\text{rank}(A) \leq m - (m - h) + n - k,$$

and simplification gives the bound claimed in the statement of the theorem.



Note that the proof of Theorem 5.3 includes an explicit elimination procedure that reveals a 0 block that is relevant to determining rank. We call this elimination scheme (that depends only on the  $h, k$ -partition that is used—there may be others)  $h, k$ -elimination. After  $h, k$ -elimination is performed, other characteristics of the resulting form may reveal further rank deficiency.

In some cases, a permutative matrix having an  $h, k$ -partition is actually permutation equivalent to a permutative matrix containing a row grouping, which means, it also has a numerical left null vector. However, in such cases it need not happen that the rank diminution be additive. So the rank may only be reduced by one, even though two phenomena are displayed. For example, swapping the second and the last row in case 9 in Section 3 gives

$$\left[ \begin{array}{cc|cc} a & b & c & d \\ a & b & d & c \\ b & a & c & d \\ b & a & d & c \end{array} \right] \Rightarrow \left[ \begin{array}{cccc} a & b & c & d \\ b & a & d & c \\ \hline b & a & c & d \\ a & b & d & c \end{array} \right],$$

where the first version has a 1,2-partition and the second one has a row grouping with first two rows and the last two rows being the 2 parts. The matrix is rank 3, not rank 2.

## 6 Hybrid Rank Deficiency via $h, k, g$ -partitions

The two phenomena that cause rank deficiency, row grouping and  $h, k$ -partitions may combine to cause rank deficiency, in a square permutative matrix, greater than either separately. One way is straightforward: one subset of the rows may be deficient in rank because of an  $h, k$ -partition within it and with  $h$  much less than  $k$ , while another, disjoint subset of rows, displays row grouping. The other way is a rather more subtle hybrid of the two: a latent row grouping shows up in part of the matrix, after  $h, k$ -elimination has been performed, as in the proof of Theorem 5.3. Parallel to Example 5.1 is the following example of the latter phenomenon.

**Example 6.1** The 8-by-8 matrix

$$A = \left[ \begin{array}{cc|cc|cc|cc} a & e & d & f & c & b & g & h \\ a & e & f & d & b & c & g & h \\ \hline b & d & c & g & f & e & a & h \\ d & b & g & c & e & f & h & a \\ \hline d & b & c & g & e & f & h & a \\ \hline e & d & f & g & c & b & a & h \\ e & d & g & f & c & b & h & a \\ \hline a & b & c & d & e & f & g & h \end{array} \right]$$

has only a 4,4-partition, as displayed, but is identically singular with  $rank(A) \leq 7$ .



After applying  $h, k$ -elimination and permuting rows and columns, we arrive at the equivalent matrix

$$A' = \begin{bmatrix} a+e & d+f & c+b & g+h & e & f & b & h \\ b+d & c+g & f+e & a+h & d & g & e & h \\ e+d & f+g & c+b & a+h & d & g & b & h \\ a+b & c+d & e+f & g+h & b & d & f & h \\ 0 & 0 & 0 & 0 & 0 & d-f & c-b & 0 \\ 0 & 0 & 0 & 0 & b-d & c-g & f-e & a-h \\ 0 & 0 & 0 & 0 & e-d & f-g & c-b & 0 \\ 0 & 0 & 0 & 0 & 0 & f-g & 0 & a-h \end{bmatrix}.$$

Now the upper left 4-by-4 block has a two-part row grouping:  $\{1, 2\}; \{3, 4\}$ , so that adding the second row to the first, the fourth to third and then subtracting the first from the third and permuting yields upper left submatrix

$$\begin{bmatrix} a+e & d+f & c+b & g+h \\ b+d & c+g & f+e & a+h \\ e+d & f+g & c+b & a+h \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This means that the first 4 columns of  $A'$  have rank at most 3 and that  $A$  has rank at most 7.

We may now formalize the idea of an  $h, k, g$ -partition.

**Definition 6.2** An  $m$ -by- $n$  permutative matrix  $A$  has an  $h, k, g$ -partition if it has an  $h, k$ -partition such that the application of  $h, k$ -elimination leaves a submatrix above the  $(m - h)$ -by- $k$  0 block that admits a  $g$ -part row grouping.

**Theorem 6.3** If an  $m$ -by- $n$  permutative matrix  $A$  has an  $h, k, g$ -partition, then

$$\text{rank}(A) \leq n + h - k - (g - 1)$$

**Proof:** After the  $h, k$ -elimination, as in the proof of Theorem 5.3, an  $(m - h)$ -by- $k$  0 submatrix is formed with an  $h$ -by- $k$  submatrix above it. If the  $h$ -by- $k$  submatrix has a  $g$ -part row grouping, the rank of the submatrix above the 0 block—the rank of the submatrix formed by the first  $k$  columns is reduced by  $(g - 1)$ . Thus, the 0 submatrix may be expanded to  $(m - h + (g - 1))$ -by- $k$ . Application of Lemma 5.2 then gives

$$\text{rank}(A) \leq m - (m - h + (g - 1)) + n - k$$

with a simplified version shown in Theorem 6.3.

□

Thus, for Example 6.1, application of Theorem 6.3 gives the rank  $8+4-4-(2-1)=7$ .

Note that permutative matrices can have multiple row grouping phenomena based on  $h, k$ -partition with  $g_1$ -fold,  $g_2$ -fold,  $g_3$ -fold individually on top of the 0 submatrix post- $h, k$ -elimination. Therefore, the general form for the rank ceiling can be

$$\text{rank}(A) \leq n + h - k - (g_1 - 1) - (g_2 - 1) - \dots$$

## 7 Further Rank Reduction via Super Grouping

"Super grouping", or groupings of groupings may occur in both pure row grouping and in  $h, k, g$ -partitions. Here, we talk about the case of pure row grouping first. Super grouping requires a permutative matrix to have multiple row groupings, and the idea is similar to row grouping. From section 4, we know that each row grouping corresponds to a numerical left null vector indicating a pair of groups of rows that have the same weighted sum. Each super grouping corresponds to a numerical left null vector as well. However, the null vector's entries do not only reflect the indices of two groups of rows. They, in fact, represent two (or more) groups of row groupings, which means those two groups of row groupings that involve many row groups themselves, have the same weighted sum of the weighted sums of each group. Therefore, another zero row can be produced from row operations, due to the super grouping, in addition to, yet encompassing all the corresponding row groupings.

Similar results apply to the  $h, k, g$ -partition when only looking at the upper left submatrix produced by  $h, k$ -elimination. Thus, the rank of the submatrix can be even less than the basic row grouping would suggest. So the rank of the permutative matrix will be less than the result would from the  $h, k, g$ -partition.

In practice, super grouping is similar to a row grouping but is based on the row sum of each row grouping rather than rows. After applying row grouping, we look at the submatrix formed by each row grouping's one remaining row of the weighted sums (remember for each row grouping, we delete all but one group's equal weighted sum). If this new submatrix has a row grouping, we can further eliminate rows and see that the rank of the submatrix, and thus the entire matrix, is lower.

## 8 Algorithms to find row groupings, $h, k$ -partitions or $h, k, g$ -partitions

An algorithm can help determine whether identical rank deficiency of a permutative matrix is due to row grouping,  $h, k$ -partitions or  $h, k, g$ -partitions.

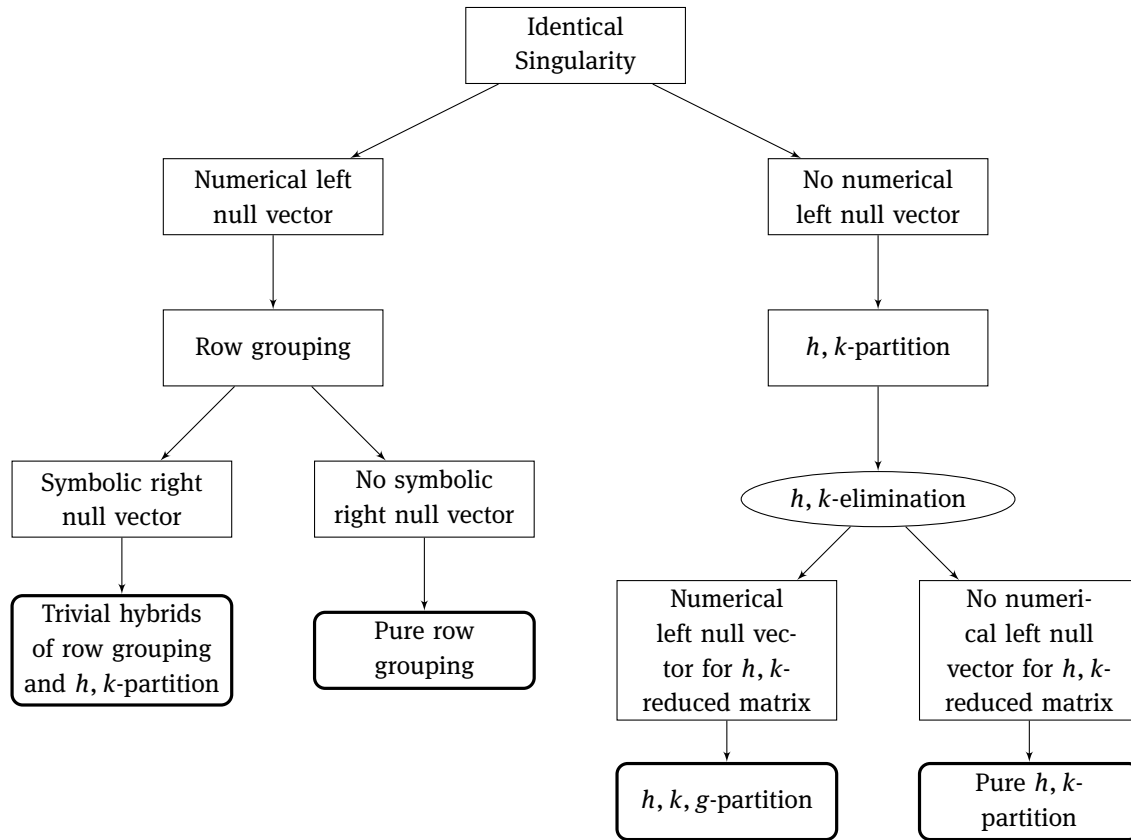
If there is a nonzero, numerical left null vector for the matrix, then the permutative matrix has a row grouping by Theorem 4.3. The indices of the positive entries in the left null vector and the indices of the negative entries in the left null vector indicate the two groups of rows with equal weighted sums, with weights being the absolute values of the corresponding entries. In general, if a matrix has row grouping with  $g > 2$ , it will have  $g - 1$  independent numerical left null vectors.

If the matrix contains no numerical left null vector, we assume its rank deficiency is caused by an  $h, k$ -partition or an  $h, k, g$ -partition. To find the  $k$  parts of the column partition in the matrix, we look at a right null vector. If the right null vector of a permutative matrix contains repeated symbolic values, the corresponding indices indicate the columns belonging to a part of the column partition. From another perspective, the number of distinct values in the right null vector gives the value of  $k$  in the partition. Summing all the columns in each of the  $k$  parts and looking at those  $k$  post-summation vectors, each different set of entries in those  $k$  columns shows a different  $h$ -part in the row partition. For example, if we have a post-summation column with  $\{1, 2\}$  and  $\{3, 4, 5, 6\}$  rows having the same values respectively and another column with  $\{1, 2, 3, 4\}$  and  $\{5, 6\}$  rows having the same values, then we will partition the rows into three sets  $\{1, 2\}, \{3, 4\}, \{5, 6\}$  and  $h = 3$ .

After the  $h, k$ -elimination process, we are left with an  $h$ -by- $k$  submatrix above the 0 block. If we can still find a numerical left null vector in the submatrix, we have an  $h, k, g$ -partition of the matrix. After the  $h, k$ -

elimination, the positive and negative entries in a left null vector of the  $h$ -by- $k$  submatrix shows the grouping imposed on a pair of groups of  $h$ -parts which have the same weighted sum of the weighted sums of every part of the row partition within each group.

The following chart shows the process of finding the causality of a symbolic permutative matrix' identical singularity.



## 9 Additional Observations

Here, we make three more observations about permutative matrices.

First, if  $A$  is square, permutative and based upon the variables  $a_1, a_2, \dots, a_n$ , then  $Ae = (a_1 + \dots + a_n)e$ , so that  $a_1 + \dots + a_n$  is an eigenvalue of  $A$ . Of course,  $\det A$  is a homogeneous polynomial in  $a_1, a_2, \dots, a_n$  (perhaps identically 0). This means

**Proposition 9.1** If  $A$  is an  $n$ -by- $n$  permutative matrix based on the variables  $a_1, \dots, a_n$ , then  $\det A$  is a homogeneous polynomial in  $a_1, a_2, \dots, a_n$  and

$$a_1 + a_2 + \dots + a_n \mid \det A.$$

If  $A$  is  $m$ -by- $n$  and permutative, we may count the number of columns in which a particular variable lies. If the variables are  $a_1, \dots, a_n$  and  $A = a_1A_1 + \dots + a_nA_n$ , let  $q_i$  be the number of columns in which  $a_i$  appears. This is the same as  $\text{rank } A_i$ . Then, define  $q(A) = \max_{1 \leq i \leq n} q_i(A)$ . We have

**Theorem 9.2** For any permutative matrix  $A$ ,  $\mathcal{P}(A)$  contains matrices of rank at least  $q(A)$ .

**Proof:** Let  $A'$  be a  $q(A)$ -by- $q(A)$  submatrix that has the variable  $a_i$ , which attains  $q(A)$ , occurring in every column. As  $a_i$  appears only once in each row, we may permute rows and columns of  $A'$  to make  $a_i$  appear in every diagonal position (and nowhere else in  $A'$ ). Now, we define a numerical matrix  $B'$  from  $A'$  (and a numerical matrix  $B$  from  $A$ , of which  $B'$  is a submatrix) by choosing positive values for each symbol in such a way that the value for  $a_i$  is greater than the sum of all other values. In this way,  $B'$  is diagonally dominant and, thus, nonsingular. This means that  $B'$  has rank  $q(A)$  and that  $B$  has rank at least  $q(A)$ . Of course,  $B$  is in  $\mathcal{P}(A)$ , which verifies the claim. □

**Corollary 9.3** If an  $m$ -by- $n$  permutative matrix has an  $h, k$ -partition, then by Theorem 5.3, we have

$$k - h \leq n - q(A)$$

Finally, we note that there are identically invertible  $n$ -by- $n$  permutative matrices for every  $n$ . It is a worthy problem to identify and/or enumerate them all. Let the *sequential transposition* permutative matrix be the one in which positions  $n - i + 1$  and  $n - i + 2$  are transposed relative to the preceding row, beginning with row 2.

$$\begin{bmatrix} a_1 & a_2 & \dots & \dots & a_{n-1} & a_n \\ a_1 & a_2 & \dots & \dots & a_n & a_{n-1} \\ a_1 & a_2 & \dots & a_n & a_{n-2} & a_{n-1} \\ \vdots & & \ddots & & & \\ a_n & a_1 & a_2 & \dots & \dots & a_{n-1} \end{bmatrix}$$

**Theorem 9.4** The  $n$ -by- $n$  sequential transposition matrix has determinant

$$-(a_1 + \dots + a_n)(a_1 - a_n)(a_2 - a_n) \dots (a_{n-1} - a_n)$$

and, therefore is identically invertible.

**Proof:** First, perform the  $n - 1$  row operations:

- subtract row  $n-1$  from row  $n$ ;
- subtract row  $n-2$  from row  $n - 1$ ;
- ⋮

and end with

- subtract row 1 from row 2,

to arrive at

$$\begin{bmatrix} a_1 & a_2 & \dots & \dots & a_{n-1} & a_n \\ 0 & 0 & \dots & \dots & a_n - a_{n-1} & a_{n-1} - a_n \\ 0 & 0 & \dots & a_n - a_{n-2} & a_{n-2} - a_n & 0 \\ \vdots & & \ddots & & & \\ a_n - a_1 & a_1 - a_n & 0 & \dots & \dots & 0 \end{bmatrix}.$$

Next, perform the  $n - 1$  column operations:

- add column  $n$  to column  $n-1$ ;
- add column  $n - 1$  to column  $n-2$ ;

to end with

- add column 2 to 1,

to arrive at

$$\begin{bmatrix} a_1 + \dots + a_n & a_2 + \dots + a_n & \dots & \dots & a_{n-1} + a_n & a_n \\ 0 & 0 & \dots & \dots & 0 & a_{n-1} - a_n \\ 0 & 0 & \dots & 0 & a_{n-2} - a_n & 0 \\ \vdots & & \ddots & & & \\ 0 & a_1 - a_n & 0 & \dots & \dots & 0 \end{bmatrix}.$$

Now, expand the determinant down the first column to arrive at the claimed formula. Since the  $a_i$ 's are distinct and positive, this cannot be 0, which completes the proof.

□

**Definition 9.5** A *reduced matrix* of a permutative matrix with an  $h, k$ -partition is an  $h$ -by- $k$  matrix with every entry being the partial row sum within each block in the  $h, k$ -partition of the original matrix. To get the reduced matrix of an  $m$ -by- $n$  permutative matrix with only a  $g$ -part row grouping means to delete the redundant rows and get a matrix of dimension  $m - (g - 1)$ -by- $n$ . Finally, to get the reduced matrix of a permutative matrix with an  $h, k, g$ -partitions means to reduce the matrix to  $h$ -by- $k$ , as in the  $h, k$ -partition case, and, to further reduce the matrix by deleting the redundant  $(g - 1)$  rows and get the resulting  $h - (g - 1)$ -by- $k$  reduced form.

For example:

$$\left[ \begin{array}{cc|cc} a & b & c & d \\ a & b & d & c \\ \hline c & d & a & b \\ c & d & b & a \end{array} \right] \Rightarrow_{\text{reduces to}} \begin{bmatrix} a & b & c + d \\ c & d & a + b \end{bmatrix}$$

It is observed that the dimension of the right null space is invariant under reduction and that each right null vector of the reduced matrix has the same set of entries as the right null vector of the original permutative matrix:

$$\begin{bmatrix} b^2 + ab - d^2 - cd \\ -(a^2 + ba - c^2 - dc) \\ ad - bc \\ ad - bc \end{bmatrix} \Rightarrow_{\text{reduces to}} \begin{bmatrix} b^2 + ab - d^2 - cd \\ -(a^2 + ba - c^2 - dc) \\ ad - bc \end{bmatrix}$$

This reduction process is analogous in  $h, k, g$ -partitions. Here we demonstrate the reduction for Example 6.1:

$$\left[ \begin{array}{cc|cc|cc} a & e & d & f & c & b & g & h \\ a & e & f & d & b & c & g & h \\ \hline b & d & c & g & f & e & a & h \\ d & b & g & c & e & f & h & a \\ d & b & c & g & e & f & h & a \\ \hline e & d & f & g & c & b & a & h \\ e & d & g & f & c & b & h & a \\ \hline a & b & c & d & e & f & g & h \end{array} \right] \Rightarrow_{\text{reduces to}} \begin{bmatrix} a + e & d + f & c + b & g + h \\ b + d & c + g & f + e & a + h \\ e + d & f + g & c + b & a + h \\ a + b & c + d & f + e & g + h \end{bmatrix}$$

which is just the  $h$ -by- $k$  submatrix above the 0 block after applying  $h, k$ -elimination. Now, applying row grouping to the  $h$ -by- $k$  reduced matrix:

$$\begin{bmatrix} a + e & d + f & c + b & g + h \\ b + d & c + g & f + e & a + h \\ e + d & f + g & c + b & a + h \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow_{\text{reduces to}} \begin{bmatrix} a + e & d + f & c + b & g + h \\ b + d & c + g & f + e & a + h \\ e + d & f + g & c + b & a + h \end{bmatrix}$$

showing that the above matrix can be seen as an underdetermined system that has one right null vector, and the right null vector of the 3-by-4 system will contain the set of entries in the right null vector of the original 8-by-8 matrix.

## 10 Conjectures/Questions

We note some natural questions that have not been resolved in the current work.

**Conjecture 1** We have identified two main ways, and variants/extensions of them, in which a permutative matrix is identically singular. We conjecture that a permutative matrix that displays none of these phenomena will, at least, be generically invertible. Equivalently, these are the only ways that identical singularity occurs.

**Conjecture 2** Clearly, an  $n$ -by- $n$  permutative matrix, with  $n \geq 2$ , always has rank at least 2. We conjecture that when  $n \geq 3$ , the rank will be at least 3. In addition, the minimum rank grows slowly and stronger lower bounds on rank would be of interest.

**Conjecture 3** It would be of interest to characterize those  $n$ -by- $n$  permutative matrices that are identically invertible, or to at least give broader sufficient conditions. The same goes for generic invertible: characterize it or give broader sufficient conditions.

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