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Essential sign change numbers of full sign pattern matrices

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Abstract: A sign pattern (matrix) is a matrix whose entries are from the set $\{+, -, 0\}$ and a sign vector is a vector whose entries are from the set $\{+, -, 0\}$. A sign pattern or sign vector is full if it does not contain any zero entries. The minimum rank of a sign pattern matrix $\mathcal{A}$ is the minimum of the ranks of the real matrices whose entries have signs equal to the corresponding entries of $\mathcal{A}$. The notions of essential row sign change number and essential column sign change number are introduced for full sign patterns and condensed sign patterns. By inspecting the sign vectors realized by a list of real polynomials in one variable, a lower bound on the essential row and column sign change numbers is obtained. Using point-line configurations on the plane, it is shown that even for full sign patterns with minimum rank 3, the essential row and column sign change numbers can differ greatly and can be much bigger than the minimum rank. Some open problems concerning square full sign patterns with large minimum ranks are discussed.

Keywords: sign pattern (matrix); full sign pattern; minimum rank; sign change number; essential row sign change number; essential column sign change number; sign vectors of a list of polynomials

MSC: 15B35, 15B36, 15A23

1 Introduction and Preliminaries

An important part of combinatorial matrix theory is the study of sign pattern matrices (see [9], [22], and the references therein). A *sign pattern (matrix)* is a matrix whose entries are from the set $\{+, -, 0\}$. A row or column sign pattern matrix is also called a *sign vector*. A sign pattern or sign vector is said to be *full* if it does not contain any zero entry. For a real matrix B, $sgn(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by + (respectively, −, 0). For a sign pattern matrix $\mathcal{A}$, the *qualitative class* of $\mathcal{A}$, denoted $Q(\mathcal{A})$, is defined as

$$Q(\mathcal{A}) = \{A \mid A \text{ is a real matrix with } \mathrm{sgn}(A) = \mathcal{A}\}.$$

A *signature sign pattern* is a square diagonal sign pattern matrix whose diagonal entries are from the set $\{+, -\}$. A square $n \times n$ sign pattern is called a *permutation sign pattern* if each row and column contains exactly one + entry and $n - 1$ zero entries.

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The product $\mathcal{P}\mathcal{D}$ of a permutation sign pattern $\mathcal{P}$ and a signature sign pattern $\mathcal{D}$ is called a *signed permutation (sign pattern)*. Two $m \times n$ sign pattern matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be *signed permutationally equivalent* if there exist signed permutation sign patterns $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{A}_2 = \mathcal{P}_1\mathcal{A}_1\mathcal{P}_2$.

The *minimum rank* of a sign pattern matrix $\mathcal{A}$, denoted $mr(\mathcal{A})$, is the minimum of the ranks of the real matrices in $Q(\mathcal{A})$. Similarly, the *rational minimum rank* of a sign pattern $\mathcal{A}$, denoted $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$, is defined to be the minimum of the ranks of the rational matrices in $Q(\mathcal{A})$. The minimum ranks of sign pattern matrices have been the focus of a large number of papers (see e.g. [1, 2, 4–8, 10–12, 14–17, 23–26, 30]), and they have important applications in areas such as communication complexity [1, 28, 29], machine learning [18], neural networks [13], combinatorics [2, 14, 31], and discrete geometry [27].

It is clear that $\mathrm{mr}(\mathcal{A}) \le \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$ for every sign pattern $\mathcal{A}$. When $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$, we say that the minimum rank of $\mathcal{A}$ can be *realized rationally*. It is known (see [4, 7, 26, 30]) that for every $m \times n$ sign pattern $\mathcal{A}$ with $\mathrm{mr}(\mathcal{A}) \le 2$ or $\mathrm{mr}(\mathcal{A}) \ge n-2$, its minimum rank can be realized rationally. Also, for every sign pattern $\mathcal{A}$ each of whose columns contains at most two zero entries, its minimum rank can be realized rationally [4]. In particular, the minimum rank of every full sign pattern can be realized rationally (and hence, over the integers). However, it is shown in [25] and [14] that there exist sign patterns with minimum rank 3 (or bigger) whose rational minimum rank is strictly greater the minimum rank.

An $m \times n$ sign pattern with minimum rank n is called an *L matrix*. Characterizations of L matrices may be found in [7, 9].

Consider a sign pattern $\mathcal{A}$. Observe that if $\mathcal{A}$ contains a zero row or zero column, then deletion of the zero row or zero column preserves the minimum rank. Similarly, if two nonzero rows (or columns) of $\mathcal{A}$ are identical or opposite (namely, are negatives of each other), then deleting such a row (or column) also preserves the minimum rank. Following [26], we say that a sign pattern is *condensed* if it does not contain any zero row or a zero column and no two rows or two columns of it are identical or opposite. Clearly, given any nonzero sign pattern $\mathcal{A}$, we can delete its zero rows and columns (if any), and delete all except the first row or column from each maximal collection of rows or columns of $\mathcal{A}$ every two of which are either equal or opposite, to get the condensed sign pattern matrix $\mathcal{A}_c$ of $\mathcal{A}$ with the same minimum rank.

For example, for the sign pattern $\mathcal{A} = \begin{bmatrix} 0 & + & + & + & + \\ + & 0 & 0 & + & + \\ - & 0 & 0 & - & - \\ + & - & - & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we have $\mathcal{A}_c = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & - & 0 \end{bmatrix}$.

Since every sign pattern and its condensed sign pattern have the same minimum rank, without loss of generality, in most of the subsequent discussions concerning the minimum rank, we may assume that the sign patterns involved are condensed.

For a full sign vector $v = [v_i]$ with n components, the it sign change number of v, denoted $\mathrm{sc}(v)$, is the cardinality of the set $\{i \mid 1 \le i \le n-1 \text{ and } v_i v_{i+1} = -\}$. Consider an $m \times n$ full sign pattern $\mathcal{A}$. The *maximal row sign change number* of $\mathcal{A}$, denoted $\mathrm{rsc}(\mathcal{A})$, is the maximum of the sign change numbers of the rows of $\mathcal{A}$; and the *maximal column sign change number* of $\mathcal{A}$, denoted $\mathrm{csc}(\mathcal{A})$, is defined similarly. Of course, $\mathrm{csc}(\mathcal{A}) = \mathrm{rsc}(\mathcal{A}^T)$. The *essential row sign change number* of $\mathcal{A}$, denoted $\mathrm{ersc}(\mathcal{A})$, is defined as

$$\mathrm{ersc}(\mathcal{A}) = \min_{\mathcal{Q} \in \mathcal{P}_n} \mathrm{rsc}(\mathcal{A}\mathcal{Q})$$

where $\mathcal{P}_n$ is the set of all signed permutations of order n. The *essential column sign change number* of $\mathcal{A}$, denoted $\mathrm{ecsc}(\mathcal{A})$, may be defined similarly and is equal to $\mathrm{ersc}(\mathcal{A}^T)$.

The sign change number of a full sign vector is extended to an arbitrary sign vector $v = [v_i]$ that may contain zero entries in [26], called the polynomial sign change number, and denoted $\mathrm{psc}(v)$. Indeed, $\mathrm{psc}(v)$ is the smallest possible degree of a real polynomial $p(t)$ such that $\mathrm{sgn}(p(i)) = v_i$ for all i. As shown in [26], for any sign vector v, $\mathrm{psc}(v)$ can be easily determined. Using psc instead of sc, we can extend the above notions of rsc, csc, ersc, and ecsc to all condensed sign patterns (possibly with some zero entries). However, in most of this paper, we concentrate on full sign patterns.

The essential row or column sign change number is a powerful tool for studying the minimum ranks of sign patterns. The following is an important result on the minimum ranks of sign patterns.

Theorem 1.1 ([3, 26]). *For every sign pattern $\mathcal{A}$, $mr(\mathcal{A}) \leq \min\{ersc(\mathcal{A}_c), ecsc(\mathcal{A}_c)\} + 1$.*

The following observation is obvious.

Observation 1.2. *For any nonzero sign pattern $\mathcal{A}$, the following three statements are equivalent.*
(i) $mr(\mathcal{A}) = 1$.
(ii) $ersc(\mathcal{A}_c) = 0$.
(iii) $ecsc(\mathcal{A}_c) = 0$.

The characterization of sign patterns with minimum rank 2 given in [26] in terms of the essential row sign change number or the essential column sign change number of their condensed sign patterns may be restated as follows.

Theorem 1.3 ([26]). *For any sign pattern $\mathcal{A}$, the following three statements are equivalent.*
(i) $mr(\mathcal{A}) = 2$.
(ii) *$\mathcal{A}_c$ has at least two rows and $ersc(\mathcal{A}_c) = 1$.*
(iii) *$\mathcal{A}_c$ has at least two columns and $ecsc(\mathcal{A}_c) = 1$.*

Note that for a full sign pattern $\mathcal{A}$, the condition that "$\mathcal{A}_c$ has at least two rows" and the condition "$\mathcal{A}_c$ has at least two columns" in the preceding theorem are redundant as they follow from $\text{ersc}(\mathcal{A}_c) = 1$. In particular, for every full sign pattern $\mathcal{A}$, $\text{ersc}(\mathcal{A}_c) = 1$ if and only if $\text{ecsc}(\mathcal{A}_c) = 1$.

A natural question concerning sign patterns with minimum rank 3 is the following.

Problem 1.4. *Are the following three conditions equivalent for all sign patterns $\mathcal{A}$?*
(i) $mr(\mathcal{A}) = 3$.
(ii) *$\mathcal{A}_c$ has at least three rows and $ersc(\mathcal{A}_c) = 2$.*
(iii) *$\mathcal{A}_c$ has at least three columns and $ecsc(\mathcal{A}_c) = 2$.*

By considering the total number of sign vectors (or full sign vectors) realized by a list real polynomials in one variable, we obtain some interesting results in the next two sections on the essential row and column sign change numbers, which show that no two of the three preceding conditions are equivalent, even for full sign patterns.

2 Sign vectors realized by a list of polynomials

Alon [2] gave upper bounds for the total number of sign vectors realized by a list of multi-variable real polynomials. For our purposes here, we restrict to single variable polynomials. Let $p_1, \ldots, p_m$ be m real polynomials of degree at most k in a real variable t. Let $S(p_1, \ldots, p_m)$ be the set of all sign vectors realized by this list of polynomials, namely,

$$S(p_1, \ldots, p_m) = \{\text{sgn}([p_1(t), \ldots, p_m(t)]^T) \mid t \in \mathbb{R}\}.$$

Similarly, let $\underline{S}(p_1, \ldots, p_m)$ denotes the set of full sign vectors realized by $p_1, \ldots, p_m$, namely,

$$\underline{S}(p_1, \ldots, p_m) = \{\text{sgn}([p_1(t), \ldots, p_m(t)]^T) \mid t \in \mathbb{R} \text{ and } p_1(t)p_2(t)\ldots p_m(t) \neq 0\}.$$

As $\text{sgn}([p_1(t), \ldots, p_m(t)]^T)$ remains invariant over every open interval where $p_1(t)p_2(t)\ldots p_m(t) \neq 0$ always, and $p_1(t)p_2(t)\ldots p_m(t)$ has at most mk real zeros, it is clear that $|\underline{S}(p_1, \ldots, p_m)|$ is bounded above by the number of connected components of the set $\{x \in \mathbb{R} \mid p_1(x)p_2(x)\dots p_m(x) \neq 0\}$, which is at most $mk+1$. Also, each real zero of $p_1(t)p_2(t)\dots p_m(t)$ yields a sign vector with at least one zero entry realized by the polynomials $p_1,\dots,p_m$. Thus we obtain the following result.

Theorem 2.1. *Let $p_1,\dots,p_m$ be m real polynomials of degree at most k in a real variable t. Then*

$$|\underline{S}(p_1,\dots,p_m)| \le mk+1, \text{ and } |S(p_1,\dots,p_m)| \le 2mk+1.$$

We now construct a full sign pattern for which the essential column sign change number is 2 but the essential row sign change number is greater than 2.

Example 2.2. *Let $\mathcal{A}$ be a full sign pattern whose columns consist of all the distinct full column sign vectors in $\{+,-\}^6$ with leading entry + and sign change number at most 2. Then $\mathcal{A}$ is a condensed full sign pattern with 6 rows and it can be seen that the number of columns of $\mathcal{A}$ is $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}=16$. It is easy to check that ecsc$(\mathcal{A})=2$ and mr$(\mathcal{A})=3$. We now show that ersc$(\mathcal{A})\ge 3$.*

Proof. Assume otherwise. Then there is a signed permutation $\mathcal{Q}$ such that rsc$(\mathcal{A}\mathcal{Q})\le 2$. Replacing $\mathcal{A}$ with $\mathcal{A}\mathcal{Q}$ if necessary, we may assume that rsc$(\mathcal{A})\le 2$. Then for each $i=1,\dots,6$, there is a real polynomial $p_i(t)$ of degree at most 2 such that the ith row of $\mathcal{A}$ is equal to sgn$([p_i(1),p_i(2),\dots,p_i(16)])$. It follows that for each $j=1,\dots,16$, the jth column of $\mathcal{A}$ is of the form sgn$([p_1(j),\dots,p_6(j)]^T)$ and hence it belongs to $\underline{S}(p_1,\dots,p_6)$. But by Theorem 2.1, $|\underline{S}(p_1,\dots,p_6)|\le 6\cdot 2+1=13$, contradicting the fact that $\mathcal{A}$ has 16 distinct columns. □

The proof of the above example may be adapted to show the following result.

Theorem 2.3. *Let $\mathcal{A}$ be any $m\times n$ condensed sign pattern. Let $k=$ ersc$(\mathcal{A})$ and $k'=$ ecsc$(\mathcal{A})$. Then $n\le 2mk+1$ and $m\le 2nk'+1$. Furthermore, if $\mathcal{A}$ is full then $n\le mk+1$ and $m\le nk'+1$.*

The last part of the preceding theorem concerning condensed full sign patterns may be restated as follows.

Corollary 2.4. *Let $\mathcal{A}$ be any $m\times n$ condensed full sign pattern. Then*

$$\mathrm{ersc}(\mathcal{A}) \ge \left\lceil \frac{n-1}{m} \right\rceil, \text{ and } \mathrm{ecsc}(\mathcal{A}) \ge \left\lceil \frac{m-1}{n} \right\rceil.$$

Of course, any 6×14 submatrix of the matrix $\mathcal{A}$ in Example 2.2 also yields a full sign pattern for which the essential row and column sign change numbers are different. More generally, we can extend the construction of the matrix in Example 2.2 to get a sequence of matrices that share this property.

Example 2.5. *For each $m\ge 6$, let $\mathcal{A}_m$ be the $m\times n$ condensed sign pattern matrix whose columns comprise all distinct full $m\times 1$ sign vectors with sign change number at most 2 and leading entry +. We assume that the columns are arranged lexicographically, with + preceding −. Note that $n=\binom{m-1}{0}+\binom{m-1}{1}+\binom{m-1}{2}=1+(m-1)+(m-1)(m-2)/2=1+(m-1)m/2$. Clearly, ecsc$(\mathcal{A}_m)=2$, but by the preceding theorem, ersc$(\mathcal{A}_m)\ge(m-1)/2$, which tends to infinity as m tends to infinity.*

As an immediate consequence of the preceding theorem, we get the following result.

Corollary 2.6. *Let $\mathcal{L}_n$ be the $2^{n-1}\times n$ sign pattern whose rows consist of all the distinct $1\times n$ full sign vectors having the leading entry + (with the rows arranged in lexicographical order). Then ersc$(\mathcal{L}_n)=n-1$ and ecsc$(\mathcal{L}_n)\ge\lceil\frac{2^{n-1}-1}{n}\rceil$.*

It is easy to see that

$$\lim_{n\to\infty}\frac{\mathrm{ecsc}(\mathcal{L}_n)}{\mathrm{ersc}(\mathcal{L}_n)}=\infty.$$

It is well known that $\mathcal{L}_n$ is an L matrix and every condensed full L matrix with n columns is signed permutationally equivalent to $\mathcal{L}_n$.

We now construct square condensed sign patterns for which the essential row and column sign change numbers differ.

Example 2.7. *Let $m \geq 8$ and let $\mathcal{A}_m$ be the $m \times n$ sign pattern given in Example 2.5, where $n = 1+(m-1)m/2 > m$. Let $\mathcal{T}_n$ be the $n \times n$ full sign pattern all of whose entries on or below the diagonal are equal to + with the remaining entries all equal to −. Clearly, $\mathcal{T}_n$ is a condensed full sign pattern with column sign change number 1. Observe that both the last row of $\mathcal{T}_n$ and the first row of $\mathcal{A}_m$ are the all + vector and $\mathrm{csc}(\begin{bmatrix} \mathcal{T}_n \\ \mathcal{A}_m \end{bmatrix}) = 3$. Since all the entries in the first column of $\mathcal{T}_n$ or $\mathcal{A}_m$ are +, no row of $\mathcal{T}_n$ is equal to the negation of any row of $\mathcal{A}_m$. We now delete all the rows of $\mathcal{T}_n$ that are equal to some row of $\mathcal{A}_m$, and delete additional rows from the top of $\mathcal{T}_n$ if necessary so that the resulting submatrix $\mathcal{B}_n$ of $\begin{bmatrix} \mathcal{T}_n \\ \mathcal{A}_m \end{bmatrix}$ is $n \times n$. It is easy to see that the square sign pattern $\mathcal{B}_n$ is full and condensed, with $\mathrm{ecsc}(\mathcal{B}_n) \leq 3$. But as $\mathcal{B}_n$ contains $\mathcal{A}_m$ as a submatrix, we have $\mathrm{ersc}(\mathcal{B}_n) \geq \mathrm{ersc}(\mathcal{A}_m) \geq \lceil (m-1)/2 \rceil$, which is at least 4 and tends to infinity as m tends to infinity.*

We have constructed many condensed full sign patterns for which the essential row and column sign change numbers differ greatly. Suppose that $\mathcal{A}$ is an $m \times n$ (with $m \leq n$) condensed full sign pattern such that $|\mathrm{ecsc}(\mathcal{A}) - \mathrm{ersc}(\mathcal{A})| \geq 3$. By changing the (i, i) entries of $\mathcal{A}$ to zeros for all $i = 1, \dots, m$, we get a condensed sign pattern $\hat{\mathcal{A}}$. Observe that by changing one entry of a full sign vector to zero, the polynomial sign change number is never decreased and either remains the same (e.g., $\mathrm{psc}([+\ 0\ -]) = \mathrm{psc}([+\ -\ -]) = 1$), or increases by 1 (e.g., $\mathrm{psc}([+\ -\ 0]) = 2 = \mathrm{psc}([+\ -\ -]) + 1$), or increases by 2 (e.g., $\mathrm{psc}([+\ 0\ +]) = 2 = \mathrm{psc}([+\ +\ +]) + 2$). Thus $\mathrm{ersc}(\mathcal{A}) \leq \mathrm{ersc}(\hat{\mathcal{A}}) \leq \mathrm{ersc}(\mathcal{A}) + 2$ and $\mathrm{ecsc}(\mathcal{A}) \leq \mathrm{ecsc}(\hat{\mathcal{A}}) \leq \mathrm{ecsc}(\mathcal{A}) + 2$. Therefore, the essential row and column sign change numbers of the condensed sign pattern $\hat{\mathcal{A}}$ are different.

3 Sign patterns $\mathcal{A}$ such that $\mathrm{mr}(\mathcal{A}) < \min\{\mathrm{ersc}(\mathcal{A}_c), \mathrm{ecsc}(\mathcal{A}_c)\} + 1$

Our main objective in this section is to construct sign patterns $\mathcal{A}$ for which $\mathrm{mr}(\mathcal{A})$ is smaller than $\min\{\mathrm{ersc}(\mathcal{A}_c), \mathrm{ecsc}(\mathcal{A}_c)\} + 1$.

In [14], a direct correspondence between condensed sign patterns with minimum rank r and point-hyperplane configurations in $\mathbb{R}^{r-1}$ is established. In particular, a point-line configuration in $\mathbb{R}^2$ consisting of m points $p_1, \dots, p_m$ and n (nonvertical) lines $L_1, \dots, L_n$ gives rise to an $m \times n$ sign pattern $\mathcal{A} = [a_{ij}]$ with minimum rank at most 3, where for each $i = 1, \dots, m$, and $j = 1, \dots, n$,

$$a_{ij} = \begin{cases} +, & \text{if } p_i \text{ is above } L_j; \\ -, & \text{if } p_i \text{ is below } L_j; \\ 0, & \text{if } p_i \text{ is on } L_j. \end{cases}$$

Consider the point-line configuration below consisting of the 6 lines $L_1, \dots, L_6$ and 16 points marked as 1 through 16. For convenience, we also include a circle that contains all the intersection points among the the 6 lines as interior points.

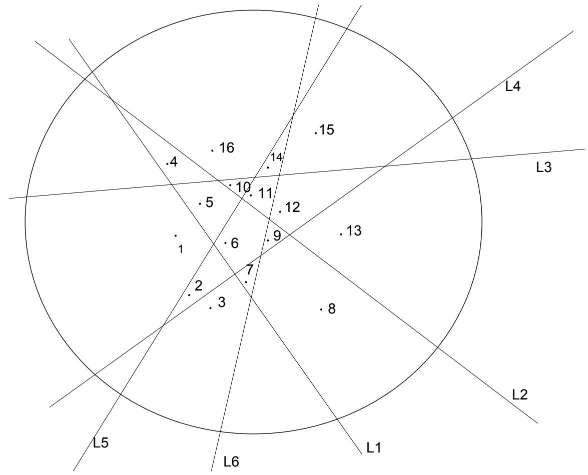

Figure 3.0

Note that the 6 lines $L_1, \dots, L_6$ in the preceding figure are in *general position* (namely, no two lines have the same slope and no three lines are concurrent). The set $\mathbb{R}^2 \setminus (\cup_{i=1}^{6} L_i)$ consists of 22 open regions. The 12 unbounded regions (which have nonempty intersections with the indicated circle) occur in opposite pairs whose relative positions to all the 6 lines are opposite. Hence, two points from any pair of opposite regions would give rise to opposite sign vectors relative to the 6 lines. Also, any two points from the same bounded region would yield the same sign vector relative to the 6 lines. In order to obtain a condensed sign pattern, we pick one point from each of the bounded regions and pick only one point from the union of any two opposite unbounded regions.

If we label the rows of the corresponding sign pattern by the points 1, ..., 16, and label the columns by the lines $L_1, \dots, L_6$, we obtain a 16×6 condensed full sign pattern $\mathcal{A}$ whose transpose is

$$\mathcal{A}^T = \begin{bmatrix} - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & + \\ - & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + \\ - & - & - & + & - & - & - & - & - & - & - & - & - & + & + & + \\ + & + & - & + & + & + & - & - & + & + & + & + & - & + & + & + \\ + & - & - & + & + & - & - & - & - & + & - & - & - & - & - & + \\ + & + & + & + & + & + & + & - & - & + & + & - & - & + & - & + \end{bmatrix}.$$

As mentioned above, $\text{mr}(\mathcal{A}) \le 3$. Observe that $\mathcal{A}[\{6, 7, 8, 9\}, \{4, 5, 6\}]$ is an L matrix with minimum rank 3. Hence, $\text{mr}(\mathcal{A}) = 3$. With the help of MATLAB, we have verified that $\text{ersc}(\mathcal{A}) = 3$. In fact, MATLAB indicates that upon negating the first four columns of $\mathcal{A}$, the resulting sign pattern has row sign change number 3. By Theorem 2.3, $\text{ecsc}(\mathcal{A}) \ge \lceil \frac{16-1}{6} \rceil = 3$. Thus this condensed full sign pattern $\mathcal{A}$ satisfies $\text{mr}(\mathcal{A}) = 3 < \min\{\text{ersc}(\mathcal{A}), \text{ecsc}(\mathcal{A})\} + 1$.

By adding more lines and points in the preceding point-line configuration, we can obtain bigger sign patterns containing $\mathcal{A}$ as a submatrix but still with minimum rank 3. Of course, the bigger sign patterns also have essential row and column sign change numbers at least 3. Indeed, it can be shown by simple induction that for $n \ge 6$, n lines in $\mathbb{R}^2$ in general position would divide $\mathbb{R}^2$ into $\frac{n(n+1)}{2} + 1$ open regions, and after deleting one from each pair of unbounded opposite regions, we have $\frac{n(n-1)}{2} + 1$ open regions and one point may be taken from each such region. This would yield a condensed full sign pattern $\mathcal{C}_n$ of minimum rank 3 and of size $(\frac{n(n-1)}{2} + 1) \times n$. It follows from Corollary 2.4 that $\text{ecsc}(\mathcal{C}_n) \ge \lceil (n-1)/2 \rceil$ (which tends to infinity as $n \to \infty$). It is conceivable (though we do not have a proof) that we may also have $\text{ersc}(\mathcal{C}_n) \to \infty$.

Further, by using the *dual transform* D_o [27], which converts a nonzero vector v in $\mathbb{R}^2$ to the line $D_o(v) = \{x \in \mathbb{R}^2 \mid \langle x, v \rangle = 1\}$ and vice versa, we can transform the $(\frac{n(n-1)}{2} + 1)$-point, n-line configuration to an n-point, $(\frac{n(n-1)}{2} + 1)$-line configuration that corresponds to the sign pattern $\mathcal{C}_n^T$. By Theorem 2.8 of [14], the square full sign pattern $\mathcal{E}_n = \begin{bmatrix} \mathcal{C}_n & + \\ - & \mathcal{C}_n^T \end{bmatrix}$ of order $\frac{n(n+1)}{2} + 1$ satisfies $\text{mr}(\mathcal{E}_n) = 3$. However, $\text{ersc}(\mathcal{E}_n) \ge \text{ersc}(\mathcal{C}_n^T) \ge$

$\lceil (n-1)/2 \rceil$ and $\text{ecsc}(\mathcal{E}_n) \geq \text{ecsc}(\mathcal{C}_n) \geq \lceil (n-1)/2 \rceil$. Thus for large n, there is a huge gap between $\text{mr}(\mathcal{E}_n) = 3$ and $1 + \min\{\text{ersc}(\mathcal{E}_n), \text{ecsc}(\mathcal{E}_n)\}$, which is at least $\lceil (n+1)/2 \rceil$.

4 Sign change numbers of full sign patterns with large minimum ranks

The following existence result of Alon et al. [1, 2] (proved using probabilistic method) is of great interest.

Theorem 4.1 ([1, 2]). *Let $r(n, n)$ be the largest possible minimum rank achieved by $n \times n$ full sign pattern matrices. Then*

$$\frac{n}{16} \leq r(n, n) \leq \frac{n}{2} + 3\sqrt{n}.$$

It is conjectured [1] that $r(n, n)$ might be asymptotically equal to $(1 + o(1))\frac{n}{2}$.

Of course, in view of Theorem 1.1, every $n \times n$ full sign patterns $\mathcal{A}$ such that $\text{mr}(\mathcal{A}) \geq \frac{n}{16}$ must satisfy $\min\{\text{ersc}(\mathcal{A}_c), \text{ecsc}(\mathcal{A}_c)\} \geq \frac{n}{16} - 1$. But constructing $n \times n$ full sign patterns with large minimum rank ($\geq \frac{n}{16}$, or even possibly close to $n/2$) is a very difficult open problem that awaits further research.

As observed in [28], it is clear that every $n \times n$ full sign pattern $\mathcal{A}$ containing the L matrix $\mathcal{L}_k$, where $k = 1 + \lfloor \log_2 n \rfloor$ has minimum rank at least k. But this obvious logarithmic lower bound k is much smaller than $\frac{n}{16}$ for large n.

A significant lower bound on the minimum rank of a full sign pattern is given by Forster [17].

Theorem 4.2 ([17]). *Let $\mathcal{A}$ be an $m \times n$ full sign pattern and let B be the $(1, -1)$ matrix in $Q(\mathcal{A})$. Then*

$$mr(\mathcal{A}) \geq \frac{\sqrt{mn}}{\|B\|_2},$$

where $\|B\|_2$ is the spectral norm of B.

Forster's lower bound yields the following result on the minimum rank of the sign pattern of a Hadamard matrix H of order n (that is, H is a $(1, -1)$ matrix of order n such that $HH^T = nI$).

Corollary 4.3. *Let H be a Hadamard matrix of order n. Then $mr(sgn(H)) \geq \sqrt{n}$.*

Possibly the sign patterns of some special Hadamard matrices of large order n (a multiple of 4) have minimum ranks exceeding $\frac{n}{16}$. But as pointed out by Noga Alon, there is no universal linear lower bound for the minimum ranks of the sign patterns of Hadamard matrices. Indeed, if B stands for a Hadamard matrix of order 4, then $\text{rank}(B) = 3$ and the Hadamard matrix $H = \otimes_{i=1}^{k} B$ of order 4^k has rank 3^k. Hence, $\text{mr}(\text{sgn}(H)) \leq 3^k$ and $\lim_{k\to\infty} \frac{\text{mr}(\text{sgn}(H))}{4^k} = 0$.

Since a necessary condition for a full sign pattern to have large minimum rank is that its essential row and column sign change numbers are large, a natural question is the following.

Problem 4.4. *What is the largest possible value of $\min\{ersc(\mathcal{A}), ecsc(\mathcal{A})\}$ for $n \times n$ full sign patterns $\mathcal{A}$ and how can one construct the extremal full sign patterns?*

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