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A Lie product type formula in Euclidean Jordan algebras

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Abstract: In this paper, we state and prove an analog of Lie product formula in the setting of Euclidean Jordan algebras.

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1 Introduction

For two $n \times n$ complex matrices A and B, the Lie product formula [5] asserts that

$$\lim_{k \to \infty} (exp(A/k)exp(B/k))^k = exp(A + B).$$

Our objective in this paper is to state and prove an analog of this formula in the setting of Euclidean Jordan algebras.

2 Preliminaries

Throughout this paper, we let V be a Euclidean Jordan algebra of rank r and $K := \{x \circ x : x \in V\}$ be the cone of squares in V. We use the notation $x \geq 0$ ($x > 0$) when $x \in K$ (respectively, $x \in K^o$ (=interior (K))). We recall some concepts, properties, and results used in this paper. Most of these can be found in [2].

A Euclidean Jordan algebra V is a finite dimensional real Hilbert space that carries an inner product $\langle \cdot, \cdot \rangle$ and a bilinear Jordan product $x \circ y : V \times V \to V$ satisfying the following conditions:
(i) $x \circ y = y \circ x$ for all $x, y \in V$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$; and
(iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in V$.

In addition, we assume that there is an element $e \in V$ (called the *unit* element) such that $x \circ e = x$ for all $x \in V$. The (trace) inner product is defined by $\langle x, y \rangle := trace(x \circ y)$ for any $x, \ y \in V$.

A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two (non-trivial) Euclidean Jordan algebras. It is well known that any nonzero Euclidean Jordan algebra is a product of simple Euclidean Jordan algebras and every simple algebra is isomorphic to one of the algebras given below:

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(i) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices with trace inner product and the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$;
(ii) The algebra $\mathcal{H}^n$ of all $n \times n$ complex Hermitian matrices with trace inner product and the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$;
(iii) The algebra $\mathcal{Q}^n$ of all $n \times n$ quaternion Hermitian matrices with (real) trace inner product and the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$;
(iv) The algebra $\mathcal{O}^3$ of all 3×3 octonion Hermitian matrices with (real) trace inner product and the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$;
(v) The Jordan spin algebra $\mathcal{L}^n$ $(n \geq 3)$ quadratic forms in R^n with standard inner product and the Jordan product

$$x \circ y := (x^T y, x_1 y_2 + y_1 x_2, \cdots, x_1 y_n + y_1 x_n)^T.$$

The spectral decomposition For $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$ such that

$$x = \lambda_1(x)e_1 + \cdots + \lambda_r(x)e_r. \tag{1}$$

The numbers $\lambda_i(x)$ are called the eigenvalues of x.

For a given $a \in V$, the *quadratic representation* $P_a : V \to V$ is defined respectively by

$$P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

Given (1), the Frobenius norm and spectral norm of x are respectively defined by

$$||x|| := \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{r} \lambda_i^2(x)} \quad \text{and} \quad ||x||_\infty = max_{1 \leq i \leq r} |\lambda_i(x)|. \tag{2}$$

Proposition 1. *(see Lemma 2.9, [6]) For any* $x,\ y \in V$, $||x \circ y|| \leq ||x||\,||y||$.

Proposition 2. *(see Theorem 2, [7]) For any* $x,\ y \in V$, $||x \circ y|| \leq ||x||_\infty\,||y||$.

(2), Proposition 1, and Proposition 2 immediately yield the following proposition.

Proposition 3. *Let* $x,\ y \in V$ *and* $n \in \mathcal{N}$. *Then*
(a) $||x \circ y||_\infty \leq ||x \circ y|| \leq ||x||_\infty\,||y|| \leq ||x||\,||y||$.
(b) $||x^n|| \leq ||x||^n$, *where* $x^n = x \circ x \circ \cdots \circ x$.
(c) $||x^n||_\infty = ||x||_\infty^n$.

Proposition 4. *(Theorem 4.1, [8], Corollary 1, [3]) Let V be any Euclidean Jordan algebra. Then*

$$||x + y||_\infty \leq ||x||_\infty + ||y||_\infty.$$

Suppose that f is analytic on $\mathbb{R}$, i.e., $f(t) = \sum_0^\infty \alpha_n t^n$ and $x = \sum_1^r \lambda_i(x)e_i$. Then

$$f(x) = \sum_1^r (\sum_{n=0}^\infty \alpha_n \lambda_i^n(x)) e_i = \sum_1^r f(\lambda_i(x)) e_i.$$

3 Majorization

Given a vector $x = (x_1, x_2, \ldots, x_r)$ in $\mathbb{R}^r$, we write $x^\downarrow := (x_1^\downarrow, x_2^\downarrow, \ldots, x_r^\downarrow)$ for the vector obtained by rearranging the components of x in the decreasing order. For two vectors $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ in

$\mathbb{R}^r$, we say that x is *majorized* by y and write $x \prec y$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad (k = 1, 2, \ldots, r-1)$$

and

$$\sum_{i=1}^{r} x_i^{\downarrow} = \sum_{i=1}^{r} y_i^{\downarrow},$$

and we say that x is *weekly majorized* by y and write $x \prec_w y$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad (k = 1, 2, \ldots, r-1, r).$$

When $x \geq 0$ and $y \geq 0$, we say that x is **log-majorized** by y and write $x \prec_{log} y$ if

$$\Pi_1^k x_i^{\downarrow} \leq \Pi_1^k y_i^{\downarrow} \quad (k = 1, 2, \ldots, r-1)$$

and

$$\Pi_1^r x_i^{\downarrow} = \Pi_1^r y_i^{\downarrow}.$$

It is well known that $x \prec_{log} y \Rightarrow x \prec_w y$.

Theorem 1. *([1], Theorem II. 3.1) For $x = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_r^{\downarrow})$ and $y = (y_1^{\downarrow}, y_2^{\downarrow}, \ldots, y_r^{\downarrow})$ in $\mathbb{R}^r$ and for any convex function $\phi : \mathbb{R} \to \mathbb{R}$, the following two conditions are equivalent:*

$$x \prec y. \tag{3}$$

$$\sum_{i=1}^{r} \phi(x_i) \leq \sum_{i=1}^{r} \phi(y_i). \tag{4}$$

If ϕ is a strictly convex on $\mathbb{R}$, then equality in (4) holds if and only if $x_i^{\downarrow} = y_i^{\downarrow}$ for all i.

4 The Main Result

Theorem 2. *Let V be a Euclidean Jordan algebra. Suppose that f and g are analytic on $\mathbb{R}$ (that is, they have power series expansions valid on all of $\mathbb{R}$) with $f(0) = 1 = g(0)$. Then, for any $a, b \in V$,*

$$\lim_{k\to\infty} (f(a/k) \circ g(b/k))^k = exp(f'(0)a + g'(0)b).$$

The proof of Theorem 2 can be given following Herzog's argument in [4].

Lemma 1. *If $a, b \in V$ and $k \in \mathbb{N}$, then $||(a+b)^k - a^k|| \leq (||a||_\infty + ||b||)^k - ||a||_\infty^k$.*

Proof. We use induction to prove this inequality. It is obvious when $k = 1$. Suppose that the inequality holds for $k-1$. Then by Proposition 3,

$$\begin{aligned}
||(a+b)^k - a^k|| &= ||a \circ [(a+b)^{k-1} - a^{k-1}] + b \circ [(a+b)^{k-1} - a^{k-1}] + b \circ a^{k-1}|| \\
&\leq ||a \circ [(a+b)^{k-1} - a^{k-1}]|| + ||b \circ [(a+b)^{k-1} - a^{k-1}]|| + ||b \circ a^{k-1}|| \\
&\leq ||a||_\infty ||(a+b)^{k-1} - a^{k-1}|| + ||b|| ||(a+b)^{k-1} - a^{k-1}|| + ||b|| ||a^{k-1}||_\infty \\
&\leq ||a||_\infty [(||a||_\infty + ||b)||)^{k-1} - ||a||_\infty^{k-1}] + ||b|| [(||a||_\infty + ||b||)^{k-1} - ||a||_\infty^{k-1}] + ||b|| ||a||_\infty^{k-1} \\
&= (||a||_\infty + ||b||)^k - ||a||_\infty^k.
\end{aligned}$$

□

Lemma 2. *Let $a \in V$. Then* $\lim_{n\to\infty}(e+\frac{a}{n})^n = exp(a)$.

Proof. Writing the spectral decomposition for a as $a = \sum_1^r \lambda_i(a)e_i$, we have

$$exp(a) = \sum_1^r exp(\lambda_i(a))e_i \text{ and } (e+\frac{a}{n})^n = \sum_i^r (1+\frac{\lambda_i(a)}{n})^n e_i.$$

Since $(1+\frac{\lambda_i(a)}{n})^n \to exp(\lambda_i(a))$ as $n\to\infty$, we have $\lim_{n\to\infty}(e+\frac{a}{n})^n = exp(a)$. □

Proof of Theorem 2. Let

$$f(t) = \sum_0^\infty \alpha_n t^n \text{ and } g(t) = \sum_0^\infty \beta_n t^n.$$

$a,\ b \in V$, for each $k \in \mathbb{N}$, we have

$$f(a/k) = e + f'(0)a/k + u_k \text{ and } g(b/k) = e + g'(0)b/k + v_k,$$

where $u_k = \sum_{n=2}^\infty \alpha_n \frac{a^n}{k^n}$ and $v_k = \sum_{n=2}^\infty \beta_n \frac{b^n}{k^n}$. Thus,

$$||k^2 u_k|| \le \sum_{n=2}^\infty |\alpha_n| \frac{||a||^n}{k^{n-2}} \le \sum_{n=2}^\infty |\alpha_n|\, ||a||^n < +\infty \text{ and } ||k^2 v_k|| \le \sum_{n=2}^\infty |\beta_n| \frac{||b||^n}{k^{n-2}} \le \sum_{n=2}^\infty |\beta_n|\, ||b||^n < +\infty.$$

Hence,

$$f(a/k)\circ g(b/k) = (e+f'(0)a/k+u_k)\circ(e+g'(0)b/k+v_k) = e + \frac{f'(0)a+g'(0)b}{k} + w_k,$$

where $w_k = u_k+v_k+\frac{1}{k}(f'(0)a\circ v_k+g'(0)b\circ u_k)+\frac{f'(0)g'(0)}{k^2}a\circ b+u_k\circ v_k$. It is easy to verify that $\{k^2 w_k\}$ is a bounded sequence in V. So, there exists a $M>0$ such that $||w_k|| \le \frac{M}{k^2}$ for all $k\in\mathbb{N}$. Now, letting $c = f'(0)a+g'(0)b$, we have

$$\begin{aligned}
||(f(a/k)\circ g(b/k))^k - (e+\frac{c}{k})^k|| &= ||(e+\frac{c}{k}+w_k)^k - (e+\frac{c}{k})^k|| \\
&\le (||e+\frac{c}{k}||_\infty + ||w_k||)^k - ||e+\frac{c}{k}||_\infty^k \\
&\le (||e+\frac{c}{k}||_\infty + \frac{M}{k^2})^k - ||e+\frac{c}{k}||_\infty^k \\
&= k(\beta(k))^{k-1}\frac{M}{k^2} \qquad (5)\\
&= \frac{M(\beta(k))^{k-1}}{k},
\end{aligned}$$

where $\beta(k) \in (||e+\frac{c}{k}||_\infty,\ ||e+\frac{c}{k}||_\infty+\frac{M}{k^2})$. Note that the first inequality is from Lemma 1 and (5) is from the Mean Value Theorem. Thus, we have

$$\begin{aligned}
0 \le (\beta(k))^{k-1} \le (||e+\frac{c}{k}||_\infty+\frac{M}{k^2})^{k-1} &\le (||e||_\infty + \frac{||c||_\infty}{k}+\frac{M}{k^2})^{k-1} \\
&\le (1+\frac{||c||_\infty+M}{k})^k \to exp(||c||_\infty+M).
\end{aligned}$$

Note that the third inequality is from Proposition 4. Hence, $\lim_{k\to\infty}\frac{M(\beta(k))^{k-1}}{k} = 0$. Therefore,

$$\lim_{k\to\infty}(f(a/k)\circ(exp(b/k))^k = \lim_{k\to\infty}(e+\frac{c}{k})^k = exp(f'(0)a+g'(0)b).$$

Note that the last equality is from Lemma 2. □

Putting $f(t) = g(t) = exp(t)$ for $t\in\mathbb{R}$, Theorem 2 immediately yields the following corollary.

Corollary 1. *For any $a,\ b \in V$,* $\lim_{k\to\infty}(exp(a/k)\circ exp(b/k))^k = exp(a+b)$.

4.1 An application

Lemma 3. *(see Proposition III 5.3, [1]) Let A be an $n \times n$ complex matrix. Then*

$$Re(\lambda^{\downarrow}(A)) \prec \lambda^{\downarrow}(Re(A)),$$

where $Re(A) = \frac{A+A^*}{2}$.

Theorem 3. *Let $V = \mathcal{S}^n$ or $\mathcal{H}^n$ or $\mathcal{Q}_n$ and $0 < x,\ s \in V$. Then*

$$\lambda^{\downarrow}((P_{x^{1/2}}(s)) \prec \lambda^{\downarrow}(x \circ s).$$

Proof. (i) When $V = \mathcal{S}^n$ or $V = \mathcal{H}^n$, $P_X(Y) = XYX$ for $X,\ Y \in V$.

Since $X \circ S = \frac{XS+SX}{2}$, by Lemma 3, we have $Re(\lambda^{\downarrow}(XS)) \prec \lambda^{\downarrow}(X \circ S)$. Since $X,\ S \succ 0$,

$$\lambda_i(XS) = \lambda_i(X^{1/2}X^{1/2}S) = \lambda_i(X^{1/2}SX^{1/2}) = \lambda_i(P_{X^{1/2}}(S)).$$

Thus, $\lambda^{\downarrow}(P_{X^{1/2}}(S)) \prec \lambda^{\downarrow}(X \circ S)$.

(ii) When $V = \mathcal{Q}^n$, $P_X(Y) = XYX$ for $X,\ Y \in \mathcal{Q}^n$

For an $n \times n$ quaternion matrix A, we write $A = A_1 + A_2 j$, where $A_1,\ A_2$ are $n \times n$ complex matrices. The complex adjoint matrix of A is defined by $\chi_A := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$. It is well known (e.g., Theorem 4.2, [10]) that χ_A is Hermitian if and only if A is Hermitian and the eigenvalues of A coincide with the eigenvalues of χ_A (see Theorem 5.4 and Corollary 5.1, [10]) when A is Hermitian. Now, $P_{X^{1/2}}(S) = X^{1/2}SX^{1/2}$. By Theorem 4.2, [10], we have $\chi_{X^{1/2}SX^{1/2}} = \chi_{X^{1/2}}\chi_S\chi_{X^{1/2}} = (\chi_X)^{1/2}\chi_S(\chi_X)^{1/2}$ and $\chi_{(XS+SX)/2} = (\chi_X\chi_S + \chi_S\chi_X)/2$. Therefore, the result follows by Case (i). □

Corollary 2. *Let $V = \mathcal{S}^n$ or $\mathcal{H}^n$ or $\mathcal{Q}_n$ and $0 < x,\ s \in V$. Then*

$$tr((x \circ s)^{2^k}) \geq tr((P_{x^{1/2}}(s))^{2^k} \text{ for } k \in \mathbb{N}.$$

Proof. Since for any fixed $k \in \mathbb{N}$, $f(t) = t^{2^k}$ is a convex function, by Theorem 3 and Theorem 1, we have

$$tr((x \circ s)^{2^k}) \geq tr((P_{x^{1/2}}(s))^{2^k}.$$

□

Lemma 4. *Let $A,\ B \in \mathcal{Q}^n$. Then*

$$\lambda^{2^k}(exp(A+B)) \prec_{log} \lambda^{2^k}(exp(A/2)exp(B)exp(A/2)) \text{ for } k \in \mathbb{N}.$$

Proof. First we show that for any $C \in \mathcal{Q}^n$ $exp(\chi_C) = \chi_{exp(C)}$. From Corollary 6.1 in [10], for $C \in \mathcal{Q}^n$, there exists a unitary U such that

$$C = U^* \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} U.$$

Thus,

$$exp(C) = U^* \begin{bmatrix} exp(\lambda_1) & 0 & \cdots & 0 \\ 0 & exp(\lambda_2) & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & exp(\lambda_n) \end{bmatrix} U \text{ and } \chi_C = \chi_U^* \chi_D \chi_U,$$

where $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$. Note that χ_D is a $2n \times 2n$ diagonal matrix with the same eigenvalues (including multiplicities) as D by item 7 in Theorem 4.2, [10]. Hence,

$$\chi_{exp(C)} = \chi_U^\star \begin{bmatrix} exp(\lambda_1) & 0 & \cdots & 0 \\ 0 & exp(\lambda_2) & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & exp(\lambda_n) \end{bmatrix} \chi_U \text{ and } exp(\chi_C) = \chi_U^\star \begin{bmatrix} exp(\lambda_1) & 0 & \cdots & 0 \\ 0 & exp(\lambda_2) & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & exp(\lambda_n) \end{bmatrix} \chi_U.$$

Thus, $exp(\chi_C) = \chi_{exp(C)}$. Now, from Theorem 2.10 in [9], we have

$$\begin{aligned} \lambda(exp(\chi_A + \chi_B)) &\prec_{log} \lambda(exp(\chi_A/2)exp(\chi_B)exp(\chi_A/2)) \Rightarrow \\ \lambda(exp(\chi_{A+B})) &\prec_{log} \lambda(\chi_{exp(A/2)}\chi_{exp(B)}\chi_{exp(A/2)}) \Rightarrow \\ \lambda(\chi_{exp(A+B)}) &\prec_{log} \lambda(\chi_{exp(A/2)exp(B)exp(A/2)}) \Rightarrow \\ \lambda(exp(A+B)) &\prec_{log} \lambda(exp(A/2)exp(B)exp(A/2)) \Rightarrow \\ \lambda^{2^k}(exp(A+B)) &\prec_{log} \lambda^{2^k}(exp(A/2)exp(B)exp(A/2)). \end{aligned}$$

We note that the second to last implication uses the fact that the eigenvalues of $C \in \mathcal{Q}^n$ coincide with the eigenvalues of χ_C (see Theorem 5.4 and Corollary 5.1, [10]). □

Since log-majorization implies weak majorization, Lemma 4 immediately yields the following corollary.

Corollary 3. *Let $A, B \in \mathcal{Q}^n$. Then*

$$tr(exp(A+B))^{2^k} \leq tr((exp(A/2)exp(B)exp(A/2))^{2^k} \text{ for } k \in \mathbb{N}.$$

Theorem 4. *Suppose that $A, B \in \mathcal{Q}^n$. Then*

$$\lim_{k\to\infty} tr((exp(A/2^{k+1})exp(B/2^k)exp(A/2^{k+1}))^{2^k} = tr(exp(A+B)).$$

Proof. Taking $exp(A/2^k)$ as x and $exp(B/2^k)$ as s In Corollary 2 and taking $exp(A/2^k)$ as A and $exp(B/2^k)$ as B in Corollary 3, we have

$$tr((exp(A/2^k) \circ exp(B/2^k))^{2^k} \geq tr((exp(A/2^{k+1})exp(B/2^k)exp(A/2^{k+1}))^{2^k} \geq tr(exp(A+B)).$$

Thus, by Corollary 1 and the Squeeze theorem,

$$\lim_{k\to\infty} tr((exp(A/2^{k+1})exp(B/2^k)exp(A/2^{k+1}))^{2^k} = tr(exp(A+B)).$$

□

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