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Some norm inequalities for special Gram matrices

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Abstract: In this paper we firstly give majorization relations between the vectors $F_n = \{f_0, f_1, \dots, f_{n-1}\}$,$L_n = \{l_0, l_1, \dots, l_{n-1}\}$ and $P_n = \{p_0, p_1, \dots, p_{n-1}\}$ which constructed with fibonacci, lucas and pell numbers. Then we give upper and lower bounds for determinants, Euclidean norms and Spectral norms of Gram matrices G_F=$\langle F_n, F_n\rangle$, G_L=$\langle L_n, L_n\rangle$,G_P=$\langle P_n, P_n\rangle$, G_{FL}=$\langle F_n, L_n\rangle$, G_{FP}=$\langle F_n, P_n\rangle$.

Keywords: Gram matrix, Matrix norms, Fibonacci, Lucas and Pell Numbers

MSC: 65D10, 92C45

1 Introduction

Special matrices is a widely studied subject in matrix analysis.Especially special matrices whose entries are well known number sequences have become a very interesting research subject in recent years and many authors have obtained some good results in this area. For example, Türkmen and Civciv have established some norm inequalities on circulant matrices with lucas numbers [1]. S.Solak has studied the norms of circulant matrices with fibonacci and lucas numbers [2], Bozkurt and Tam have obtained some results belong to determinants and inverses of r-circulant matrices associated with a number sequence [3] and S.Shen and J.Cen have made a similar study by using the same special matrix with k-fibonacci and k-lucas numbers [4]. For more properties, formulas belong to the Fibonacci, Lucas and Pell numbers (see, e.g., [11, 12]).

In this study we obtain some inequalities of determinants, euclidean norms and spectral norms in view of Gram matrices whose entries are fibonacci, lucas and pell numbers. The Gram matrix [7] has many applications namely system identification [8], modeling of power density spectrum [9] and model reduction [10]. In addition, Gram matrices are naturally positive semi definite matrices and their eigenvalues are non-negative real numbers. So, many good results can be obtained in terms of matrix norms by using Gram matrices.

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2 Some Preliminaries

The Gram matrix of a set of vectors $y_1, y_2, \ldots, y_n$ in an inner product space is the Hermitian matrix of linear products, whose entries are given by $g_{ij} = \langle y_i, y_j \rangle$ [6].

$$G = \begin{bmatrix} \langle y_1, y_1 \rangle & \ldots & \langle y_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle y_n, y_1 \rangle & \ldots & \langle y_n, y_n \rangle \end{bmatrix} \tag{1}$$

In view of (1) we obtain some special 2×2 Gram matrix such that

$$G_F = \begin{bmatrix} \langle F_n, F_n \rangle & \langle F_n, F_n \rangle \\ \langle F_n, F_n \rangle & \langle F_n, F_n \rangle \end{bmatrix} \tag{2}$$

$g_{ij} = \langle y_i, y_i \rangle = \langle F_n, F_n \rangle$ where $F_n = \{f_0, f_1, \ldots, f_{n-1}\}$ denotes $n-$ dimensional real vector whose entries are Fibonacci numbers,

$$G_L = \begin{bmatrix} \langle L_n, L_n \rangle & \langle L_n, L_n \rangle \\ \langle L_n, L_n \rangle & \langle L_n, L_n \rangle \end{bmatrix} \tag{3}$$

$g_{ij} = \langle y_i, y_i \rangle = \langle L_n, L_n \rangle$ where where $L_n = \{l_0, l_1, \ldots, l_{n-1}\}$ denotes $n-$ dimensional real vector whose entries are Lucas numbers,

$$G_P = \begin{bmatrix} \langle P_n, P_n \rangle & \langle P_n, P_n \rangle \\ \langle P_n, P_n \rangle & \langle P_n, P_n \rangle \end{bmatrix} \tag{4}$$

$g_{ij} = \langle y_i, y_i \rangle = \langle P_n, P_n \rangle$ where $P_n = \{p_0, p_1, \ldots, p_{n-1}\}$ denotes $n-$dimensional real vector whose entries are Pell numbers,

$$G_{FL} = \begin{bmatrix} \langle F_n, F_n \rangle & \langle F_n, L_n \rangle \\ \langle L_n, F_n \rangle & \langle L_n, L_n \rangle \end{bmatrix} \tag{5}$$

$g_{ij} = \langle y_i, y_j \rangle = \langle F_n, L_n \rangle$ where $F_n = \{f_0, f_1, \ldots, f_{n-1}\}$ and $L_n = \{l_0, l_1, \ldots, l_{n-1}\}$ denote $n-$dimensional real vectors whose entries are Fibonacci and Lucas numbers respectively,

$$G_{FP} = \begin{bmatrix} \langle F_n, F_n \rangle & \langle F_n, P_n \rangle \\ \langle P_n, F_n \rangle & \langle P_n, P_n \rangle \end{bmatrix} \tag{6}$$

$g_{ij} = \langle y_i, y_j \rangle = \langle F_n, P_n \rangle$ where $F_n = \{f_0, f_1, \ldots, f_{n-1}\}$ and $P_n = \{p_0, p_1, \ldots, p_{n-1}\}$ denote $n-$dimensional real vectors whose entries are Fibonacci and Pell numbers respectively.

The singular values of A is the square roots of the eigenvalues of the Hermitian, positive semidefinite matrix $(A^{\star}A)$ where $A^{\star}$ is stand for the transpose conjugate of A. Spectral norm of A is the largest singular values of A induced by $\|A\|_2 = s_1(A)$, where $s_i(A) = \lambda_i(A^{\star}A)^{1/2}$, $i = 1, 2, \ldots, n$. If A is positive semi definite, the spectral norm of matrix A is the largest eigenvalue of matrix A. The Euclidean norm of A $n-$square matrix defined as

$$\|A\|_F = \sqrt{tr(A^{\star}A)} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$$

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then the $l_p norm$ of vector x is defined as

$$||x||_p = \left(\sum_{i=1}^{k} |x_i|^p \right)^{\frac{1}{p}}, p \geq 1.$$

The l_2 norm is *Euclidean norm* and induced by $\|.\|$ in this work.

3 Main results

Let $x = (x_1, x_2, \dots, x_n) \in R^n$. We rearrange its components as $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ in decreasing order and induced by $x^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \dots, x_n^{\downarrow})$. For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, k = 1, 2, \dots, n,$$

we say that x is weakly majorized by y, denoted $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ holds, then we say that x is majorized by y and denote $x \prec y$.

Lemma 1. *[5] If f is convex, then*

$$x \prec y \Rightarrow f(x) \prec_w f(y)$$

if f is increasing and convex, then

$$x \prec_w y \Rightarrow f(x) \prec_w f(y).$$

In particular, for $f(x) = x^2$, on $\mathbb{R}_+$

$$\sum_{i=1}^{k} x_i^2 \leq \sum_{i=1}^{k} y_i^2, \quad \text{for } k = 1, 2, \dots, n$$

Theorem 2. *Let $F_n = \{f_0, f_1, \dots, f_{n-1}\}$ and $L_n = \{l_0, l_1, \dots, l_{n-1}\}$ be two vectors whose entries are Fibonacci and Lucas numbers respectively. Then following inequalities hold.*

$$\|F_n\|_p \leq \|L_n\|_p$$

Proof. If we rearrange the components of F_n and L_n in decreasing order

$F_n^{\downarrow} = \{f_{n-1}, f_{n-2}, \dots, 1, 1, 0\}$, $L_n^{\downarrow} = \{l_{n-1}, l_{n-2}, \dots, 3, 2, 1\}$ and use well known formula $l_n = f_n + 2f_{n-1}$ we get

$$\sum_{i=0}^{n-1} f_i^{\downarrow} \leq \sum_{i=0}^{n-1} f_i^{\downarrow} + 2.f_{i-1}^{\downarrow}$$

$$\sum_{i=0}^{n-1} f_i^{\downarrow} \leq \sum_{i=0}^{n-1} l_i^{\downarrow}$$

$$F_n \prec_w L_n$$

for a convex and increasing function $f(x) = x^p, p > 1$

$$\sum_{i=0}^{n-1} f_i^p \leq \sum_{i=0}^{n-1} l_i^p$$

$$\left(\sum_{i=0}^{n-1} f_i^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=0}^{n-1} l_i^p\right)^{\frac{1}{p}}$$

$$\|F_n\|_p \leq \|L_n\|_p$$

□

Corollary 3. *Let $F_n = \{f_0, f_1, ..., f_{n-1}\}$ and $L_n = \{l_0, l_1, ..., l_{n-1}\}$ be two vectors whose entries are Fibonacci and Lucas numbers respectively. Then*

$$\|F_n\| \leq \|L_n\|$$

Proof. The proof can be easily obtained by taking $p = 2$ in the proof of Theorem 2. For $p = 2$ we get

$$\sum_{i=0}^{n-1} f_i^2 \leq \sum_{i=0}^{n-1} l_i^2$$

$$\left(\sum_{i=0}^{n-1} f_i^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=0}^{n-1} l_i^2\right)^{\frac{1}{2}}$$

$$\|F_n\| \leq \|L_n\|$$

□

Theorem 4. *Let $F_n = \{f_0, f_1, \ldots, f_{n-1}\}$ and $P_n = \{p_0, p_1, \ldots, p_{n-1}\}$ be $n-$ dimensional real vectors constructed with Fibonacci and Pell numbers respectively. Then following majorization relation hold.*

$$\|F_n\|_p \leq \|P_n\|_p$$

Proof. Let U_n be any vector whose entries satisfy the recurrence relation $u_{n+1} = k.u_n + q.u_{n-1}$ with initial conditions $u_0 = 0, u_1 = 1$. If we take $k = q = 1$, then $U_n = F_n$. If we take $k = 2$ and $q = 1$, then $U_n = P_n$. If we rearrange the components of vector u in decreasing order we get

$$\sum_{i=0}^{n-1} u_i^{\downarrow} + u_{i-1}^{\downarrow} \leq \sum_{i=0}^{n-1} u_i^{\downarrow} + u_i^{\downarrow} + u_{i-1}^{\downarrow}$$

$$\sum_{i=0}^{n-1} f_i^{\downarrow} \leq \sum_{i=0}^{n-1} 2.u_i^{\downarrow} + u_{i-1}^{\downarrow}$$

$$\sum_{i=0}^{n-1} f_i^{\downarrow} \leq \sum_{i=0}^{n-1} p_i^{\downarrow}$$

$$F_n \prec_w P_n$$

for a convex and increasing function $f(x) = x^p, p > 1$

$$\sum_{i=0}^{n-1} f_i^p \leq \sum_{i=0}^{n-1} p_i^p$$

$$\left(\sum_{i=0}^{n-1} f_i^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=0}^{n-1} p_i^p\right)^{\frac{1}{p}}$$

$$\|F_n\|_p \leq \|P_n\|_p$$

□

Corollary 5. *Let $F_n = \{f_0, f_1, \ldots, f_{n-1}\}$ and $P_n = \{p_0, p_1, \ldots, p_{n-1}\}$ be $n-$dimensional real vectors constructed with Fibonacci and Pell numbers respectively. Then*

$$\|F_n\| \leq \|P_n\|$$

Proof. The proof can be easily obtained by taking $p = 2$ in the proof of Theorem 4. For $p = 2$ we get

$$\sum_{i=0}^{n-1} f_i^2 \le \sum_{i=0}^{n-1} p_i^2$$

$$\left(\sum_{i=0}^{n-1} f_i^2\right)^{\frac{1}{2}} \le \left(\sum_{i=0}^{n-1} p_i^2\right)^{\frac{1}{2}}$$

$$\|F_n\| \le \|P_n\|$$

□

Theorem 6. *Let* 2×2 *Gram matrix* G_F *be as in (2). Then following inequalities hold.*

$$\|G_F\|_F = \|G_F\|_2 = 2f_n f_{n-1}$$

Proof. The matrix G_F can be written as

$$G_F = \begin{bmatrix} \|F_n\|^2 & \|F_n\|^2 \\ \|F_n\|^2 & \|F_n\|^2 \end{bmatrix} \tag{7}$$

by using well known formula $\cos\theta = \frac{\langle u,v\rangle}{|u|.|v|}$.

$$\|G_F\|_F = \left(\sum_{i,j=1}^{2} g_{ij}^2\right)^{\frac{1}{2}} = \sqrt{4.\left(\|F_n\|^2\right)^2} = 2\|F_n\|^2$$

In addition, since $\|F_n\|^2 = f_n f_{n-1}$ we get $\|G_F\|_F = 2f_n f_{n-1}$.

The characteristic polynomial of matrix G_F can be written as

$$\Delta_{G_F}(\lambda) = \lambda^2 - 2\|F_n\|^2 \lambda$$

The roots of characteristic polynomial $\Delta_{G_F}(\lambda) = \lambda^2 - 2\|F_n\|^2 \lambda$ are

$$\lambda^2 - 2\|F_n\|^2 \lambda = 0$$

$$\lambda(\lambda - 2\|F_n\|^2) = 0$$

$$\lambda_1 = 0, \lambda_2 = 2\|F_n\|^2 .$$

Since G_F is positive semi definite, it is clear that $\|G_F\|_2 = 2\|F_n\|^2 = 2f_n f_{n-1}$. So it can easily be seen that

$$\|G_F\|_F = \|G_F\|_2 = 2f_n f_{n-1}.$$

If we take the Gram matrix as in (3), we have

$$\|G_L\|_F = \|G_L\|_2 = 2\|L_n\|^2 = 2(l_n l_{n-1} + 2)$$

If we take the Gram matrix as in(4),then analogously in previous section, we have

$$\|G_P\|_F = \|G_P\|_2 = 2\|P_n\|^2 = p_n p_{n-1}.$$

□

Theorem 7. *Let* 2×2 *Gram matrix* G_{FY} *be as in* (5,6) *and* Y_n *be any vector satisfying the inequality* $\|F_n\| \le \|Y_n\|$. *Then following inequality hold.*

$$0 \le |G_{FY}| \le \|F_n\|^2 . \|Y_n\|^2$$

Proof. The matrix G_{FY} can be written as

$$G_{FY} = \begin{bmatrix} \|F_n\|^2 & \|F_n\| \,.\, \|Y_n\| \,.\, \cos\theta \\ \|Y_n\| \,.\, \|F_n\| \,.\, \cos\theta & \|F_n\|^2 \end{bmatrix}$$

by using well known formula $\cos\theta = \frac{\langle u,v\rangle}{|u|.|v|}$, where θ denotes the angel between F_n and Y_n. Then

$$|G_{FY}| = \|F_n\|^2 \,.\, \|Y_n\|^2 - \|F_n\|^2 \,.\, \|Y_n\|^2 \cos^2\theta$$

$$|G_{FY}| = \|F_n\|^2 \,.\, \|Y_n\|^2 \,.(1 - \cos^2\theta)$$

$$|G_{FY}| = \|F_n\|^2 \,.\, \|Y_n\|^2 \,.(\sin^2\theta).$$

Owing to the fact that $0 \le \sin^2\theta \le 1$ we get

$$0 \le |G_{FY}| \le \|F_n\|^2 \,.\, \|Y_n\|^2$$

From Corollary 3 and Corollary 5, since $\|F_n\| \le \|L_n\|$ and $\|F_n\| \le \|P_n\|$ we can take vector Y_n as L_n and P_n.

□

Corollary 8. *Let* 2×2 *matrix* G_{FY} *be as in (5). Then, following inequality hold.*

$$0 \le |G_{FL}| \le f_n f_{n-1}\,(l_n l_{n-1} + 2)$$

Proof. The proof can easily be obtained by taking $\|Y_n\| = \|L_n\|$ in the proof of Theorem 7. If we take $Y_n = L_n$ we get

$$0 \le |G_{FL}| \le \|F_n\|^2 \,.\, \|L_n\|^2$$

since $\|F_n\|^2 = f_n f_{n-1}$ and $\|L_n\|^2 = l_n l_{n-1} + 2$ we get

$$0 \le |G_{FL}| \le f_n f_{n-1}\,(l_n l_{n-1} + 2)$$

□

Corollary 9. *Let* 2×2 *matrix* G_{FP} *be as in (6). Then, following inequality hold.*

$$0 \le |G_{FP}| \le \frac{f_n f_{n-1} . p_n p_{n-1}}{2}$$

Proof. The proof can easily be obtained by taking $\|Y_n\| = \|P_n\|$ in the proof of Theorem 7. If we take $Y_n = P_n$ we get

$$0 \le |G_{FP}| \le \|F_n\|^2 \,.\, \|P_n\|^2$$

since $\|F_n\|^2 = f_n f_{n-1}$ and $\|P_n\|^2 = \frac{p_n p_{n-1}}{2}$ we get

$$0 \le |G_{FP}| \le \frac{f_n f_{n-1} . p_n p_{n-1}}{2}$$

□

Theorem 10. *Let* 2×2 *Gram matrix* G_{FY} *be as in* (5,6) *and* Y_n *be any vector satisfying the inequality* $\|F_n\| \le \|Y_n\|$. *Then following inequality hold.*

$$\sqrt{\|Y_n\|^4 + \|F_n\|^4} \le \|G_{FY}\|_F \le \|F_n\|^2 + \|Y_n\|^2$$

Proof. The Euclidean norm of G_{FY} is defined as

$$\|G_{FY}\|_F = \left(\sum_{i,j=1}^{2} g_{ij}^2\right)^{\frac{1}{2}}$$

$$\|G_{FY}\|_F = \sqrt{\left(\|F_n\|^2\right)^2 + 2.\|F_n\|^2.\|Y_n\|^2.\cos^2\theta + \left(\|Y_n\|^2\right)^2}$$

$$\|G_{FY}\|_F = \sqrt{\|F_n\|^4 + \|Y_n\|^4 + 2.\|F_n\|^2.\|Y_n\|^2.\cos^2\theta}$$

$$\sqrt{\|F_n\|^4 + \|Y_n\|^4} \leq \|G_{FY}\|_F \leq \|F_n\|^2 + \|Y_n\|^2$$

□

Corollary 11. *Let* 2×2 *Gram matrix* G_{FL} *be as in (5). Then, following inequality hold.*

$$\sqrt{(f_n f_{n-1})^2 + (l_n l_{n-1} + 2)^2} \leq \|G_{FL}\|_F \leq f_n f_{n-1} + l_n l_{n-1} + 2$$

Proof. The proof can easily be obtained by taking $Y_n = L_n$ in the proof of Theorem 10. If we take $Y_n = L_n$, we get

$$\sqrt{\|F_n\|^4 + \|L_n\|^4} \leq \|G_{FL}\|_F \leq \|F_n\|^2 + \|L_n\|^2$$

since $\|F_n\|^2 = f_n f_{n-1}$ and $\|L_n\|^2 = l_n l_{n-1} + 2$ we get

$$\sqrt{(f_n f_{n-1})^2 + (l_n l_{n-1} + 2)^2} \leq \|G_{FL}\|_F \leq f_n f_{n-1} + l_n l_{n-1} + 2$$

□

Corollary 12. *Let* 2×2 *Gram matrix* G_{FP} *be as in (6). Then, following inequality hold.*

$$\sqrt{(f_n f_{n-1})^2 + \frac{(p_n p_{n-1})^2}{4}} \leq \|G_{FP}\|_F \leq f_n f_{n-1} + \frac{p_n p_{n-1}}{2}$$

Proof. The proof can easily be obtained by taking $Y_n = P_n$ in the proof of Theorem 10. If we take $Y_n = P_n$ we get

$$\sqrt{\|F_n\|^4 + \|P_n\|^4} \leq \|G_{FP}\|_F \leq \|F_n\|^2 + \|P_n\|^2$$

since $\|F_n\|^2 = f_n f_{n-1}$ and $\|P_n\|^2 = \frac{p_n p_{n-1}}{2}$ we get

$$\sqrt{(f_n f_{n-1})^2 + \frac{(p_n p_{n-1})^2}{4}} \leq \|G_{FP}\|_F \leq f_n f_{n-1} + \frac{p_n p_{n-1}}{2}$$

□

Theorem 13. *Let* 2×2 *Gram matrix* G_{FY} *be as in (5,6) and* Y_n *be any vector satisfying the inequality* $\|F_n\| \leq \|Y_n\|$. *Then following inequality hold.*

$$\|Y_n\|^2 \leq \|G_{FY}\|_2 \leq \|F_n\|^2 + \|Y_n\|^2$$

Proof. The characteristic polynomial of matrix G_{FY} is

$$\Delta_{G_{FY}}(\lambda) = \lambda^2 - (\|F_n\|^2 + \|Y_n\|^2).\lambda + \|F_n\|^2.\|Y_n\|^2.\sin^2\theta$$

The roots of this polynomial are eigenvalues of matrix G_{FY}. Since G_{FY} is positive semidefinite, maximum eigenvalue of matrix G_{FY} is equal to spectral norm of matrix G_{FY}. The eigenvalues of G_{FY} are:

$$\lambda_{1,2} = \frac{\|F_n\|^2 + \|Y_n\|^2 \pm \sqrt{\left(\|F_n\|^2 + \|Y_n\|^2\right)^2 - 4.\|F_n\|^2.\|Y_n\|^2.\sin^2\theta}}{2}$$

Let λ_1 be maximum eigenvalue of matrix G_{FY}. Then

$$\frac{\|F_n\|^2 + \|Y_n\|^2 + \sqrt{\left(\|F_n\|^2 - \|Y_n\|^2\right)^2}}{2} \leq \lambda_1 \leq \frac{\|F_n\|^2 + \|Y_n\|^2 + \sqrt{\left(\|F_n\|^2 + \|Y_n\|^2\right)^2}}{2}$$

$$\frac{\|F_n\|^2 + \|Y_n\|^2 + \left|\|F_n\|^2 - \|Y_n\|^2\right|}{2} \le \lambda_1 \le \|F_n\|^2 + \|Y_n\|^2$$

from corollary 3 and 5, since $\|F_n\| \le \|Y_n\|$, we get

$$\|Y_n\|^2 \le \lambda_1 \le \|F_n\|^2 + \|Y_n\|^2$$

$$\|Y_n\|^2 \le \|G_{FY}\|_2 \le \|F_n\|^2 + \|Y_n\|^2$$

□

Corollary 14. *Let* 2×2 *Gram matrix* G_{FL} *be as in (5). Then, following inequality hold:*

$$l_n l_{n-1} + 2 \le \|G_{FL}\|_2 \le f_n f_{n-1} + l_n l_{n-1} + 2$$

Proof. The proof can easily be obtained by taking $Y_n = L_n$ in the proof of Theorem 13. If we take $Y_n = L_n$ we get

$$\|L_n\|^2 \le \|G_{FL}\|_2 \le \|F_n\|^2 + \|L_n\|^2$$

since $\|F_n\|^2 = f_n f_{n-1}$ and $\|L_n\|^2 = l_n l_{n-1} + 2$ we get

$$l_n l_{n-1} + 2 \le \|G_{FL}\|_2 \le f_n f_{n-1} + l_n l_{n-1} + 2$$

□

Corollary 15. *Let* 2×2 *Gram matrix* G_{FP} *be as in (6). Then, following inequality hold:*

$$\frac{p_n p_{n-1}}{2} \le \|G_{FP}\|_2 \le f_n f_{n-1} + \frac{p_n p_{n-1}}{2}$$

Proof. The proof can easily be obtained by taking $Y_n = P_n$ in the proof of Theorem 13. If we take $Y_n = P_n$ we get

$$\|P_n\|^2 \le \|G_{FP}\|_2 \le \|F_n\|^2 + \|P_n\|^2$$

since $\|F_n\|^2 = f_n f_{n-1}$ and $\|P_n\|^2 = p_n p_{n-1}/2$ we get

$$\frac{p_n p_{n-1}}{2} \le \|G_{FP}\|_2 \le f_n f_{n-1} + \frac{p_n p_{n-1}}{2}$$

□

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