

## Research Article

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# A class of cyclic $(v; k_1, k_2, k_3; \lambda)$ difference families with $v \equiv 3 \pmod{4}$ a prime

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**Abstract:** We construct several new cyclic  $(v; k_1, k_2, k_3; \lambda)$  difference families, with  $v \equiv 3 \pmod{4}$  a prime and  $\lambda = k_1 + k_2 + k_3 - (3v - 1)/4$ . Such families can be used in conjunction with the well-known Paley-Todd difference sets to construct skew-Hadamard matrices of order  $4v$ . Our main result is that we have constructed for the first time the examples of skew Hadamard matrices of orders  $4 \cdot 239 = 956$  and  $4 \cdot 331 = 1324$ .

## 1 Introduction

Let  $\mathbf{Z}_v = \{0, 1, \dots, v - 1\}$  be the ring of integers modulo an integer  $v > 1$ . Let  $k_1, \dots, k_t$  be nonnegative integers,  $\lambda$  an integer such that

$$\lambda(v - 1) = \sum_{i=1}^t k_i(k_i - 1), \quad (1)$$

and let  $X_1, \dots, X_t$  be subsets of  $\mathbf{Z}_v$  such that  $|X_i| = k_i$ ,  $i \in \{1, 2, \dots, t\}$ .

**Definition 1.** We say that  $X_1, \dots, X_t$  are supplementary difference sets (SDS) or a difference family with parameters  $(v; k_1, \dots, k_t; \lambda)$ , if for every  $c \in \mathbf{Z}_v \setminus \{0\}$  there are exactly  $\lambda$  ordered triples  $(a, b, i) \in \mathbf{Z}_v \times \mathbf{Z}_v \times \{1, 2, \dots, t\}$  such that  $\{a, b\} \subseteq X_i$  and  $a - b \equiv c \pmod{v}$ .

In the context of an SDS  $(v; k_1, \dots, k_t; \lambda)$ , it is convenient to introduce an additional parameter, order  $n$ , defined by

$$n = k_1 + \dots + k_t - \lambda. \quad (2)$$

We refer to the sets  $X_i$  as the *base blocks* of the SDS. In the case  $t = 1$  the SDSs are called *cyclic difference sets*, see [1], [11], [14]. In some cases we have to replace some of the base blocks with their complements in  $\mathbf{Z}_v$ . This gives a new SDS with different parameter  $\lambda$ . However, the parameter  $n$  remains the same. Thus we can easily compute the new value of  $\lambda$ .

There are exactly twelve integers  $n < 500$  for which the existence question for Hadamard matrices of order  $4n$  is still undecided (see [6]):

$$167, 179, 223, 283, 311, 347, 359, 419, 443, 479, 487, 491.$$

All of these integers are primes congruent to 3 modulo 4. This motivates the search for special methods to construct Hadamard matrices for these orders.

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One of the powerful methods is based on the Goethals-Seidel array (GS-array), see e.g. [13]:

$$\begin{bmatrix} Z_0 & Z_1R & Z_2R & Z_3R \\ -Z_1R & Z_0 & -Z_3^T R & Z_2^T R \\ -Z_2R & Z_3^T R & Z_0 & -Z_1^T R \\ -Z_3R & -Z_2^T R & Z_1^T R & Z_0 \end{bmatrix},$$

where the  $Z_i$  are suitable  $\{1, -1\}$ -circulant matrices and  $R$  is the square matrix with ones on the back-diagonal and zeros elsewhere.

If  $X \subseteq \mathbf{Z}_v$ , then the *associated sequence* of  $X$  is the  $\{\pm 1\}$ -sequence  $\xi_0, \xi_1, \dots, \xi_{v-1}$  with  $\xi_i = -1$  if  $i \in X$  and  $\xi_i = 1$  otherwise.

Usually, to construct a Hadamard matrix of order  $4v$  via the GS-array, we require an SDS, say  $X_0, X_1, X_2, X_3$ , with parameters  $(v; k_0, k_1, k_2, k_3; \lambda_0)$  and order  $n_0 = v$ , i.e., such that  $\lambda_0 = k_0 + k_1 + k_2 + k_3 - v$ . Given such SDS, let  $A_0, A_1, A_2, A_3$  be the sequences associated to the base blocks  $X_0, X_1, X_2, X_3$ , and let  $Z_0, Z_1, Z_2, Z_3$  be the circulant matrices with the first rows  $A_0, A_1, A_2, A_3$ , respectively. By plugging these circulants into the GS-array, we obtain a Hadamard matrix. This will be a skew-Hadamard matrix if the block  $X_0$  is of skew type. This means that  $X_0$  has the property that  $i \in X_0$  if and only if  $-i \notin X_0$ . In particular,  $v$  must be odd,  $0 \notin X_0$  and  $|X_0| = (v-1)/2$ .

In this paper we are interested in the special case of this construction where  $v \equiv 3 \pmod{4}$  is a prime and  $k_0 = (v-1)/2$ . Moreover, except for the last section, we shall assume that we have chosen  $X_0$  to be the Paley-Todd difference set, see [15, Theorem 27.4, p. 234]. Then the base blocks  $X_1, X_2, X_3$  form an SDS with the parameter set  $(v; k_1, k_2, k_3; \lambda)$  and order  $n = (3v-1)/4$ . Thus we have

$$\lambda = k_1 + k_2 + k_3 - \frac{3v-1}{4}. \quad (3)$$

Without any loss of generality, we may assume that

$$v/2 > k_1 \geq k_2 \geq k_3 \geq 0. \quad (4)$$

We denote by  $\mathcal{P}$  the collection of parameter sets  $(v; k_1, k_2, k_3; \lambda)$ , with  $v \equiv 3 \pmod{4}$  a prime, and satisfying the conditions (1), (3) and (4).

**Conjecture 1.** *For each parameter set in  $\mathcal{P}$ , there exists at least one SDS.*

We claim that the family  $\mathcal{P}$  is infinite. This follows from the following stronger result.

**Proposition 1.** *For each prime number  $v \equiv 3 \pmod{4}$ , there exist nonnegative integers  $k_1, k_2, k_3$  and  $\lambda$  such that  $(v; k_1, k_2, k_3; \lambda) \in \mathcal{P}$ .*

*Proof.* Recall the famous result of Gauss that every positive integer is a sum of at most three triangular numbers. This fact is equivalent to the assertion that every positive integer congruent to 3 modulo 8 is a sum of three odd squares (see [9]).

As  $(3; 1, 1, 0; 0) \in \mathcal{P}$ , we may assume that  $v > 3$ . Since  $v$  is odd, we have  $4v-1 \equiv 3 \pmod{8}$ . Hence, there exist positive odd integers  $s_1, s_2, s_3$  such that  $\sum s_i^2 = 4v-1$ . Note that  $s_i < \sqrt{4v-1} < v$ , and so the integers  $k_i = (v-s_i)/2$  are positive and less than  $v/2$ . By permuting the  $k_i$ , we may assume that  $k_1 \geq k_2 \geq k_3$ . It is now easy to verify that  $(v; k_1, k_2, k_3; \lambda) \in \mathcal{P}$ , where  $\lambda = \sum k_i - (3v-1)/4$ .  $\square$

In section 2 (and in the appendix) we list all parameter sets in  $\mathcal{P}$  with  $v \leq 131$  and for each of them we provide at least one SDS whenever such SDS is known. Although there remain several undecided cases, these computational results suggest that Conjecture 1 is true.

In section 3 we give three non-equivalent SDSs with the parameter set  $(239; 119, 112, 106; 158)$  and  $n = 179$ . The importance of these three SDSs is that each of them, in conjunction with the Paley-Todd difference set (the set of nonzero squares in the finite field  $\mathbf{Z}_{239}$ ), gives a skew-Hadamard matrix of size  $4 \cdot 239$ .

Finally, for the parameter set  $(331; 165, 155, 155, 155; 299)$ , which is not in  $\mathcal{P}$ , we construct in section 4 six non-equivalent SDSs in which the block of size 165 is of skew type. Thus we obtain six skew-Hadamard matrices of order  $4 \cdot 331 = 1324$ .

To the best of our knowledge, skew-Hadamard matrices of the two orders mentioned above were not known previously. See [12, Table 1]) for the list of 98 odd positive integers  $m < 500$  for which no skew-Hadamard matrix of order  $4m$  was known at that time. Subsequently, skew-Hadamard matrices of order  $4m$  were constructed for  $m = 109, 145, 247$  [4] and  $m = 213$  [6]. To update the above mentioned list, one should delete from it the six integers 109, 145, 213, 239, 247, 331. The first 22 entries (those with  $4m < 1000$ ) in the updated list are:

$$69, 89, 101, 107, 119, 149, 153, 167, 177, 179, 191, 193, \\ 201, 205, 209, 223, 225, 229, 233, 235, 245, 249.$$

Only five of these values, namely  $v = 107, 167, 179, 191, 223$ , admit the parameter sets in  $\mathcal{P}$ .

## 2 Difference families with parameters in $\mathcal{P}$ and $v \leq 131$

In this section we provide evidence for Conjecture 1. In the appendix we list all parameter sets  $(v; k_1, k_2, k_3; \lambda)$  with  $v \leq 131$  which belong to the family  $\mathcal{P}$ . For each of them we either give a reference to papers where the SDSs have been constructed or give explicit examples of SDSs that we have constructed. If no SDSs are known, we indicate by a question mark that the existence question remains undecided.

In some cases we use the known D-optimal SDSs to construct the desired difference family with three base blocks. This works only when  $k_1 = (v - 1)/2$ . Assume that there exists an SDS  $(X_2, X_3)$  with the parameter set  $(v; k_2, k_3; \lambda')$ , where  $\lambda' = k_2 + k_3 - (v - 1)/2$ . If  $X_1$  is the Paley-Todd difference set in  $\mathbf{Z}_v$ , then  $(X_1, X_2, X_3)$  is an SDS with parameters  $(v; k_1, k_2, k_3; \lambda) \in \mathcal{P}$ . In that case we say that the SDS  $(X_1, X_2, X_3)$  is constructed from the D-optimal SDS  $(X_2, X_3)$ . A list of known D-optimal SDSs  $(v; k_2, k_3; \lambda')$  with  $v < 100$  and examples of the corresponding DO-designs are given in [8].

The multiplicative group  $\mathbf{Z}_v^*$  of the prime field  $\mathbf{Z}_v$ , acts on  $\mathbf{Z}_v$  by multiplication modulo  $v$ . In many cases we construct the base blocks  $X_1, X_2, X_3$  of an SDS as the union of orbits of a nontrivial subgroup  $H$  of  $\mathbf{Z}_v^*$ . Let  $H = \langle h \rangle$ , i.e.,  $h$  is a generator of  $H$ . The orbit containing the integer  $j \in \mathbf{Z}_v$  is written as  $H \cdot j$ . In particular, when  $j = 0$  we obtain the trivial orbit  $H \cdot 0 = \{0\}$ . In all of our computations, the order of  $H$  is a small odd prime divisor  $q$  of  $v - 1$ . So, an  $H$ -orbit has size 1 or  $q$ . If it is of size 1, say  $\{x\}$ , then  $(h - 1)x = 0$ . As  $v$  is a prime, it follows that  $x = 0$ . Hence, all nontrivial orbits of  $H$  are of size  $q$ .

We write the blocks  $X_1, X_2, X_3$  as

$$X_1 = \bigcup_{j \in J} H \cdot j, \quad X_2 = \bigcup_{k \in K} H \cdot k, \quad X_3 = \bigcup_{l \in L} H \cdot l. \tag{5}$$

Here, the set  $J$  is a set of representatives of the  $H$ -orbits comprising  $X_1$ , etc. Thus, in order to specify the blocks  $X_1, X_2, X_3$ , it suffices to specify the subgroup  $H$  and the corresponding sets of representatives  $J, K, L \subseteq \mathbf{Z}_v$ .

In order to be able to use this method it is necessary that  $H$  be chosen so that  $q$  divides either  $k_i$  or  $v - k_i$  for each  $i$ . This explains why we were able to find an SDS for  $v = 239$  but not for some smaller values of  $v$ , say for  $v = 107$ .

Table 1 summarizes what we presently know about the existence of SDSs with parameters in  $\mathcal{P}$  for  $v \leq 131$ . The entry “yes” means that the SDS exists, while “?” means that the existence question remains undecided.

## 3 Three skew-Hadamard matrices of order $4 \cdot 239$

According to the list [12, Table 1] no skew-Hadamard matrix of order  $4 \cdot 239$  is known. We construct such a matrix by using the method presented in the Introduction.

Table 1

$\nu$	$k_1$	$k_2$	$k_3$	$\lambda$	yes/?	$\nu$	$k_1$	$k_2$	$k_3$	$\lambda$	yes/?
3	1	1	0	0	yes	71	34	32	28	41	?
7	3	3	1	2	yes	71	31	31	30	39	yes
7	2	2	2	1	yes	79	39	37	31	48	yes
11	4	4	3	3	yes	79	38	35	32	46	?
19	9	7	6	8	yes	79	37	34	33	45	yes
19	7	7	7	7	yes	83	39	37	34	48	?
23	11	10	7	11	yes	83	37	37	35	47	?
31	15	15	10	17	yes	103	51	48	42	64	yes
31	13	12	12	14	yes	103	51	46	43	63	yes
43	21	21	15	25	yes	103	49	49	42	63	yes
43	21	18	16	23	yes	103	46	46	45	60	yes
43	20	17	17	22	yes	107	49	48	46	63	?
43	19	19	16	22	yes	127	61	58	54	78	?
47	22	22	17	26	yes	127	60	57	56	77	?
47	21	19	19	24	yes	127	57	57	57	76	yes
59	29	28	22	35	yes	131	65	61	55	83	yes
67	31	28	28	37	yes	131	64	58	57	81	?
67	30	30	27	37	yes	131	61	61	56	80	yes

For this construction we need a difference family with three base blocks  $X_1, X_2, X_3$  having the parameters  $(239; 119, 112, 106; 158)$ . We have constructed three such families by using the subgroup  $H = \{1, 10, 24, 44, 98, 100, 201\}$  of  $\mathbf{Z}_{239}^*$ . In each case, the base blocks  $X_1, X_2, X_3$  are unions of orbits of  $H$  as in (5). These three families are pairwise non-equivalent. Their sets  $J, K, L$  are:

$$\begin{aligned} J_1 &= \{1, 3, 5, 6, 15, 17, 19, 28, 34, 38, 39, 57, 58, 63, 85, 95, 107\}, \\ K_1 &= \{1, 3, 4, 5, 15, 16, 17, 18, 19, 21, 23, 29, 35, 45, 58, 63\}, \\ L_1 &= \{0, 1, 4, 6, 7, 8, 13, 16, 18, 34, 35, 45, 47, 58, 63, 95\}; \end{aligned}$$

$$\begin{aligned} J_2 &= \{1, 3, 9, 13, 14, 15, 16, 18, 23, 28, 29, 38, 42, 45, 58, 85, 107\}, \\ K_2 &= \{4, 6, 7, 8, 9, 13, 14, 17, 28, 39, 45, 47, 57, 58, 95, 107\}, \\ L_2 &= \{0, 1, 4, 5, 6, 9, 13, 18, 21, 34, 35, 39, 42, 57, 85, 95\}; \end{aligned}$$

$$\begin{aligned} J_3 &= \{1, 2, 3, 4, 5, 6, 7, 8, 14, 16, 18, 19, 34, 39, 57, 58, 95\}, \\ K_3 &= \{5, 9, 14, 15, 17, 18, 21, 23, 29, 34, 35, 39, 45, 47, 57, 58\}, \\ L_3 &= \{0, 1, 6, 7, 8, 13, 15, 16, 17, 23, 28, 35, 57, 63, 85, 107\}, \end{aligned}$$

respectively.

## 4 Six skew-Hadamard matrices of order $4 \cdot 331$

According to the list [12, Table 1] no skew-Hadamard matrix of order  $4 \cdot 331$  is known. Although there are six parameters sets in  $\mathcal{P}$  with  $\nu = 331$ , we did not succeed to construct an SDS for any of them. However

we were able to construct six difference families with base blocks  $X_0, X_1, X_2, X_3$  having the parameter set  $(331; 165, 155, 155, 155; 299)$ . In all six cases  $X_0$  is of skew type (but not a difference set). Hence, by plugging the corresponding four circulant matrices into the GS-array, we obtain a skew-Hadamard matrix of order  $4 \cdot 331 = 1324$ . We have verified that the six difference families constructed above are pairwise non-equivalent.

We have constructed these 6 families by using the subgroup

$$H = \{1, 74, 80, 85, 111, 120, 167, 180, 270, 274, 293\}$$

of  $\mathbf{Z}_{331}^*$  of order 11. All four base blocks are unions of orbits of  $H$  as in (5).

For the first two families we use the sets of orbit representatives  $M, J, K, L$  and  $M, J', K, L$ , where

$$\begin{aligned} M &= \{2, 4, 8, 10, 14, 16, 20, 28, 38, 31, 32, 37, 56, 62, 64\}, \\ J &= \{0, 1, 2, 5, 10, 13, 19, 22, 28, 31, 37, 53, 56, 64, 101\}, \\ J' &= \{0, 7, 8, 13, 14, 19, 28, 31, 37, 38, 49, 53, 62, 73, 76\}, \\ K &= \{0, 1, 5, 7, 8, 10, 11, 16, 31, 38, 41, 56, 62, 64, 73\}, \\ L &= \{0, 1, 7, 8, 10, 13, 14, 19, 22, 32, 37, 44, 62, 73, 76\}. \end{aligned}$$

For the remaining four families we use the sets  $M, J, K, L$ ;  $M, J, K, L'$ ;  $M, J, K', L$ , and  $M, J, K', L'$ , where

$$\begin{aligned} M &= \{4, 13, 14, 16, 22, 32, 37, 38, 41, 49, 53, 56, 62, 64, 76\}, \\ J &= \{0, 2, 4, 11, 14, 19, 20, 32, 37, 41, 56, 73, 76, 88, 101\}, \\ K &= \{0, 2, 10, 11, 20, 22, 31, 32, 37, 38, 53, 62, 64, 76, 101\}, \\ K' &= \{0, 1, 2, 4, 5, 7, 8, 13, 14, 19, 28, 41, 49, 76, 88\}, \\ L &= \{0, 4, 8, 11, 13, 16, 19, 20, 22, 31, 37, 38, 49, 64, 101\}, \\ L' &= \{0, 1, 4, 7, 8, 10, 11, 13, 14, 19, 22, 31, 37, 41, 44\}. \end{aligned}$$

(In the last four cases the block  $X_1$  is symmetric.)

## 5 Conclusions

One of the standard methods of construction of Hadamard matrices of order  $4v$  uses the Goethals–Seidel array. This method requires supplementary difference sets (SDSs) with 4 basic blocks and special parameter sets, namely  $(v; k_0, k_1, k_2, k_3; \lambda_0)$  with order  $n_0 = v$ .

In this paper we investigate a special subclass of such SDSs which are obtained from simpler SDSs having only 3 base blocks. Their parameter sets have the form  $(v; k_1, k_2, k_3; \lambda)$  with  $v \equiv 3 \pmod{4}$  a prime, and have the order  $n = (3v - 1)/4$ . We assume that they are normalized in the sense that  $v/2 > k_1 \geq k_2 \geq k_3$ . We denote by  $\mathcal{P}$  this infinite family of parameter sets (see Proposition 1). If we add the Paley-Todd difference set in  $\mathbf{Z}_v$ , we obtain an SDS with four base blocks, whose order  $n_0$  is equal to  $v$ . Consequently, the Hadamard matrix constructed by using this new SDS is a skew-Hadamard matrix.

We have conjectured that, for each parameter set in  $\mathcal{P}$ , there exists at least one SDS.

To provide evidence to this conjecture, we performed many computations to construct such SDSs. Table 1 gives a summary of the previously known and the new SDSs belonging to  $\mathcal{P}$  in the range  $v \leq 131$ . Among the 36 parameter sets in this range, the SDSs are not known in 8 cases.

Moreover, we have constructed three non-equivalent SDSs for the parameter set  $(239; 119, 112, 106; 158)$ , which belongs to  $\mathcal{P}$  but has  $v = 239 > 131$ . Each of them gives a skew-Hadamard matrix of order 956. These are the first examples of such matrices.

As an aside, we also record six SDSs with four base blocks which give skew-Hadamard matrices of order 1324. Again, these are the first examples of such matrices.

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## A Appendix: SDSs for parameter sets in $\mathcal{P}$ with $v \leq 131$

$(3; 1, 1, 0; 0), \quad n = 2$   
 $\{0\}, \{0\}, \emptyset$

$(7; 3, 3, 1; 2), \quad n = 5$   
 $\{0, 1, 3\}, \{0, 1, 3\}, \{0\}$

$(7; 2, 2, 2; 1), \quad n = 5$   
 $\{0, 1\}, \{0, 2\}, \{0, 3\}$

$(11; 4, 4, 3; 3), \quad n = 8$   
 $\{0, 1, 3, 5\}, \{0, 1, 4, 5\}, \{0, 2, 5\}$

$(19; 9, 7, 6; 8), \quad n = 14$   
 $\{0, 1, 2, 3, 5, 7, 12, 13, 16\}, \{0, 1, 2, 4, 5, 10, 13\}, \{0, 1, 4, 6, 8, 13\}$

$(19; 7, 7, 7; 7), \quad n = 14$   
 $\{0, 1, 3, 4, 7, 8, 13\}, \{0, 1, 2, 5, 8, 10, 13\}, \{0, 1, 2, 5, 7, 9, 11\}$

$(23; 11, 10, 7; 11), \quad n = 17$   
 $\{0, 1, 2, 3, 5, 7, 8, 11, 12, 15, 17\}, \{0, 1, 2, 3, 6, 8, 10, 11, 14, 18\}, \{0, 1, 2, 5, 7, 11, 14\}$

$(31; 15, 15, 10; 17), \quad n = 23$   
 $\{0, 1, 3, 4, 6, 7, 8, 9, 13, 15, 17, 18, 19, 24, 27\}, \{0, 1, 2, 3, 4, 5, 8, 10, 11, 14, 16, 17, 21, 23, 25\},$   
 $\{0, 1, 4, 5, 6, 9, 11, 16, 19, 23\}$

$(31; 13, 12, 12; 14), \quad n = 23$   
 $\{0, 1, 2, 3, 5, 6, 8, 13, 15, 16, 19, 22, 25\}, \{0, 1, 2, 5, 7, 9, 13, 14, 15, 17, 22, 27\},$   
 $\{0, 1, 3, 4, 5, 9, 10, 11, 13, 16, 20, 24\}$

$(43; 21, 21, 15; 25), \quad n = 32$   
 $\{0, 1, 2, 4, 8, 9, 10, 11, 12, 14, 15, 16, 19, 21, 24, 27, 28, 30, 32, 33, 37\},$   
 $\{0, 1, 3, 4, 5, 7, 8, 9, 12, 13, 16, 17, 19, 22, 24, 26, 28, 33, 34, 36, 37\},$   
 $\{0, 1, 2, 4, 6, 7, 10, 12, 13, 15, 20, 26, 27, 28, 33\}$

$(43; 21, 18, 16; 23), \quad n = 32$   
 $\{0, 1, 2, 4, 8, 9, 10, 11, 12, 14, 15, 16, 19, 21, 24, 27, 28, 30, 32, 33, 37\},$   
 $\{0, 1, 2, 5, 6, 7, 10, 12, 14, 16, 17, 20, 21, 23, 24, 30, 31, 32\},$   
 $\{0, 1, 2, 5, 6, 8, 10, 14, 17, 20, 22, 23, 25, 27, 33, 36\}$

$(43; 20, 17, 17; 22), \quad n = 32$   
 $\{0, 2, 4, 5, 6, 8, 9, 11, 12, 14, 17, 19, 21, 22, 25, 26, 31, 32, 33, 35\},$   
 $\{0, 1, 2, 4, 7, 9, 11, 12, 15, 16, 19, 20, 24, 25, 26, 32, 34\},$

$\{0, 1, 2, 3, 4, 7, 8, 9, 14, 17, 19, 20, 23, 25, 29, 32, 36\}$

$(43; 19, 19, 16; 22), \quad n = 32$

$\{0, 1, 2, 3, 4, 6, 8, 11, 12, 14, 16, 19, 24, 25, 29, 31, 34, 35, 40\},$

$\{0, 1, 2, 3, 4, 5, 7, 10, 11, 13, 15, 18, 19, 20, 24, 25, 27, 31, 32\},$

$\{0, 1, 2, 3, 7, 9, 10, 12, 15, 28, 29, 32, 36\}$

$(47; 22, 22, 17; 26), \quad n = 35$

$\{0, 1, 2, 3, 4, 6, 7, 8, 9, 11, 16, 17, 18, 19, 22, 25, 27, 31, 36, 37, 39, 43\},$

$\{0, 1, 3, 4, 5, 6, 7, 8, 10, 14, 15, 18, 19, 22, 25, 27, 28, 30, 31, 34, 38, 40\},$

$\{0, 1, 2, 5, 6, 7, 11, 13, 15, 18, 22, 24, 29, 32, 34, 39, 40\}$

Two other solutions were constructed in [3, Proposition 2.1].

$(47; 21, 19, 19; 24), \quad n = 35$

$\{0, 1, 2, 3, 4, 6, 7, 10, 11, 12, 14, 15, 17, 21, 23, 24, 30, 31, 36, 39, 41\},$

$\{0, 1, 3, 4, 6, 7, 12, 14, 16, 17, 21, 22, 26, 28, 29, 33, 35, 37, 41\},$

$\{0, 1, 2, 3, 4, 5, 7, 9, 15, 18, 19, 20, 23, 25, 26, 30, 34, 39, 42\}$

Two other solutions were constructed in [3, Proposition 2.2].

$(59; 29, 28, 22; 35), \quad n = 44$

Three non-equivalent SDSs with these parameters can be constructed from the D-optimal SDS with parameters  $(59; 28, 22; 21)$  found in [10] and the two additional D-optimal SDSs found in [8]. We have constructed very recently yet another D-optimal SDS with the same parameters, not equivalent to any of the three SDSs mentioned above. Its two base blocks are

$X_2 = \{0, 2, 3, 4, 5, 6, 7, 8, 9, 13, 15, 17, 19, 20, 23, 25, 28, 29, 31, 32, 33, 36, 40, 43, 44, 45, 50, 54\},$

$X_3 = \{0, 1, 2, 4, 7, 8, 9, 14, 15, 17, 18, 21, 24, 26, 28, 33, 34, 36, 39, 44, 45, 54\}.$

In the following two cases we use the subgroup  $H = \{1, 29, 37\}$  of  $\mathbf{Z}_{67}^*$ . The base blocks  $X_1, X_2, X_3$  of the SDS are given by the formulas (5) with  $J, K, L$  given below.

$(67; 31, 28, 28; 37), \quad n = 50$

$J = \{0, 1, 4, 5, 6, 8, 10, 16, 18, 23, 27\},$

$K = \{0, 1, 2, 3, 4, 5, 12, 15, 32, 34\},$

$L = \{0, 2, 3, 4, 5, 8, 10, 18, 23, 30\}.$

$(67; 30, 30, 27; 37), \quad n = 50$

$J = \{8, 12, 15, 16, 17, 25, 27, 32, 34, 41\},$

$K = \{1, 3, 9, 12, 15, 23, 25, 32, 34, 36\},$

$L = \{1, 4, 6, 8, 10, 18, 23, 25, 34\}.$

$(71; 34, 32, 28; 41), \quad n = 53 \quad ?$

$(71; 31, 31, 30; 39), \quad n = 53$

$H = \{1, 5, 25, 54, 57\}$  is the subgroup of order 5 of the group  $\mathbf{Z}_{71}^*$ . The base blocks  $X_1, X_2, X_3$  of the SDS are

given by the formulas (5) with

$$J = \{0, 1, 6, 7, 11, 14, 27\},$$

$$K = \{0, 1, 6, 9, 11, 13, 27\},$$

$$L = \{1, 2, 6, 13, 14, 42\}.$$

(79; 39, 37, 31; 48),  $n = 59$

Use the D-optimal SDS with parameters (79; 37, 31; 29) constructed in [2].

(79; 38, 35, 32; 46),  $n = 59$  ?

(79; 37, 34, 33; 45),  $n = 59$

$H = \{1, 23, 55\}$  is the subgroup of order 3 of the group  $\mathbf{Z}_{79}^*$ . The base blocks  $X_1, X_2, X_3$  of the SDS are given by the formulas (5) with

$$J = \{0, 2, 5, 6, 9, 18, 20, 22, 27, 30, 33, 34, 40\},$$

$$K = \{0, 1, 6, 15, 18, 22, 24, 30, 33, 34, 41, 47\},$$

$$L = \{1, 2, 3, 4, 5, 10, 11, 20, 27, 33, 41\}.$$

(83; 39, 37, 34; 48),  $n = 62$  ?

(83; 37, 37, 35; 47),  $n = 62$  ?

(103; 51, 48, 42; 64),  $n = 77$

Use the D-optimal SDSs with parameters (103; 48, 42; 39) constructed in [7].

(103; 51, 46, 43; 63),  $n = 77$

Use the D-optimal SDSs with parameters (103; 46, 43; 38) constructed in [7].

In the following two cases we use the subgroup  $H = \{1, 46, 56\}$  of  $\mathbf{Z}_{103}^*$ . The base blocks  $X_1, X_2, X_3$  of the SDS are given by the formulas (5) with  $J, K, L$  given below.

(103; 49, 49, 42; 63),  $n = 77$

$$J = \{0, 1, 3, 4, 6, 7, 10, 12, 15, 17, 20, 23, 29, 38, 40, 42, 44\},$$

$$K = \{0, 2, 4, 5, 6, 7, 10, 12, 14, 17, 19, 21, 31, 40, 49, 53, 55\},$$

$$L = \{1, 3, 7, 8, 12, 19, 20, 21, 38, 44, 47, 49, 53, 60\}.$$

(103; 46, 46, 45; 60),  $n = 77$

$$J = \{0, 3, 5, 6, 7, 10, 11, 14, 17, 22, 29, 31, 33, 40, 49, 55\},$$

$$K = \{0, 1, 2, 7, 10, 14, 15, 17, 19, 20, 29, 33, 44, 47, 49, 55\},$$

$$L = \{1, 4, 5, 8, 10, 12, 15, 20, 22, 29, 33, 49, 51, 53, 62\}.$$

(107; 49, 48, 46; 63),  $n = 80$  ?

(127; 61, 58, 54; 78),  $n = 95$  ?

(127; 60, 57, 56; 77),  $n = 95$  ?



$(127; 57, 57, 57; 76), \quad n = 95$

An SDS with these parameters has been constructed in [5].

For  $v = 131$  there are three parameter sets in  $\mathcal{P}$  as listed below. We have constructed an SDS for the first and the third parameter set by using the subgroup  $H = \{1, 53, 58, 61, 89\}$  of  $\mathbf{Z}_{131}^*$ . The base blocks  $X_1, X_2, X_3$  are given by the formulas (5) with  $J, K, L$  specified below.

$(131; 65, 61, 55; 83), \quad n = 98$

$$J = \{2, 3, 4, 7, 8, 9, 11, 18, 19, 33, 36, 38, 51\},$$

$$K = \{0, 1, 2, 6, 7, 8, 9, 14, 22, 27, 36, 42, 44\},$$

$$L = \{1, 2, 14, 17, 21, 22, 29, 38, 42, 51, 79\}.$$

$(131; 64, 58, 57; 81), \quad n = 98?$

$(131; 61, 61, 56; 80), \quad n = 98$

$$J = \{0, 2, 8, 11, 12, 14, 18, 19, 23, 27, 29, 33, 79\},$$

$$K = \{0, 2, 3, 4, 8, 9, 14, 17, 27, 33, 42, 44, 79\},$$

$$L = \{0, 2, 4, 7, 8, 14, 17, 19, 21, 27, 29, 79\}.$$

## References

- [1] L. D. Baumert, Cyclic difference sets. Lecture Notes in Mathematics, Vol. 182 Springer-Verlag, Berlin-New York 1971.
- [2] D. Ž. Đoković, Some new D-optimal designs. Australas. J Combin. 15 (1997), 221–231.
- [3] D. Ž. Đoković, Skew-Hadamard matrices of orders 188 and 388 exist. International Mathematical Forum, 22 (2008), 1063–1068
- [4] D. Ž. Đoković, Skew-Hadamard matrices of orders 436, 580, and 988 exist, J. Combin. Designs, 16 (2008), 493–498.
- [5] D. Ž. Đoković, Supplementary difference sets with symmetry for Hadamard matrices. Operators and Matrices, 3 (2009), 557–569.
- [6] D. Ž. Đoković, O. Golubitsky and I. S. Kotsireas, Some new orders of Hadamard and skew-Hadamard matrices. J. Combin. Designs, 22 (2014), 270–277.
- [7] D. Ž. Đoković and I. S. Kotsireas, New results on D-optimal matrices. J. Combin. Designs, 20 (2012), 278–289.
- [8] D. Ž. Đoković and I. S. Kotsireas, D-optimal matrices of orders 118, 138, 150, 154 and 174. In: C. J. Colbourn (ed.) Algebraic Design Theory and Hadamard Matrices, pp. 71–82, ADTHM, Lethbridge, Alberta, Canada, July 2014. Springer Proceedings in Mathematics & Statistics, vol. 133. Springer 2015.
- [9] W. Duke, Some old problems and new results about quadratic forms, Not. Amer. Math. Soc. 44 (1997), 190–196.
- [10] R. J. Fletcher, C. Koukouvinos and J. Seberry, New skew-Hadamard matrices of order 4.59 and new D-optimal designs of order 2.59, Discrete Math. 286 (2004), 251–253.
- [11] D. Jungnickel, A. Pott, K. W. Smith, Difference sets, in Handbook of combinatorial designs. Edited by C. J. Colbourn and J. H. Dinitz. Second edition. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007.
- [12] C. Koukouvinos, S. Stylianou, On skew-Hadamard matrices, Discrete Math. 308 (2008), 2723–2731.
- [13] J. Seberry, M. Yamada, Hadamard matrices, sequences, and block designs. In Contemporary design theory, 431–560, Wiley-Intersci. Ser. Discrete Math. Optim., Wiley, New York, 1992,
- [14] D. R. Stinson, Combinatorial designs. Constructions and analysis. Springer-Verlag, New York, 2004.
- [15] J. H. van Lint, R. M. Wilson, A course in combinatorics. Cambridge University Press, Cambridge, 1992.