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On numerical range of $\mathfrak{sp}(2n, \mathbb{C})$

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Abstract: In this paper we studied the classical numerical range of matrices in $\mathfrak{sp}(2n, \mathbb{C})$. We obtained some result on the relationship between the numerical range of a matrix in $A = \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^\top \end{pmatrix} \in \mathfrak{sp}(2n, \mathbb{C})$ and that of its diagonal block, the singular values of its off-diagonal block A_2.

Keywords: Numerical range, elliptical disk.

MSC: 15A45, 15A18

1 Introduction

Let $A \in \mathbb{C}_{n\times n}$ be a complex matrix. The numerical range of A is the set of all complex numbers, denoted by $W(A)$, of the following form:

$$W(A) := \{x^* Ax : x \in \mathbb{C}^n \text{ and } \|x\| = 1\}.$$

By the well-known Toeplitz-Hausdorff theorem $W(A)$ is convex [1, 5, 6]. When $n = 2$, the numerical range $W(A)$ is the closed elliptical disk with foci the eigenvalues λ_1 and λ_2 and the minor axis length $\sqrt{\operatorname{tr} A^*A - |\lambda_1|^2 - |\lambda_2|^2}$ [4], where tr A^*A is the trace of the matrix A^*A. There are many generalizations of the classical numerical range. Among all of these generalizations, Tam[7] studied the numerical range that is associated with Lie algebras. In this paper, we focused on the classical numerical range of matrices in $\mathfrak{sp}(2n, \mathbb{C})$, a Lie algebra of type C.

2 Numerical range of matrices in $\mathfrak{sp}(2n, \mathbb{C})$

Consider the complex symplectic Lie algebra [3, p.3] which is simple for $n \geq 1$:

$$\begin{aligned} \mathfrak{sp}(2n, \mathbb{C}) &:= \mathfrak{sp}(2n) \oplus i\mathfrak{sp}(2n) \\ &= \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^\top \end{pmatrix} : A_1, A_2, A_3 \in \mathbb{C}_{n\times n}, A_2^\top = A_2, A_3^\top = A_3 \right\}. \end{aligned}$$

The compact group $K = \mathrm{Sp}(2n, \mathbb{C}) \cap U(2n)$ [3] consists of the matrices

$$\begin{pmatrix} U & -\overline{V} \\ V & \overline{U} \end{pmatrix} \in \mathrm{U}(2n).$$

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It is known that for any $B \in \mathfrak{sp}(2n, \mathbb{C})$, there is $U \in K$ such that $UBU^\star \in \mathfrak{b} \subset \mathfrak{sp}(2n, \mathbb{C})$, where

$$\mathfrak{b} := \left\{ \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^\top \end{pmatrix}, \ A_1 \in \mathbb{C}_{n\times n} \text{ is upper triangular}, A_2^\top = A_2 \right\} \tag{2.1}$$

is a Borel subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$. The eigenvalues of $A \in \mathfrak{sp}(2n, \mathbb{C})$ occur in pairs but opposite in sign as we can see it from (2.1). By Toeplitz-Hausdorff theorem, the numerical range $W(A)$ of a matrix $A \in \mathfrak{sp}(2n, \mathbb{C})$ is convex. When $n = 1$, $W(A)$ is the elliptical disk with foci $\pm\lambda$ and the minor axis length $\frac{1}{2}|a_2|$, where $\pm\lambda$ are the eigenvalues of A.

Lemma 2.1. If $A \in \mathfrak{sp}(2n, \mathbb{C})$, then $0 \in W(A)$.

Proof. Let $A \in \mathfrak{sp}(2n, \mathbb{C})$. The eigenvalues $\pm\lambda_1, \dots, \pm\lambda_n$ of A are in the numerical range $W(A)$. Since the set $W(A)$ is convex,

$$0 = \frac{1}{2}\lambda_1 + \frac{1}{2}(-\lambda_1) \in W(A).$$

□

Now we explore the relationship between the numerical range of A and that of A_1, the singular values of A_2.

Theorem 2.2. Let $A = \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^\top \end{pmatrix} \in \mathfrak{b}$. If the largest singular value of A_2 is s , then the circular disk $D_{s/2}$ centered at the origin and with the radius $r = \frac{s}{2}$ is a subset of the numerical range $W(A)$, i.e., $D_{s/2} \subset W(A)$.

Proof. By Takagi's factorization of complex symmetric matrices, $A_2 = U\mathrm{diag}\,(s_1, \dots, s_n)U^\top$ with $U \in U(n)$ and $s_1 \geq \cdots \geq s_n$ the singular values of A_2. Then

$$U^\star A_2 \overline{U} = \mathrm{diag}\,(s_1, \dots, s_n).$$

Let X_1 be the first column of U and $X = \frac{1}{\sqrt{2}}\begin{pmatrix} X_1 \\ \overline{X}_1 \end{pmatrix}$. Direct computation shows that

$$\begin{aligned} W(A) \ni X^\star AX &= \frac{1}{2}(X_1^\star, X_1^\top)\begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^\top \end{pmatrix}\begin{pmatrix} X_1 \\ \overline{X}_1 \end{pmatrix} \\ &= \frac{1}{2}(X_1^\star A_1 X_1 - X_1^\top A_1^\top \overline{X}_1 + X_1^\star A_2 \overline{X}_1) \\ &= \frac{1}{2}[X_1^\star A_1 X_1 - (X_1^\star A_1 X_1)^\top + X_1^\star A_2 \overline{X}_1] \\ &= \frac{1}{2}X_1^\star A_2 \overline{X}_1 \\ &= \frac{1}{2}s_1. \end{aligned}$$

Replace X_1 with $e^{-i\theta/2}X_1$, $\theta \in [0, 2\pi]$ in above equation, we can see that $\frac{1}{2}s_1 e^{i\theta} \in W(A)$. So the circle $\{\frac{1}{2}e^{i\theta}s_1 : \theta \in [0, 2\pi]\} \subset W(A)$. Since $W(A)$ is convex, $D_{s/2} \subset W(A)$. □

In Theorem 2.2, if $n = 1$, $A = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. The singular value of the $(2, 1)$ block is $|b|$ and the numerical range $W(A)$ is the elliptical disk with foci a, $-a$ and minor axis length $|b|$. Clearly the circular disk $D_{|b|/2}$ is a subset of $W(A)$.

Lemma 2.3. Let $A = \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^\top \end{pmatrix} \in \mathfrak{sp}(2n, \mathbb{C})$. The numerical range of A is the union of all elliptical disks with foci $X^\star A_1 X \in W(A_1)$ and $-Y^\star A_1 Y \in -W(A_1)$ and minor axis length $|X^\star A_2 \overline{Y}|$, where $X, Y \in \mathbb{C}^n$ are unit vectors.

Proof. For any unit vector $Z \in \mathbb{C}^{2n}$, let $Z = \begin{pmatrix} e^{i\alpha} \sin\theta X \\ e^{i\beta} \cos\theta \overline{Y} \end{pmatrix}$, where X, Y are unit vectors in $\mathbb{C}^n$.

$$\begin{aligned} Z^\star AZ &= X^\star A_1 X \sin^2\theta + e^{-i(\alpha-\beta)} X^\star A_2 \overline{Y} \sin\theta\cos\theta - Y^\top A_1^\top \overline{Y} \cos^2\theta \\ &= \begin{pmatrix} e^{i\alpha}\sin\theta \\ e^{i\beta}\cos\theta \end{pmatrix}^\star \begin{pmatrix} X^\star A_1 X & X^\star A_2 \overline{Y} \\ 0 & -(Y^\star A_1 Y)^\top \end{pmatrix} \begin{pmatrix} e^{i\alpha}\sin\theta \\ e^{i\beta}\cos\theta \end{pmatrix} \end{aligned}$$

which generates the required ellipse containing $Z^\star AZ$ if we let θ, α, β run over all the values in $[0, 2\pi]$. By the convexity property of numerical range, the union of all these closed elliptical disks is the numerical range of A. □

Actually, Lemma 2.3 holds for any matrix $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where A_1 and A_3 are square matrix.

For any square matrix A and $s > 0$, define the set

$$W(A, s) := \{a + re^{i\theta} : a \in W(A), 0 \le \theta \le 2\pi, \text{ and } 0 \le r \le s\}.$$

Clearly $W(A) = W(A, 0) \subset W(A, s)$. The set $W(A, s)$ is the region obtained by expanding the region $W(A)$ by s units in all directions. The following is the main result of this paper.

Theorem 2.4. For any $A = \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^\top \end{pmatrix} \in \mathfrak{b}$, define $B = A_1 \oplus (-A_1^\top)$. Then

1. $W(A) \subseteq W(B, \frac{1}{2}s)$, where s is the largest singular values of A_2.
2. If A_1 is skew symmetric and $A_2 = sI_n$, then $W(A) = W(B, \frac{1}{2}s)$.
3. If $W(A) = W(B, \frac{1}{2}s)$, then $W(A_1) = W(-A_1)$, furthermore for any point x on the boundary $\partial W(A_1)$ that is not convex combination of any other two points in $W(A_1)$, there is $X \in \mathbb{C}^n$ such that $x = X^\star A_1 X = -X^\star A_1^\top X$.

Proof. *(1)* Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathbb{C}^{2n}$ be any unit vector with $X_1 = (x_1, \dots, x_n)^\top$ and $X_2 = (x_{n+1}, \dots, x_{2n})^\top$. Then

$$\begin{aligned} X^\star AX &= X^\star BX + X_1^\star A_2 X_2 \\ &= X^\star BX + X_1^\star A_2 X_2 \\ &= X^\star BX + (UX_1)^\star UA_2U^\top (\overline{U}X_2) \text{ for any } U \in U(n) \end{aligned} \tag{2.2}$$

Since A_2 is symmetric, there exists a $U \in U(n)$ such that $UA_2U^\top = \operatorname{diag}(s_1, \dots, s_n)$, where $s_1 \ge \cdots \ge s_n$ are singular values of A_2. From now on, we always assume that $A_2 = \operatorname{diag}(s_1, \dots, s_n)$ unless otherwise specified. Let $Y_1 = (y_1, \dots, y_n)^\top = UX_1$ and $Y_2 = (y_{n+1}, \dots, y_{2n})^\top = \overline{U}X_2$. In (2.2),

$$\begin{aligned} |(UX_1)^\star UA_2U^\top \overline{U}X_2| &= |Y_1^\star \operatorname{diag}(s_1, \dots, s_n) Y_2| \\ &\le \sum_{k=1}^n |y_k|\, s_k\, |y_{n+k}| \\ &\le \sum_{k=1}^n \frac{1}{2} s_k (|y_k|^2 + |y_{n+k}|^2) \\ &\le \frac{1}{2} s_1 \sum_{k=1}^n (|y_k|^2 + |y_{n+k}|^2) \\ &= \frac{1}{2} s_1 \sum_{k=1}^n (|x_k|^2 + |x_{n+k}|^2) = \frac{1}{2} s. \end{aligned} \tag{2.3}$$

Thus $X^\star AX \in W(B, \frac{1}{2}s)$. Hence $W(A) \subseteq W(B, \frac{1}{2}s)$. The equality in (2.3) holds if and only if

$$Y_2 = e^{i\beta}Y_1 \text{ for some } \beta \text{ and } s_j = s \text{ for all } 1 \le j \le n \text{ where } y_j \ne 0.$$

(2) Assume that A_1 is skew symmetric and $A_2 = sI_n$, we will show that the boundary of $W(B, \frac{1}{2}s)$ is a subset of $W(A)$. Let $p = Z^\star BZ + \frac{1}{2}se^{i\theta}$ be any point on the boundary of $W(B, s)$, where $Z = (\sin\alpha X_1^\top, \cos\alpha X_2^\top)^\top \in \mathbb{C}^{2n}$ for some unit vectors $X_1, X_2 \in \mathbb{C}^n$ and $\alpha \in [0, 2\pi]$.

$$p = \sin^2\alpha X_1^\star A_1 X_1 + \cos^2\alpha X_2^\star A_1 X_2 + \frac{1}{2}se^{i\theta}.$$

Since the numerical range $W(A_1)$ is convex, the convex combination $\sin^2\alpha X_1^\star A_1 X_1 + \cos^2\alpha X_2^\star A_1 X_2 = X_3^\star A_1 X_3 \in W(A_1)$ for some unit vector $X_3 \in \mathbb{C}^n$. Let $Z_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\theta/2}X_3 \\ e^{i\theta/2}X_3 \end{pmatrix}$. Then

$$W(A) \ni Z_1^\star AZ_1 = \frac{1}{2}X_3^\star A_1 X_3 + \frac{1}{2}X_3^\star A_1 X_3 + \frac{1}{2}e^{i\theta}X_3^\star sI_n X_3 = p$$

So the boundary of $W(B, \frac{1}{2}s)$ is a subset of $W(A)$. Thus $W(B, \frac{1}{2}s) \subset W(A)$ since $W(A)$ is convex. Therefore $W(A) = W(B, \frac{1}{2}s)$.

(3) Suppose that $W(A) = W(B, \frac{1}{2}s)$. Assume that $W(A_1) \ne W(-A_1)$. Since $W(A_1)$ and $W(-A_1)$ are convex, we can pick a point $p = X_1^\star A_1 X_1$, $\|X_1\| = 1$ on the boundary of $W(A_1)$ that satisfies the following: the supporting line ℓ of $W(A_1)$ through p does not intersects $W(-A_1)$ and both $W(A)$ and $W(-A_1)$ are on the same side of ℓ. Choose a vector $pq = \frac{1}{2}se^{i\theta}$ that is perpendicular to ℓ and $q \notin W(A_1)$. So $q = p + \frac{1}{2}se^{i\theta}$. Because $W(B) = \operatorname{conv}\{W(A_1) \cup W(-A_1)\}$, the convex hull containing $W(A_1)$ and $W(-A_1)$, q is on the boundary of $W(B, \frac{1}{2}s)$ and the distance between q and $W(B)$ is $\frac{1}{2}s$. Since $W(A) = W(B, \frac{1}{2}s)$, there exists a unit vector $Z = \begin{pmatrix} \sin\alpha Y_1 \\ \cos\alpha Y_2 \end{pmatrix} \in \mathbb{C}^{2n}$ such that

$$q = Z^\star AZ = \sin^2\alpha Y_1^\star A_1 Y_1 + \cos^2\alpha(-Y_2^\star A_1^\top Y_2) + \frac{1}{2}\sin(2\alpha)Y_1^\star A_2 Y_2 \in W(A) \tag{2.4}$$

Because $\sin^2\alpha Y_1^\star A_1 Y_1 + \cos^2\alpha(-Y_2^\star A_1^\top Y_2)$ is a convex combination of $Y_1^\star A_1 Y_1 \in W(A_1)$ and $-Y_2^\star A_1^\top Y_2 \in W(-A_1)$, hence it is in $W(B)$. Thus the distance between $q = Z^\star AZ$ and $W(B)$,

$$\begin{aligned} d(q, W(B)) &\le \left|\frac{1}{2}\sin(2\alpha)Y_1^\star A_2 Y_2\right| \\ &\le \frac{1}{2}s\sin(2\alpha) \quad \text{by inequalities (2.2) and (2.3)} \end{aligned}$$

For that the equality holds, $\sin 2\alpha = 1$, $Y_2 = e^{i\beta}Y_1$ and $\sin^2\alpha Y_1^\star A_1 Y_1 + \cos^2\alpha(-Y_2^\star A_1^\top Y_2) = p$. But if this is the case, then $Y_1^\star A_1 Y_1 = -Y_2^\star A_1^\top Y_2 = p$, that means p is also in $W(-A_1)$, which is not true. So $W(A_1) = W(-A_1)$. Furthermore, if $x = p$ is a point on the boundary of $W(A_1) = W(-A_1)$ that is not a convex combination of any other points in $W(A_1)$, let q be the point in Equation (2.4). Since the distance $d(q, W(B)) = \frac{1}{2}s$, $Y_1 = e^{i\beta}Y_2$ and $Y_1^\star A_1 Y_1 = -Y_2^\star A_1^\top Y_2 = p$. Let $X = Y_1$, then $X^\star A_1 X = -X^\star A_1^\top X = p = x$. □

Theorem 2.4 holds for more general matrices. Let $A = \begin{pmatrix} B_{m\times m} & C_{m\times n} \\ 0 & D_{n\times n} \end{pmatrix}$ be any $m + n$ square matrix with $m \le n$. Let $C = U_1 \Sigma U_2^\star$ be the singular value decomposition of C. So we can assume that the block $C = \Sigma$. If $W(A) = W(B, \frac{1}{2}s)$, then $W(B) = W(D)$ and for each x on the boundary of $W(B)$ that is not a convex combination of another two points of $W(B)$, then there exists $X \in \mathbb{C}^m$ and $Y = \begin{pmatrix} X \\ 0 \end{pmatrix} \in \mathbb{C}^n$ such that $x = X^\star BX = Y^\star DY$. Clearly if $B = D$ and $C = sI_n$, then $W(A) = W(B, \frac{1}{2}s)$. We conjecture that $W(A) = W(B, \frac{1}{2}s)$ if and only if A is unitarily similar to a matrix of the following form

$$\begin{pmatrix} B_1 & 0 & sI_k & 0 \\ 0 & B_2 & 0 & \Sigma_1 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & D_2 \end{pmatrix},$$

where $W(B_2) \cup W(D_2) \subset W(B_1)$ and the singular values of Σ_1 are less than or equal to s.

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