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Characterizations of the distribution of the Demmel condition number of real Wishart matrices

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Abstract: The Demmel condition number is an indicator of the matrix condition, and its properties have recently found applications in many practical problems, such as in MIMO communication systems, in the analytical prediction of level-crossing and fade duration statistics of Rayleigh channels, and in spectrum sensing for cognitive radio systems, among others. As the characterizations of a probability distribution play an important role in probability and statistics, in this paper we study the characterizations of the distribution of the Demmel condition number of real Wishart matrices by truncated first moment. Since the truncated distributions arise in practical statistics where the ability of record observations is limited to a given threshold or within a specified range, we hope that these characterizations will be quite useful for practitioners and researchers in the fields of probability, statistics, and other applied sciences, such as actuarial science, linear algebra, multivariate statistics, physics, wireless communications, among others.

Keywords: Characterization, Demmel condition number, truncated first moment, Wishart matrices

MSC: 15A12, 15A52

1 Introduction

Random matrix theory (RMT) deals with the statistical properties of certain random matrices, which appear in many fields of pure and applied sciences, such as, probability, statistics, multivariate statistics, linear algebra, operator algebra theory, actuarial science, physics, wireless communications, among others. The study of random matrices started with the pioneering works of Wishart (1928) in Statistics, and Wigner (1955) in Physics. Further development continued with the contributions of many other researchers and authors, see, for example, Haggerup and Thorbjornsen (2003), Edelman and Rao (2005), and Edelman and Wang (2013), among others. In recent years, many researchers have studied the distributions of the Demmel (or the scaled) condition numbers (DCN) of both real and complex Wishart matrices, their properties and applications in various branches of pure and applied sciences and engineering, for which the interested readers are referred to Demmel (1988), Edelman (1988, 1992), Chen and Dongarra (2005), Anderson and Wells (2009), Wei et al. (2011), Zhong et al. (2011) and Zhang et al. (2016), among others.

Wei et al. (2011) derived the probability density function (pdf) of Demmel condition number (DCN) for complex Wishart matrices using Mellin transformation. In this paper we have considered the pdf of DCN for real Wishart matrices. Since the truncated distributions arise in practical statistics where the ability of record observations is limited to a given threshold or within a specified range, there has been a great interest, in recent years, in the characterizations of probability distributions by truncated moments. Before a particular

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probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given continuous probability distribution satisfies the underlying requirements by its characterization. Thus the characterizations of probability distributions by truncated moments may play an important part in the determination of distributions by using certain properties in the given data. For example, the development of the general theory of the characterizations of probability distributions by truncated moment began with the work of Galambos and Kotz (1978). Further development on the characterizations of probability distributions by truncated moments continued with the contributions of many authors and researchers, among them Kotz and Shanbhag (1980), Glänzel et al. (1984), and Glänzel (1987) are notable. Most of these characterizations are based on a simple relationship between two different moments truncated from the left at the same point. As pointed out by Glänzel (1987), these characterizations by truncated moments may serve as a basis for parameter estimation, and may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. As pointed out by Kim and Jeon (2013), in actuarial science, the credibility theory proposed by Buhlmann (1967) allows actuaries to estimate the conditional mean loss for a given risk to establish an adequate premium to cover the insured's loss. In their paper, Kim and Jeon (2013) have proposed a credibility theory based on truncation of the loss data, or the trimmed mean, which also contains the classical credibility theory of Buhlmann (1967) as a special case. In this paper, therefore, motivated by the importance of the distribution of the DCN of real Wishart matrices in many practical problems, we have studied its several distributional properties. Based on these distributional properties, we have established some new characterizations of the distribution of the DCN of real Wishart matrices by truncated first moment.

The organization of this paper is as follows. Section 2 discusses the properties of the distribution of the DCN of real Wishart matrices. The characterizations of the distribution of the DCN of real Wishart matrices are provided in section 3. The concluding remarks are provided in Section 4.

2 Distributional Properties of the DCN of Real Wishart Matrices

In this section, we discuss the properties of the distribution of the DCN of real Wishart matrices.

2.1 Distribution of the DCN of Real Wishart Matrices

Let A be a real random $n \times n$ matrix whose elements are independent and identically distributed (i.i.d.) standard normal random variables. Let A^T denote the transpose of A. Then, the $n \times n$ random matrix $W_{n\times n} = AA^T$ is said to be a real Wishart matrix or to have the Wishart distribution with parameters (n, n). That is, $W_{n\times n}\left(= AA^T\right) \sim W(n, n)$. Further, let $\lambda_{\max} = \lambda_1 > \lambda_2 \cdots > \lambda_n = \lambda_{\min} > 0$ and $\sigma_1 > \sigma_2 \cdots > \sigma_n > 0$ denote the distinct eigenvalues of the matrix $W_{n\times n} = AA^T$ and singular values of $A_{n\times n}$, respectively. Note that the squares of the singular values of $A_{n\times n}$ are the eigenvalues of the Wishart matrix $W_{n\times n} = AA^T$. Then, the random quantity $\kappa_2(A) = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} = \frac{\sigma_1}{\sigma_n}$, the square root of the ratio of largest to smallest eigenvalues of the real Wishart matrix $W_{n\times n} = AA^T$, is defined as the 2-norm (or the standard) condition number of $A_{n\times n}$, see, for example, Edelman (1988). Similar to the standard condition number, the Demmel (or the scaled) condition numbers (DCN) of the real random matrix of $A_{n\times n}$ is a random quantity $\kappa_D(A)$, which is defined as follows: $\kappa_D(A) = \sqrt{\frac{\sum_{i=1}^{n}\lambda_i}{\lambda_n}}$, where $\sum_{i=1}^{n}\lambda_i$, is defined as the trace of the matrix $W_{n\times n} = AA^T$. Note that $\sqrt{n} \le \kappa_D(A)$. As stated above, the objective of this paper is to study the characterizations of the distribution of the DCN, $\kappa_D(A)$, of the real random matrix $A_{n\times n}$ (or, equivalently, of the real Wishart matrix $W_{n\times n} = AA^T$), for which we shall need the expressions for the pdf and cumulative distribution function (cdf) of $\kappa_D(A)$. For the sake of simplicity, we shall follow Edelman (1992) for the explicit expression of the pdf of real Wishart matrices, from which we have derived independently the cdf in subsection 2.3. The pdf, $f(x)$, of the distribution of the

DCN, $\kappa_D(A) = X$, of the real Wishart matrices, is given by

$$f(x) = \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right), \tag{2.1}$$

where $x \geq \sqrt{n}$, $\sqrt{n} \leq \kappa_D(A)$, $\mu = \frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)}$, where $\Gamma(.)$ denotes the gamma function, and ${}_2F_1$ denotes the Gauss hypergeometric function, see, for example, Abramowitz and Stegun (1970), Gradshteyn and Ryzhik (1980), Prudnikov et al. (1986), and Oldham et al. (2009), among others.

2.2 Shapes of the Probability Density Function

Using Maple, the shapes of the pdf (2.1), are provided in Figure 1 below for some selected values of the parameters. The effects of the parameters are obvious from these figures; also, the distribution of the DCN appears to be unimodal and right skewed.

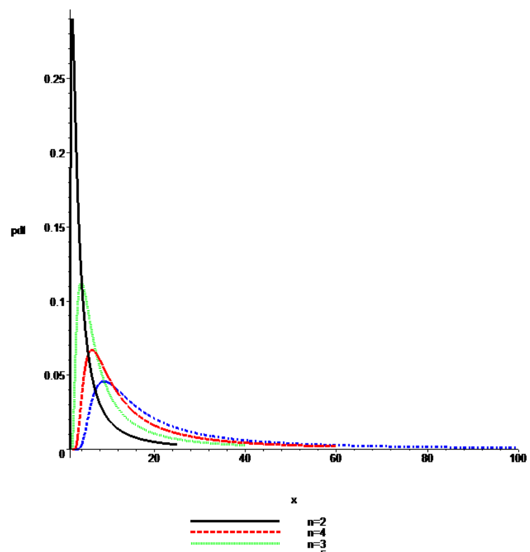

Figure 1: Plots of the pdf (2.1) for $n = 2, 3, 4, 5$

2.3 Cumulative Distribution Function

The cdf, $F(x)$, of $\kappa_D(A) = X$, $f(x)$, of the distribution of the DCN, $\kappa_D(A) = X$, of the real Wishart matrices, is given by

$$\begin{aligned} F(x) &= \int_{\sqrt{n}}^{x} f_X(t)dt \\ &= \int_{\sqrt{n}}^{x} \mu t^{1-n^2}\left(t^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\tfrac{n}{2}-\tfrac{1}{2}, \tfrac{n}{2}+1; \tfrac{n^2}{2}+\tfrac{n}{2}-1; -\left(t^2-n\right)\right)dt. \end{aligned} \tag{2.2}$$

Letting $t^2 - n = u$ in Eq. (2.2), and simplifying, we have

$$F(x) = \frac{\mu}{2}\int_{0}^{x^2-n} u^{\left[\frac{n(n+1)}{2}-1\right]-1}(u+n)^{\frac{-n^2}{2}} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -u\right)du,$$

which, using the binomial expansion for $(u+n)^{\frac{-n^2}{2}}$, can be expressed as

$$F(x)=\frac{\mu}{2n^{\frac{n^2}{2}}}\sum_{j=0}^{\infty}\left[\int_0^{x^2-n}(-1)^j\frac{\left(\frac{n^2}{2}\right)_j\left(\frac{1}{n}\right)^j}{j!}u^{\left[\frac{n(n+1)}{2}-1+j\right]-1}{}_2F_1\left(\frac{n}{2}-\frac{1}{2},\frac{n}{2}+1;\frac{n^2}{2}+\frac{n}{2}-1;-u\right)du\right] \tag{2.3}$$

where $\left(\frac{n^2}{2}\right)_j$ represent Pochhammer symbol. Thus, using Prudnikov, et al. (1986), Vol. 3, Eq. (2.21.1.4), Page 314, for the integral in the above Eq. (2.3) , and simplifying, we obtain the cdf, $F(x)$, as follows:

$$F(x)=\frac{\mu}{2n^{\frac{n^2}{2}}}\sum_{j=0}^{\infty}\left[(-1)^j\frac{\left(\frac{n^2}{2}\right)_j\left(\frac{1}{n}\right)^j}{j!}B\left(\frac{n(n+1)}{2}-1+j,1\right)\left(x^2-n\right)^{\frac{n(n+1)}{2}-1+j}\right.$$
$$\left.\times{}_3F_2\left(\frac{n}{2}-\frac{1}{2},\frac{n}{2}+1,\frac{n(n+1)}{2}-1+j;\frac{n^2}{2}+\frac{n}{2}-1,\frac{n(n+1)}{2}+j;-\left(x^2-n\right)\right)\right], \tag{2.4}$$

where $\mu=\frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)}$, $B(.,.)$ denotes the beta function, and ${}_3F_2$ denotes the generalized hypergeometric function of order $(3,2)$. Note: We would like to point out that (2.4) represents the cdf of DCN for real Wishart matrices, and is different from those in Wei et al. (2011), Zhong et al. (2011) and Zhang et al. (2016). To see the difference, we refer the interested reader to these papers.

Using Maple, the shapes of the cdf (2.4) are provided in Figure 2 below for some selected values of the parameters.

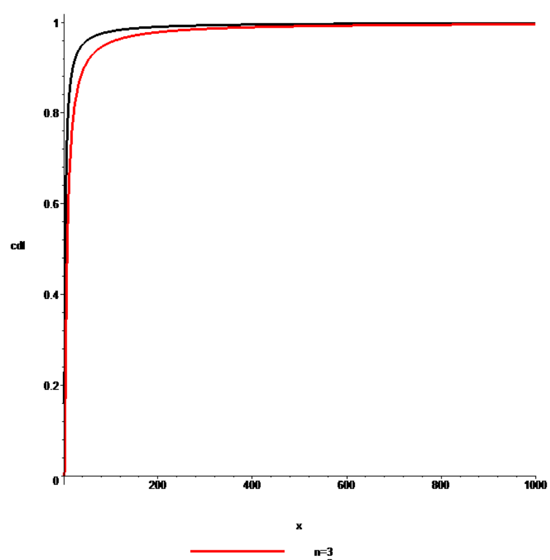

Figure 2: Plots of the cdf for $n=2$, $\sqrt{2}\le x\le 1000$ and $n=3$, $\sqrt{3}\le x\le 1000$

2.4 Survival and Hazard Functions

The survival and hazard functions are respectively given by

$$S(x) = 1 - F_X(x) = 1 - \frac{\mu}{2n^{\frac{n^2}{2}}} \sum_{j=0}^{\infty} \left[(-1)^j \frac{\left(\frac{n^2}{2}\right)_j \left(\frac{1}{n}\right)^j}{j!} B\left(\frac{n(n+1)}{2} - 1 + j, 1\right) \left(x^2 - n\right)^{\frac{n(n+1)}{2} - 1 + j} \right.$$
$$\left. \times {}_3F_2\left(\frac{n}{2} - \frac{1}{2}, \frac{n}{2} + 1, \frac{n(n+1)}{2} - 1 + j; \frac{n^2}{2} + \frac{n}{2} - 1, \frac{n(n+1)}{2} + j; -\left(x^2 - n\right)\right) \right], \quad (2.5)$$

and

$$h(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{\mu x^{1-n^2} \left(x^2 - n\right)^{\frac{n(n+1)}{2} - 2} {}_2F_1\left(\frac{n}{2} - \frac{1}{2}, \frac{n}{2} + 1; \frac{n^2}{2} + \frac{n}{2} - 1; -\left(x^2 - n\right)\right)}{1 - \frac{\mu}{2n^{\frac{n^2}{2}}} \sum_{j=0}^{\infty} \left[(-1)^j \frac{\left(\frac{n^2}{2}\right)_j \left(\frac{1}{n}\right)^j}{j!} B\left(\frac{n(n+1)}{2} - 1 + j, 1\right) \left(x^2 - n\right)^{\frac{n(n+1)}{2} - 1 + j} \times {}_3F_2\left(\frac{n}{2} - \frac{1}{2}, \frac{n}{2} + 1, \frac{n(n+1)}{2} - 1 + j; \frac{n^2}{2} + \frac{n}{2} - 1, \frac{n(n+1)}{2} + j; -\left(x^2 - n\right)\right) \right]}. \quad (2.6)$$

The possible shapes of the survival function (2.5) and the hazard rate function (2.6) are provided for some selected values of the parameter n in the following Figures 3 and 4 respectively.

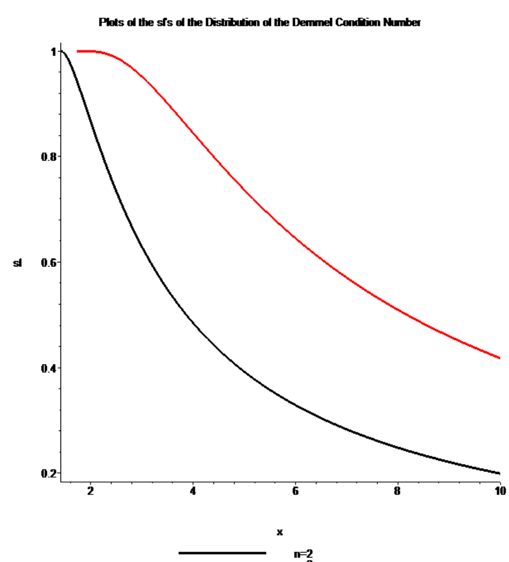

Figure 3: Plots of the survival function, $S(x)$, for $n = 2$, $\sqrt{2} \le x \le 1000$ and $n = 3$, $\sqrt{3} \le x \le 1000$

The effects of the parameters can easily be seen from these graphs. The decreasing and upside down bathtub shape behaviors of the failure rate function, $h(x)$, are also evident from these Figures.

2.5 Moment

As pointed out by Edelman (1988). "the result of directly averaging the condition number is known to be infinite." Instead of finding the expected value of the condition number, that is, $E(\kappa_D(A))$, Edelman (1988) has obtained the expected value, $E(\ln \kappa_D(A))$, of the random quantity $\ln(\kappa_D(A))$, stating the fact that "the quantity $\ln(\kappa_D(A))$ is the measure of the loss of numerical precision;" see, for example, Blum and Schub (1986). However, in this paper, in what follows, for the random variable $\kappa_D(A) = X$ having the pdf $f(x)$ as in

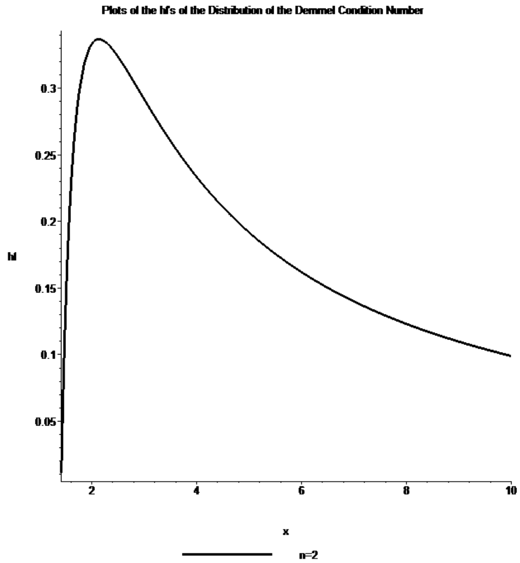

Figure 4: Plot of the hazard rate function, $h(x)$, **for** $n = 2$, $\sqrt{2} \le x \le 1000$

Eq. (2.1), we derive the moment, $E\left(X^k\right)$, when $0 < k < 1$ or $k < 0$. We have

$$E\left(X^k\right) = \int_{\sqrt{n}}^{\infty} x^k f_X(x)\,dx = \int_{\sqrt{n}}^{\infty} x^k \left[\mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right] dx, \tag{2.7}$$

Letting $x^2 - n = u$ in Eq. (2.7), and simplifying, we have

$$E\left(X^k\right) = \frac{\mu}{2}\int_0^{\infty} u^{\left[\frac{n(n+1)}{2}-1\right]-1}(u+n)^{\frac{-\left(n^2-k\right)}{2}} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -u\right) du. \tag{2.8}$$

Thus, using Gradshteyn and Ryzhik (1980), Eq. 7.512.10, Page 849, for the integral in Eq. (2.8) , and simplifying, we obtain the moment, $E\left(X^k\right)$, where $0 < k < 1$ or $k < 0$, as follows:

$$E\left(X^k\right) = \frac{n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)\Gamma\left(\frac{1-k}{2}\right)\Gamma\left(2-\frac{k}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n^2-k}{2}\right)\Gamma\left(\frac{n+3-k}{2}\right)} {}_2F_1\left(\frac{1-k}{2}, 1-\frac{k}{2}; \frac{n+3-k}{2}; 1-n\right) \tag{2.9}$$

In view of the fact that the gamma function $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$ converges for all values of $\alpha > 0$, and the generalized hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ is convergent for all $|z| < 1$, it is evident from Eq. (2.9) that the moment, $E\left(X^k\right)$, exists when $0 < k < 1$, because the gamma function, $\Gamma\left(\frac{1-k}{2}\right)$, is defined when $k < 1$. Also, we note that, since $n > 1$, $x^k < 1$ for $k < 0$ and $x > 1$. Thus, from Eq. (2.7), it follows that, for $k < 0$,

$$E\left(X^k\right) = \int_{\sqrt{n}}^{\infty} x^k \left[\mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right] dx$$
$$< \int_{\sqrt{n}}^{\infty} \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right) dx = \int_{\sqrt{n}}^{\infty} f_X(x)\,dx = 1.$$

Thus, it follows from the above that, for the random variable $\kappa_D(A) = X$ having the pdf $f(x)$ as in Eq. (2.1), the moment, $E\left(X^k\right)$, is finite when $0 < k < 1$ or $k < 0$. As pointed out by Ahsanullah, et al. (2013), one of the earliest examples in which non-integer moments (NIM) were calculated related to the spans of random walks, but more recently the properties of non-integer moments have found application in the study of random resistor networks, chaos, and diffusion-limited aggregation, see Weiss et al. (1989), and references therein. Also, see Cottone and Paola (2009), and Cottone et al. (2010), for recent developments on fractional moments. For a discussion on the existence of the first negative moment of a continuous random variable and its applications, the interested readers are referred to Khuri (1993), Section 6.9.1, P. 242, and references therein. Also, see Shakil et al. (2007). Using the software Maple, and Eq. (2.4), the graphs of the moment, $E\left(X^k\right)$, when $k < 0$ and $0 < k < 1$, for some values of the parameter, n, are sketched in Figures 5 and 6 respectively.

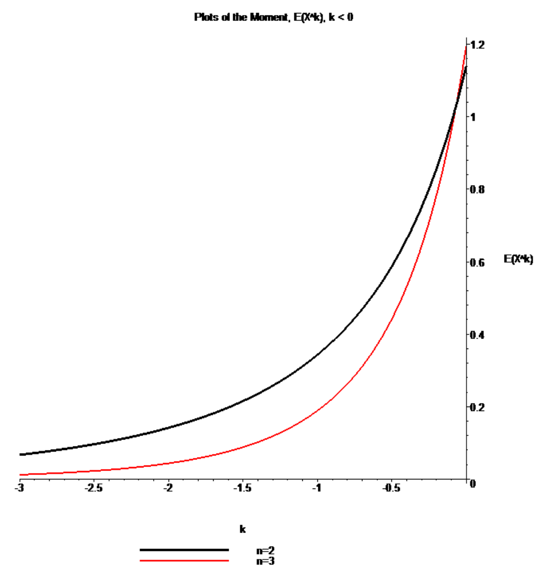

Figure 5: Plots of the Moment, $E\left(X^k\right)$, $k < 0$, for $n = 2$, $\sqrt{2} \le x \le 1000$ and $n = 3$, $\sqrt{3} \le x \le 1000$

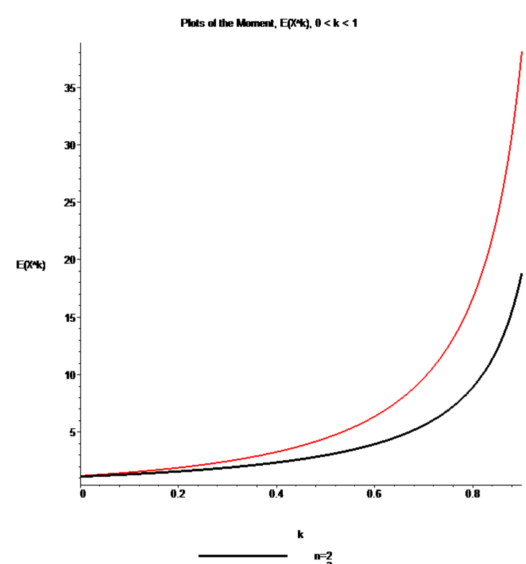

Figure 6: Plots of the Moment, $E\left(X^k\right)$, $0 < k < 1$, for $n = 2$, $\sqrt{2} \le x \le 1000$ and $n = 3$, $\sqrt{3} \le x \le 1000$

From Figures 5 and 6, it is observed that, for both cases, $k < 0$ and $0 < k < 1$, the moment, $E\left(X^k\right)$, is a monotonic increasing and concave up function of k, for the given values of parameter, n, and, obviously, as $k \to 1$, $E\left(X^k\right) \to \infty$.

3 Characterizations of the Distribution of the Demmel Condition Number

The truncated distributions arise in practical statistics where the ability of record observations is limited to a given threshold or within a specified range. In this section, we will establish some new characterizations of the distribution of DCN for real Wishart matrices by truncated first moment. For this, we need the assumption that $\frac{df(x)}{dx}$ exits for all x, $\sqrt{n} < x < \infty$ in the Theorem 3.1.

3.1 Characterization by Truncated First Moment

In what follows, we will establish some characterization results in Theorems 3.1 and 3.2 by truncated first moment, that is, by considering a product of reverse hazard rate and another function of the truncated point. For this, we will need the following Lemmas 3.1 and 3.2, which we will prove first.

Lemma 3.1: Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\gamma = \inf\{x|F(x) > 0\}$, $\delta = \sup\{x|F(x) < 1\}$, $f(x)$ is a differentiable function of x, and $h_1(x)$ is a continuous function of x, where $\gamma < x < \delta$, with $E(h_1(x))$ as finite. If $E(h_1(X)|X \le x) = g_1(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g_1(x)$ is a continuous differentiable function of x with the condition that $\int_\gamma^x \frac{h_1(u)-g_1^{/}(u)}{g_1(u)}du$ is finite for $x > \gamma$, then $f(x) = ce^{\int \frac{h_1(x)-g_1^{/}(x)}{g_1(x)}dx}$, where c is a constant determined by the condition $\int_\gamma^\delta f(x)dx = 1$.

Proof. Suppose that $E(h_1(X)|X \le x) = g_1(x)\tau(x)$. Then, since $E(h_1(X)|X \le x) = \frac{\int_\gamma^x h_1(u)f(u)du}{F(x)}$ and $\tau(x) = \frac{f(x)}{F(x)}$, we have $g_1(x) = \frac{\int_\gamma^x h_1(u)f(u)du}{f(x)}$, that is, $\int_\gamma^x h_1(u)f(u)\,du = g_1(x)f(x)$.

Differentiating both sides of the above equation with respect to x, we obtain

$$h_1(x)f(x) = f(x)g_1^{/}(x) + f^{/}(x)g_1(x).$$

From the above equation, on simplification, we obtain

$$\frac{f^{/}(x)}{f(x)} = \frac{h_1(x) - g_1^{/}(x)}{g_1(x)}$$

Thus, on integrating the above equation with respect to x, we have

$$f(x) = ce^{\int \frac{h_1(x)-g_1^{/}(x)}{g_1(x)}dx},$$

where c is obtained by the condition $\int_\gamma^\delta f(x)dx = 1$. This completes the proof of Lemma 3.1. □

Lemma 3.2: Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\gamma = \inf\{x|F(x) > 0\}$, $\delta = \sup\{x|F(x) < 1\}$, $f(x)$ is a differentiable function of x, and $h_2(x)$ is a continuous function of x, where $\gamma < x < \delta$, with $E(h_2(x))$ as finite. If $E(h_2(X)|X \ge x) = g_2(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $g_2(x)$ is a continuous differentiable function of x with the condition that $\int_x^\delta \frac{h_2(x)+g_2^{/}(x)}{g_2(x)}dx$ finite for $x < \delta$, then $f(x) = ce^{\int -\frac{h_2(x)+g_2^{/}(x)}{g_2(x)}dx}$, where c is a constant determined by the condition $\int_\gamma^\delta f(x)dx = 1$.

Proof. Suppose that $E(h_2(X)|X \geq x) = g_2(x)r(x)$. Then, since $E(h_2(X)|X \geq x) = \frac{\int_x^\delta h_2(u)f(u)du}{1-F(x)}$ and $r(x) = \frac{f(x)}{1-F(x)}$, we have $g_2(x) = \frac{\int_x^\delta h_2(u)f(u)du}{f(x)}$, that is, $\int_x^\delta h_2(u)f(u)\,du = g_2(x)f(x)$.

Differentiating both sides of the above equation with respect to x, we obtain

$$-h_2(x)f(x) = g_2^{/}(x)f(x) + g_2(x)f^{/}(x)$$

From the above equation, on simplification, we obtain

$$\frac{f^{/}(x)}{f(x)} = -\frac{h_2(x) + g_2^{/}(x)}{g_2(x)}$$

Thus, on integrating the above equation with respect to x, we have

$$f(x) = ce^{\int -\frac{h_2(x)+g_2^{/}(x)}{g_2(x)}dx},$$

where c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$. This completes the proof of Lemma 3.2. □

Theorem 3.1: Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\sqrt{n} = \inf\{x|F(x) > 0\}$, $\infty = \sup\{x|F(x) < 1\}$, and $E\left(X^k\right)$ is finite for all x, where $\sqrt{n} < x < \infty$, with $0 < k < 1$ or $k < 0$. Then, $E\left(X^k|X \leq x\right) = g_1(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and

$$g_1(x) = \frac{A_1(x)}{A_0(x)} - \frac{B_1(x)}{A_0(x)},$$

where

$$A_0(x) = x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

$$A_1(x) = \sum_{j=0}^{\infty}\left[\frac{(-1)^j}{2(j!)}\left(\frac{n^2}{2}\right)_j (n)^{-\left(\frac{n^2}{2}+j\right)} x^k B\left(\frac{n(n+1)}{2}-1+j, 1\right)\left(x^2-n\right)^{\frac{n(n+1)}{2}-1+j}\right.$$
$$\left.\times {}_3F_2\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1, \frac{n(n+1)}{2}-1+j; \frac{n^2}{2}+\frac{n}{2}-1, \frac{n(n+1)}{2}+j; -\left(x^2-n\right)\right)\right],$$

$$B_1(x) = \int_{\sqrt{n}}^{x}\frac{k}{u}A_1(u)\,du,$$

if and only if

$$f(x) = \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

where $x \geq \sqrt{n}$, and $\mu = \frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)}$.

Proof. Suppose that we have

$$f(x) = \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

where $x \geq \sqrt{n}$, and $\mu = \frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)}$. Then, since $E\left(X^k|X \leq x\right) = \frac{\int_{\sqrt{n}}^x u^k f(u)du}{F(x)}$ and $\tau(x) = \frac{f(x)}{F(x)}$, we have $g_1(x) = \frac{\int_{\sqrt{n}}^x u^k f(u)du}{f(x)} = \frac{u^k F(u)|_{\sqrt{n}}^x}{f(x)} - \frac{k\int_{\sqrt{n}}^x u^{k-1}F(u)du}{f(x)} = \frac{A_1(x)}{A_0(x)} - \frac{B_1(x)}{A_0(x)}$, where

$$A_0(x) = x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

$$A_1(x) = \sum_{j=0}^{\infty}\left[\frac{(-1)^j}{2(j!)}\left(\frac{n^2}{2}\right)_j (n)^{-\left(\frac{n^2}{2}+j\right)} x^k B\left(\frac{n(n+1)}{2}-1+j, 1\right)\left(x^2-n\right)^{\frac{n(n+1)}{2}-1+j}\right.$$
$$\left.\times\, {}_3F_2\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1, \frac{n(n+1)}{2}-1+j; \frac{n^2}{2}+\frac{n}{2}-1, \frac{n(n+1)}{2}+j; -\left(x^2-n\right)\right)\right],$$

$$B_1(x) = \int_{\sqrt{n}}^{x} \frac{k}{u} A_1(u)\, du.$$

Conversely, suppose that

$$g_1(x) = \frac{A_1(x)}{A_0(x)} - \frac{B_1(x)}{A_0(x)},$$

where $A_0(x)$, $A_1(x)$, and $B_1(x)$ are given as above.

Then, differentiating both sides of the above equation with respect to x, we obtain

$$g_1^{/}(x) = x^k + \frac{k}{\mu}\frac{x^{k-1}F(x)}{A_0(x)} - \frac{A_1(x)}{A_0(x)}\frac{A_0^{/}(x)}{A_0(x)} - \frac{k}{\mu}\frac{x^{k-1}F(x)}{A_0(x)} + \frac{B_1(x)}{A_0(x)}\frac{A_0^{/}(x)}{A_0(x)} = x^k - \frac{A_1(x)}{A_0(x)}\frac{A_0^{/}(x)}{A_0(x)} + \frac{B_1(x)}{A_0(x)}\frac{A_0^{/}(x)}{A_0(x)},$$

where

$$\frac{A_0^{/}(x)}{A_0(x)} = \frac{1-n^2}{x} + \left(\frac{n(n+1)}{2}-2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}.$$

Thus,

$$g_1^{/}(x) = x^k - g_1(x)\left[\frac{1-n^2}{x} + \left(\frac{n(n+1)}{2}-2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}\right],$$

or,

$$\frac{x^k - g_1^{/}(x)}{g_1(x)} = \frac{1-n^2}{x} + \left(\frac{n(n+1)}{2}-2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}.$$

Thus, by Lemma 3.1, we have

$$\frac{f^{/}(x)}{f(x)} = \frac{1-n^2}{x} + \left(\frac{n(n+1)}{2}-2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}.$$

On integrating the above equation with respect to x, we obtain

$$f(x) = cx^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

where c is a constant to be determined. Thus, integrating the above equation with respect to x from $\sqrt{n}$ to ∞, using the condition $\int_{\sqrt{n}}^{\infty} f(x)\, dx = 1$, and following Edelman (1992), after simplification, we obtain $c = \frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)} = \mu$ (say), and hence

$$f(x) = \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right).$$

This completes the proof of Theorem 3.1. □

Theorem 3.2: Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that

$\sqrt{n} = \inf\{x|F(x) > 0\}$, $\infty = \sup\{x|F(x) < 1\}$, and $E\left(X^k\right)$ is finite for all x, where $\sqrt{n} < x < \infty$, with $0 < k < 1$ or $k < 0$. Then, $E\left(X^k|X \geq x\right) = g_2(x)\, r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $g_2(x) = \frac{C_1(x)}{C_0(x)} - \frac{D_1(x)}{C_0(x)}$, where

$$C_0(x) = x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

$$C_1(x) = x^k \left[\begin{array}{l} 1 - \sum_{j=0}^{\infty}\left\{\frac{(-1)^j}{2(j!)}\left(\frac{n^2}{2}\right)_j (n)^{-\left(\frac{n^2}{2}+j\right)} B\left(\frac{n(n+1)}{2}-1+j, 1\right)\left(x^2-n\right)^{\frac{n(n+1)}{2}-1+j}\right. \\ \left.\quad \times {}_3F_2\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1, \frac{n(n+1)}{2}-1+j; \frac{n^2}{2}+\frac{n}{2}-1, \frac{n(n+1)}{2}+j; -\left(x^2-n\right)\right)\right\} \end{array}\right],$$

$$D_1(x) = \int_x^{\infty}\frac{k}{u} C_1(u)\, du,$$

if and only if

$$f(x) = \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

where $x \geq \sqrt{n}$, and $\mu = \frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)}$.

Proof. Suppose that we have

$$f(x) = \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

where $x \geq \sqrt{n}$, and $\mu = \frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)}$. Then, since $E\left(X^k|X \geq x\right) = \frac{\int_x^{\infty} u^k f(u) du}{1-F(x)}$ and $r(x) = \frac{f(x)}{1-F(x)}$, we have

$$g_2(x) = \frac{\int_x^{\infty} u^k f(u)\, du}{f(x)} = \frac{-u^k(1-F(u))|_x^{\infty}}{f(x)} + \frac{k\int_x^{\infty} u^{k-1}(1-F(u))\, du}{f(x)} = \frac{C_1(x)}{C_0(x)} - \frac{D_1(x)}{C_0(x)},$$

where $C_0(x) = x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)$,

$$C_1(x) = x^k \left[\begin{array}{l} 1 - \sum_{j=0}^{\infty}\left\{\frac{(-1)^j}{2(j!)}\left(\frac{n^2}{2}\right)_j (n)^{-\left(\frac{n^2}{2}+j\right)} B\left(\frac{n(n+1)}{2}-1+j, 1\right)\left(x^2-n\right)^{\frac{n(n+1)}{2}-1+j}\right. \\ \left.\quad \times {}_3F_2\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1, \frac{n(n+1)}{2}-1+j; \frac{n^2}{2}+\frac{n}{2}-1, \frac{n(n+1)}{2}+j; -\left(x^2-n\right)\right)\right\} \end{array}\right],$$

$$D_1(x) = \int_x^{\infty}\frac{k}{u} C_1(u)\, du.$$

Conversely, suppose that

$g_2(x) = \frac{C_1(x)}{C_0(x)} - \frac{D_1(x)}{C_0(x)}$, where $C_0(x)$, $C_1(x)$, and $D_1(x)$ are given as above.

Then, differentiating both sides of the above equation with respect to x, we obtain

$$\begin{aligned} g_2^{/}(x) &= -x^k + \frac{k}{\mu}\frac{x^{k-1}(1-F(x))}{C_0(x)} - \frac{C_1(x)}{C_0(x)}\frac{C_0^{/}(x)}{C_0(x)} - \frac{k}{\mu}\frac{x^{k-1}(1-F(x))}{C_0(x)} + \frac{D_1(x)}{C_0(x)}\frac{C_0^{/}(x)}{C_0(x)} \\ &= -x^k - \frac{C_1(x)}{C_0(x)}\frac{C_0^{/}(x)}{C_0(x)} + \frac{D_1(x)}{C_0(x)}\frac{C_0^{/}(x)}{C_0(x)}, \end{aligned}$$

where

$$\frac{C_0^{/}(x)}{C_0(x)} = \frac{1-n^2}{x} + \left(\frac{n(n+1)}{2}-2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}.$$

Thus,

$$g_2^{/}(x) = -x^k - g_2(x)\left[\frac{1-n^2}{x} + \left(\frac{n(n+1)}{2} - 2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}\right],$$

or,

$$-\frac{x^k + g_2^{/}(x)}{g_2(x)} = \frac{1-n^2}{x} + \left(\frac{n(n+1)}{2} - 2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}.$$

Thus, by Lemma 3.2, we have

$$\frac{f^{/}(x)}{f(x)} = \frac{1-n^2}{x} + \left(\frac{n(n+1)}{2} - 2\right)\frac{2x}{x^2-n} + \frac{\frac{d}{dx}\left({}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)\right)}{{}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right)}.$$

On integrating the above equation with respect to x, we obtain

$$f(x) = cx^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right),$$

where c is a constant to be determined. Thus, integrating the above equation with respect to x from $\sqrt{n}$ to ∞, using the condition $\int_{\sqrt{n}}^{\infty} f(x)\,dx = 1$, and following Edelman (1992), after simplification, we obtain $c = \frac{2n\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n^2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n(n+1)}{2}-1\right)} = \mu$ (say), and hence

$$f(x) = \mu x^{1-n^2}\left(x^2-n\right)^{\frac{n(n+1)}{2}-2} {}_2F_1\left(\frac{n}{2}-\frac{1}{2}, \frac{n}{2}+1; \frac{n^2}{2}+\frac{n}{2}-1; -\left(x^2-n\right)\right).$$

This completes the proof of Theorem 3.2. □

4 Concluding Remarks

The distribution of the Demmel condition number (DCN) of real Wishart matrices occurs in a wide variety of applications in many branches of pure and applied sciences, and engineering. Based on the distributional properties of the distribution of the DCN, we have established some new characterizations of the distribution of the DCN of real Wishart matrices by truncated first moment. The reason for using the truncated first moment for characterizations is that the truncated distributions arise in practical statistics where the ability of record observations is limited to a given threshold or within a specified range. We hope the findings of our paper may be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions, and, as pointed out above, these characterizations may serve as a basis for parameter estimation. For example, in actuarial science, the credibility theory proposed by Buhlmann (1967) allows actuaries to estimate the conditional mean loss for a given risk to establish an adequate premium to cover the insured's loss. In their paper, Kim and Jeon (2013) have proposed a credibility theory based on truncation of the loss data, or the trimmed mean, which also contains the classical credibility theory of Buhlmann (1967) as a special case. Since the characterizations of probability distributions by truncated moments play an important part in the determination of distributions by using certain properties in the given data, it is hoped that the findings of our paper, combined with the proposed credibility theory of Kim and Jeon (2013) based on truncation of the loss data, or the trimmed mean, may be useful for researchers in the fields of probability, statistics, and other applied sciences. In view of these assertions, the interested readers are strongly recommended to the papers of Buhlmann (1967) and Kim and Jeon (2013) for some numerical illustration based on truncation of the loss data, or the trimmed mean, and how our proposed characterizations may be applied to these problems in credibility theory.

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