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On Birkhoff–James and Roberts orthogonality

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Abstract: In this paper we present some recent results on characterizations of the Birkhoff–James and the Roberts orthogonality in C^* -algebras and Hilbert C^* -modules.

Keywords: normed linear space, Birkhoff–James orthogonality, Roberts orthogonality, C^* -algebra, numerical range, Davis–Wielandt shell

1 Introduction and preliminaries

Let X be a complex inner product space. We say that two elements $x, y \in X$ are *orthogonal*, we write $x \perp y$, if their inner product is zero. If $\|\cdot\|$ is the norm induced by the inner product $(\cdot, \cdot)$ on X , then the relation $(x, y) = 0$ can be written in terms of this norm in different ways, for example, as:

$$\|x + \lambda y\| \geq \|x\|, \quad \forall \lambda \in \mathbb{C}, \quad (1)$$

or as

$$\|x + \lambda y\| = \|x - \lambda y\|, \quad \forall \lambda \in \mathbb{C}. \quad (2)$$

Since these relations make sense in every normed linear space, not necessarily an inner product space, we can use them to extend the concept of orthogonality to normed linear spaces. Some of the most important concepts of orthogonality in normed linear spaces are the Birkhoff–James orthogonality and the Roberts orthogonality, and they are defined by the above relations: if x, y are elements of a complex normed linear space X that satisfy (1), we say that x is *Birkhoff–James orthogonal* to y [14, 21–23], and if they satisfy (2) then we say that x and y are *Roberts orthogonal* [38]. We write $x \perp_B y$ and $x \perp_R y$, respectively.

A classification of different types of orthogonality in normed linear spaces, their main properties, and the relations between them can be found in e.g. survey papers [2, 3]. In this paper, we focus on two types of orthogonality: the Birkhoff–James and the Roberts orthogonality. We only recall some basic properties of these two types of orthogonality in arbitrary normed linear spaces. A relationship between the Birkhoff–James orthogonality and the triangle equality for elements of a normed linear space is also discussed. Then we survey results on characterizations of these orthogonalities in some special kinds of complex normed linear spaces, that is, in the algebra $\mathbb{B}(H)$ of bounded linear operators on a Hilbert space H , and in a more general setting of C^* -algebras. Thereby, the Birkhoff–James orthogonality is described for arbitrary elements a and b of a C^* -algebra $\mathcal{A}$, while the Roberts orthogonality is considered in a special case, when a (or b) is the unit in $\mathcal{A}$. We also summarize some known results on the Birkhoff–James orthogonality in Hilbert C^* -modules, that is, in the context which generalizes both Hilbert spaces and C^* -algebras.

In the rest of the section we introduce some notation and definitions we shall need in the sequel.

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Recall that a C^* -algebra $\mathcal{A}$ is a Banach $*$ -algebra with the norm satisfying the C^* -condition $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. By $\operatorname{Re} a$ and $\operatorname{Im} a$ we denote the *real* and the *imaginary part* of $a \in \mathcal{A}$, respectively, that is,

$$\operatorname{Re} a = \frac{1}{2}(a + a^*), \quad \operatorname{Im} a = \frac{1}{2i}(a - a^*).$$

By $\sigma(a)$ we denote the spectrum of $a \in \mathcal{A}$. An element $a \in \mathcal{A}$ is *positive*, in short, $a \geq 0$, if a is self-adjoint and $\sigma(a) \subseteq [0, \infty)$. By $\mathcal{A}'$ we denote the dual space of $\mathcal{A}$. A *positive linear functional* on $\mathcal{A}$ is a map $\varphi \in \mathcal{A}'$ such that $\varphi(a) \geq 0$ whenever $a \geq 0$. A *state* of $\mathcal{A}$ is a positive linear functional on $\mathcal{A}$ of norm 1. The set of all states of $\mathcal{A}$ is denoted by $S(\mathcal{A})$. For comprehensive study of C^* -algebras we refer the reader to [19, 33, 36].

By $\mathbb{B}(X)$ we denote the algebra of all bounded linear operators acting on some normed linear space X . The algebra of all complex $n \times n$ matrices is denoted by $\mathbb{M}_n(\mathbb{C})$. The identity operator on X , as well as the identity matrix, will be denoted by I . We denote by $\operatorname{conv}(S)$ the convex hull of a subset S of $\mathbb{C}^n$.

2 The Birkhoff–James orthogonality

Let $(X, (\cdot, \cdot))$ be a complex inner product space and $\perp$ the orthogonality defined by inner product. It follows directly from the properties of the inner product that $\perp$ is:

- nondegenerate: $x \perp x \Leftrightarrow x = 0$;
- homogeneous: $x \perp y \Rightarrow \lambda x \perp \mu y, \forall \lambda, \mu \in \mathbb{C}$;
- symmetric: $x \perp y \Leftrightarrow y \perp x$;
- right additive: $(x \perp y \text{ and } x \perp z) \Rightarrow x \perp (y + z)$;
- left additive: $(x \perp y \text{ and } z \perp y) \Rightarrow (x + z) \perp y$;
- existent: for any two elements x and y there is $\lambda \in \mathbb{C}$ such that $x \perp (\lambda x + y)$.

Whenever we consider some type of orthogonality in a normed linear space, it is interesting to see which of these properties remain true. In an arbitrary complex normed linear space X , the Birkhoff–James orthogonality is nondegenerate, homogeneous, existent. The first two properties are obvious, while the existence property, proved in [22, Corollary 2.2], is an immediate consequence of the following theorem (see [22, Theorem 2.1]).

Theorem 2.1. *Let X be a normed linear space and $x, y \in X$. Then $x \perp_B y$ if and only if there is a norm one linear functional f on X such that $f(x) = \|x\|$ and $f(y) = 0$.*

Let us mention that the proof of the preceding theorem follows directly from the Hahn–Banach theorem, since the relation $x \perp_B y$ actually means that the distance from x to the space spanned by y is $\|x\|$.

The Birkhoff–James orthogonality is neither symmetric nor additive. For example, if we take $x = (1, 1)$ and $y = (-1, 0)$ in the space $\mathbb{C}^2$ with the max-norm, then $x \perp_B y$ but $y \not\perp_B x$. For nonadditivity, we consider again the space $\mathbb{C}^2$ with the max-norm, and if we take $x = (1, 1)$, $y = (-1, 0)$ and $z = (0, -1)$ then $x \perp_B y$ and $x \perp_B z$ but $x \not\perp_B (y + z)$, and also $x \perp_B y$ and $z \perp_B y$ but $(x + z) \not\perp_B y$.

Remark 2.2. *It is well known that in every normed linear space X the triangle inequality*

$$\|x + y\| \leq \|x\| + \|y\| \tag{3}$$

holds for all $x, y \in X$. The problem when the equality in (3) holds has been studied for certain types of normed linear spaces (see e.g. [1, 6, 10, 29, 34]). By the Hahn–Banach theorem, it can be easily shown that for elements x and y of a general normed linear space X , the equality is attained in (3) precisely when there is a norm one linear functional f on X such that $f(x) = \|x\|$ and $f(y) = \|y\|$. Comparing this to Theorem 2.1, one gets a characterization of the case of equality in (3) in terms of the Birkhoff–James orthogonality - it was proved in [5, Proposition 4.1] that the following statements are equivalent for two elements x and y of a normed linear space X :

- (a) $\|x + y\| = \|x\| + \|y\|$;

- (b) $x \perp_B (\|y\|x - \|x\|y)$;
 (c) $y \perp_B (\|y\|x - \|x\|y)$.

A similar result was obtained in Theorem 2.4 of [32] where the Birkhoff–James orthogonality was expressed in terms of the norm-parallelism.

Theorem 2.1 holds in a general normed linear space X . If X is a normed space of a special form, then we can obtain some further characterizations.

Let $X = \mathbb{B}(H)$ with the operator norm, that is, the algebra of all bounded linear operators on a complex Hilbert space H . The following result is the content of Theorem 1.1 and Remark 3.1 of [12].

Theorem 2.3 ([12]). *Let $A, B \in \mathbb{B}(H)$. Then $A \perp_B B$ if and only if there is a sequence of unit vectors $(\xi_n)_n$ in H such that*

$$\lim_{n \rightarrow \infty} \|A\xi_n\| = \|A\| \quad \text{and} \quad \lim_{n \rightarrow \infty} (A\xi_n, B\xi_n) = 0.$$

In particular, if $\dim H < \infty$, then $A \perp_B B$ if and only if there is a unit vector $\xi \in H$ such that $\|A\xi\| = \|A\|$ and $(A\xi, B\xi) = 0$.

Observe that the finite-dimensional case of the preceding theorem can be also stated as follows: $A \perp_B B$ if and only if there is a unit vector $\xi \in H$ such that

$$\|A\xi\| = \|A\| \quad \text{and} \quad \|A\xi + \lambda B\xi\| \geq \|A\xi\|, \quad \forall \lambda \in \mathbb{C}. \quad (4)$$

Since (4) is stated without use of the inner product on H , it is natural to ask whether the same characterization holds if a finite-dimensional normed linear space X is regarded with respect to an arbitrary norm $\|\cdot\|$ on X , and linear operators A and B on X with respect to the operator norm induced by that norm. In Example 4.3 of [28], it was shown that this is not true for the space $\mathbb{M}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, regarded with the operator norm induced by ℓ_p norm with $p \neq 2$. The same question was discussed in [13] for real normed linear spaces, where the following result was proved.

Theorem 2.4 ([13]). *A real finite-dimensional normed linear space X is an inner product space if and only if, for $A, B \in \mathbb{B}(X)$ it holds that $A \perp_B B$ (i.e., $\|A + \lambda B\| \geq \|A\|$ for all $\lambda \in \mathbb{R}$) if and only if there exists a unit vector $\xi \in X$ such that*

$$\|A\xi\| = \|A\| \quad \text{and} \quad \|A\xi + \lambda B\xi\| \geq \|A\xi\|, \quad \forall \lambda \in \mathbb{R}.$$

A lot of work has been done in a similar direction, see e.g. [39] and the references therein. We refer the reader to [11, 25] for different approaches to Theorem 2.3. A generalization in another direction to Theorem 2.3 can be seen in [20].

We proceed with a characterization of the Birkhoff–James orthogonality of elements of a C^* -algebra. First characterizations of the Birkhoff–James orthogonality of elements of a unital C^* -algebra $\mathcal{A}$ were obtained by Stampfli and Williams in [40, 41]; in these papers the authors described, in terms of the numerical range, those elements of a C^* -algebra which are orthogonal to the unit or to which the unit is orthogonal.

The *numerical range* of $a \in \mathcal{A}$ is defined as the set

$$V(a) = \{\varphi(a) : \varphi \in S(\mathcal{A})\}.$$

It is well known that $V(a)$ is a convex compact set which contains $\sigma(a)$. If $a \in \mathcal{A}$ is normal, then $V(a) = \text{conv}(\sigma(a))$ (see [40]). The *maximal numerical range* of $a \in \mathcal{A}$ is the subset of $V(a)$ defined as

$$V_{\max}(a) = \{\varphi(a) : \varphi \in S(\mathcal{A}), \varphi(a^*a) = \|a\|^2\}.$$

For details about numerical ranges we refer to [15, 16].

Theorem 2.5 ([40, 41]). *Let $\mathcal{A}$ be a C^* -algebra with the unit e . Then*

1. $e \perp_B a \Leftrightarrow 0 \in V(a)$;

$$2. \quad a \perp_B e \Leftrightarrow 0 \in V_{\max}(a).$$

Theorem 2.5 was generalized for arbitrary elements of $\mathcal{A}$ in [5], where a characterization of the Birkhoff–James orthogonality was obtained in terms of states acting on $\mathcal{A}$.

Theorem 2.6 ([5]). *Let $\mathcal{A}$ be a C^* -algebra. Let $a, b \in \mathcal{A}$. Then $a \perp_B b$ if and only if there is $\varphi \in S(\mathcal{A})$ such that $\varphi(a^*a) = \|a\|^2$ and $\varphi(a^*b) = 0$.*

The next step was to extend the above results in the context of Hilbert C^* -modules. Hilbert C^* -modules generalize Hilbert spaces by allowing the inner product to take values in a general C^* -algebra. This generalization, in the case of commutative C^* -algebras, appeared in the paper [24] of Kaplansky, while the noncommutative case was first considered in the papers of Paschke [35] and Rieffel [37]. By definition, a *Hilbert C^* -module V over a C^* -algebra $\mathcal{A}$* (or a (right) *Hilbert $\mathcal{A}$ -module*) is a (right) $\mathcal{A}$ -module equipped with an $\mathcal{A}$ -valued inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathcal{A}$ with the following properties:

- (1) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for $x, y, z \in V$, $\alpha, \beta \in \mathbb{C}$,
- (2) $\langle x, ya \rangle = \langle x, y \rangle a$ for $x, y \in V$, $a \in \mathcal{A}$,
- (3) $\langle x, y \rangle^* = \langle y, x \rangle$ for $x, y \in V$,
- (4) $\langle x, x \rangle \geq 0$ for $x \in V$; if $\langle x, x \rangle = 0$ then $x = 0$;

and such that V is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

Note that Hilbert spaces can be regarded as Hilbert $\mathbb{C}$ -modules. Every C^* -algebra $\mathcal{A}$ can be regarded as a Hilbert C^* -module over itself, where the inner product is defined by $\langle a, b \rangle = a^*b$; so the corresponding norm is just the norm on $\mathcal{A}$ because of the C^* -condition (i.e., $\|a^*a\| = \|a\|^2$, $a \in \mathcal{A}$). For comprehensive study of Hilbert C^* -modules the reader is referred to e.g. the books [26, 30, 42].

A characterization of the Birkhoff–James orthogonality in Hilbert $\mathcal{A}$ -modules was obtained in [5] (see also [11]), and it is given in terms of states acting on the underlying C^* -algebra $\mathcal{A}$.

Theorem 2.7 ([5]). *Let V be a Hilbert C^* -module over a C^* -algebra $\mathcal{A}$. Let $x, y \in V$. Then $x \perp_B y$ if and only if there is $\varphi \in S(\mathcal{A})$ such that $\varphi(\langle x, x \rangle) = \|x\|^2$ and $\varphi(\langle x, y \rangle) = 0$.*

Remark 2.8. *As a consequence of Theorem 2.7 and the equivalence $\|x + y\| = \|x\| + \|y\| \Leftrightarrow x \perp_B (\|y\|x - \|x\|y)$ from Remark 2.2, one can easily get a characterization of the triangle equality for elements x and y of a Hilbert C^* -module V in terms of the numerical range of $\langle x, y \rangle$, that is,*

$$\|x + y\| = \|x\| + \|y\| \Leftrightarrow \|x\|\|y\| \in V(\langle x, y \rangle). \quad (5)$$

Let us say that the characterization (5) was first obtained in [6, Theorem 2.1] by using a different technique. We also remark here that the equivalence $\|x + y\| = \|x\| + \|y\| \Leftrightarrow x \perp_B (\|y\|x - \|x\|y)$ is not sufficient to derive Theorem 2.7 from (5). Namely, if $x \perp_B z$, then it can happen that there does not exist y such that $z = \|y\|x - \|x\|y$.

For example, let us consider the elements $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and I of $\mathbb{M}_2(\mathbb{C})$ (regarded as a Hilbert C^* -module over itself). Then we have $I \perp_B A$. Suppose there is $B \in \mathbb{M}_2(\mathbb{C})$ such that $A = \|B\|I - B$. Then

$$B = \|B\|I - A = \begin{pmatrix} \|B\| + 1 & 0 \\ 0 & \|B\| - 1 \end{pmatrix},$$

from which it follows that $\|B\| + 1 \in \sigma(B)$, which cannot be true since $\|B\| + 1 > \|B\|$.

Remark 2.9. *In Hilbert C^* -modules the role of scalars is played by the elements of the underlying C^* -algebra. The strong Birkhoff–James orthogonality is a generalization of the Birkhoff–James orthogonality which involves a modular structure of Hilbert C^* -modules: an element x of a Hilbert $\mathcal{A}$ -module V is strongly Birkhoff–James orthogonal to $y \in V$ if $\|x + ya\| \geq \|x\|$ for every $a \in \mathcal{A}$. This type of orthogonality was introduced and considered in [7] (see also [8, 31]).*

3 The Roberts orthogonality

Let us first observe that the Roberts orthogonality is stronger than the Birkhoff–James orthogonality; namely, if x and y are elements of a complex normed linear space X such that $x \perp_R y$, then for all $\lambda \in \mathbb{C}$ we have

$$2\|x\| = \|(x + \lambda y) + (x - \lambda y)\| \leq \|x + \lambda y\| + \|x - \lambda y\| = 2\|x + \lambda y\|,$$

so $x \perp_B y$. Since (2) is a symmetric relation, we also have that $y \perp_B x$.

The Roberts orthogonality is evidently nondegenerate, homogeneous, and symmetric. It is neither additive nor existent, as we shall see later (Examples 3.3 and 3.4).

While the Birkhoff–James orthogonality was described in full generality for arbitrary elements of a C^* -algebra, the only known characterization of the Roberts orthogonality in a C^* -algebra is when one of the elements is the unit of a C^* -algebra. In the rest of this section, $\mathcal{A}$ will denote a C^* -algebra with the unit e . As we have mentioned before, for $a \in \mathcal{A}$ it holds $e \perp_B a \Leftrightarrow 0 \in V(a)$. Since $a \perp_R e$ implies $e \perp_B a$, it follows that the numerical range $V(a)$ of a contains zero if $a \perp_R e$. It is natural to ask ourselves whether in this special case, when one of the elements is the unit of $\mathcal{A}$, the Roberts orthogonality can also be described in terms of the numerical range.

Let us start with an easy case, when $a \in \mathcal{A}$ is normal. Then $w(a) = \|a\|$, where $w(a)$ denotes the *numerical radius* of a , that is,

$$w(a) = \max\{|z| : z \in V(a)\}.$$

Since, $a + \lambda e$ is a normal element of $\mathcal{A}$ for every $\lambda \in \mathbb{C}$, we also have $w(a + \lambda e) = \|a + \lambda e\|$ for every $\lambda \in \mathbb{C}$. It means that the Roberts orthogonality can be described in terms of the numerical radius, that is,

$$a \perp_R e \Leftrightarrow w(a + \lambda e) = w(a - \lambda e), \forall \lambda \in \mathbb{C}. \quad (6)$$

Suppose that $V(a)$ is a symmetric set with respect to the origin, that is, $V(a) = -V(a)$. Then we have $V(a + \lambda e) = -V(a - \lambda e)$ for every $\lambda \in \mathbb{C}$, and therefore $w(a + \lambda e) = w(a - \lambda e)$ for every $\lambda \in \mathbb{C}$. By (6), we conclude that $a \perp_R e$.

What about the converse: does $a \perp_R e$ imply $V(a) = -V(a)$? It is easy to check that this implication is true when a is self-adjoint. In this case, we have

$$V(a) = \text{conv}(\sigma(a)) = [\alpha, \beta] \subseteq [-\|a\|, \|a\|],$$

where $\alpha = -\|a\|$ or $\beta = \|a\|$. Suppose that $a \perp_R e$. Then for every $\lambda \in \mathbb{C}$

$$\max\{|\alpha + \lambda|, |\beta + \lambda|\} = w(a + \lambda e) = w(a - \lambda e) = \max\{|\alpha - \lambda|, |\beta - \lambda|\},$$

from which it follows that $\alpha = -\beta$. Thus, $V(a) = [-\|a\|, \|a\|]$, and therefore $V(a) = -V(a)$. In Proposition 2.1 of [4], it was proved that this implication is also true in general, that is, the following result holds.

Proposition 3.1 ([4]). *Let $\mathcal{A}$ be a C^* -algebra with the unit e , and $a \in \mathcal{A}$. If $a \perp_R e$, then $V(a) = -V(a)$.*

By Proposition 3.1, and our previous discussion, the following result on characterization of the Roberts orthogonality $a \perp_R e$ for normal $a \in \mathcal{A}$ immediately follows.

Proposition 3.2. *Let $\mathcal{A}$ be a C^* -algebra with the unit e , and $a \in \mathcal{A}$ normal. Then $a \perp_R e$ if and only if $V(a) = -V(a)$. In particular, if a is self-adjoint, then $a \perp_R e$ if and only if $\pm\|a\| \in \sigma(a)$.*

We shall now use Proposition 3.2 to construct two examples which illustrate that the Roberts orthogonality is neither additive nor existent. The first one shows that $\perp_R$ is not additive.

Example 3.3. *Consider the C^* -algebra $C([-1, 1])$ of all continuous functions on $[-1, 1]$ with respect to max norm. Let $f_1, f_2 \in C([-1, 1])$ be defined as $f_1(x) = x$ and $f_2(x) = |x| - \frac{1}{2}$. Then f_1 and f_2 are self-adjoint elements of $C([-1, 1])$ such that $\sigma(f_1) = [-1, 1]$ and $\sigma(f_2) = [-\frac{1}{2}, \frac{1}{2}]$.*

By Proposition 3.2, $f_1 \perp_R 1$ and $f_2 \perp_R 1$, where 1 stands for the constant function on $[-1, 1]$ (which is the unit element of the regarded C^* -algebra). However, $(f_1 + f_2) \not\perp_R 1$, since $\sigma(f_1 + f_2) = [-\frac{1}{2}, \frac{3}{2}]$.

The following example shows that the Roberts orthogonality is not existent.

Example 3.4. Let a be a normal element of a C^* -algebra $\mathcal{A}$ such that $\sigma(a) = \{0, 2, i\}$. Then $V(a) = \text{conv}(\sigma(a))$, which is the triangle with vertices at $0, 2, i$. Note that for every $\lambda \in \mathbb{C}$ the set $V(\lambda e + a) = \lambda + V(a)$ is not symmetric with respect to the origin. Therefore, by Proposition 3.2, $(\lambda e + a) \not\perp_R e$ for all $\lambda \in \mathbb{C}$.

Proposition 3.1 states that in the general case (when a is not necessarily normal) the symmetry of the numerical range with respect to the origin is a necessary condition for the Roberts orthogonality of a and e . However, as the following example shows, it is not a sufficient condition.

Example 3.5. Using [17, Theorem 2] we obtain that the numerical range $V(A)$ of a matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

is a circular disk centered at the origin. However, by direct calculation we see that $\|A + I\|$ is 2.1617, and $\|A - I\|$ is 2.1366, both rounded to 4 decimal places. Therefore, $A \not\perp_R I$.

Let us see what happens in the general case. Let $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. There is $\varphi \in S(\mathcal{A})$ such that

$$\|a + \lambda e\|^2 = \|(a + \lambda e)^*(a + \lambda e)\| = \varphi((a + \lambda e)^*(a + \lambda e)).$$

Further, for each state ψ of $\mathcal{A}$ we have $\|a + \lambda e\|^2 \geq \psi((a + \lambda e)^*(a + \lambda e))$, so we get

$$\varphi(a^*a) \geq \psi(a^*a) + 2\text{Re}(\bar{\lambda}(\psi(a) - \varphi(a)))$$

for all $\psi \in S(\mathcal{A})$. Let us denote $\mu := \varphi(a)$. Then

$$\varphi(a^*a) \geq \psi(a^*a) \text{ for all } \psi \in S(\mathcal{A}) \text{ such that } \psi(a) = \mu. \quad (7)$$

Therefore,

$$\varphi(a^*a) = \max \mathcal{L}_\mu(a),$$

where

$$\mathcal{L}_\mu(a) = \{\psi(a^*a) : \psi \in S(\mathcal{A}), \psi(a) = \mu\}$$

is the set which is connected with the Davis–Wielandt shell of a .

Recall that the *Davis–Wielandt shell* of $a \in \mathcal{A}$ is defined as the set

$$DV(a) = \{(\varphi(a), \varphi(a^*a)) : \varphi \in S(\mathcal{A})\}.$$

Note that the projection of $DV(a)$ on the first coordinate is the set $V(a)$.

Since $S(\mathcal{A})$ is a weak*-compact and convex subset of $\mathcal{A}'$, and the map $\varphi \mapsto (\varphi(a), \varphi(a^*a))$ is weak*-continuous on $\mathcal{A}'$, we conclude that $DV(a)$ is a compact convex subset of $\mathbb{C} \times \mathbb{R}$.

The *upper boundary* of $DV(a)$ is the set

$$DV_{ub}(a) = \{(\mu, r) \in DV(a) : r = \max \mathcal{L}_\mu(a)\}.$$

So, if $\varphi \in S(\mathcal{A})$ is such that $\|a + \lambda e\|^2 = \varphi((a + \lambda e)^*(a + \lambda e))$ for some $\lambda \in \mathbb{C}$, then it follows from (7) that $(\varphi(a), \varphi(a^*a)) \in DV_{ub}(a)$. So, it was natural to expect that the characterization of the Roberts orthogonality of a and e in general case can be stated in terms of the Davis–Wielandt shell and, as the case of normal elements suggests, some kind of its symmetry. It turns out that the following theorem holds.

Theorem 3.6 ([4]). *Let $\mathcal{A}$ be a C^* -algebra with the unit e and $a \in \mathcal{A}$. Then $a \perp_R e$ if and only if $DV_{ub}(a) = DV_{ub}(-a)$.*

In the rest of the paper we consider the case when $\mathcal{A} = \mathbb{B}(H)$ is the C^* -algebra of all bounded linear operators acting on a complex Hilbert space H . The *Davis–Wielandt shell* of $A \in \mathbb{B}(H)$ is defined as the set

$$DW(A) = \{((A\xi, \xi), (A^*A\xi, \xi)) : \xi \in H, \|\xi\| = 1\}.$$

Since $A = \operatorname{Re} A + i\operatorname{Im} A$, identifying $\mathbb{C} \times \mathbb{R}$ with $\mathbb{R}^3$, we have

$$DW(A) = \{(((\operatorname{Re} A)\xi, \xi), ((\operatorname{Im} A)\xi, \xi), (A^*A\xi, \xi)) : \xi \in H, \|\xi\| = 1\},$$

which is a *joint numerical range* of self-adjoint operators $\operatorname{Re} A$, $\operatorname{Im} A$ and A^*A . The set $DW(A)$ is compact if $\dim H < \infty$, and it is convex if $\dim H \geq 3$ (see [9]). It is known that the set of all states of a unital C^* -algebra $\mathbb{B}(H)$ is a weak*-closed convex hull of the set of all vector states of $\mathbb{B}(H)$, i.e., the states of $\mathbb{B}(H)$ of the form $T \mapsto (T\xi, \xi)$ for some unit vector ξ in H . Thus, for $A \in \mathbb{B}(H)$, we have $DV(A) \subseteq \overline{\operatorname{conv}(DW(A))}$. On the other hand, since for every unit vector $\xi \in H$, the map $T \mapsto (T\xi, \xi)$ is a state of $\mathbb{B}(H)$, it holds $DW(A) \subseteq DV(A)$. Then the convexity and compactness of $DV(A)$ imply $\overline{\operatorname{conv}(DW(A))} \subseteq DV(A)$. Hence

$$DV(A) = \overline{\operatorname{conv}(DW(A))}.$$

Therefore $DV(A) = \overline{DW(A)}$ if H is infinite-dimensional, and $DV(A) = DW(A)$ if $3 \leq \dim H < \infty$. Thus, by Theorem 3.6, $A \perp_R I$ if and only if $DV_{ub}(A) = DV_{ub}(-A)$, where

$$DV_{ub}(A) = \{(\mu, r) \in \overline{DW(A)} : r = \max \mathcal{L}_\mu(A)\},$$

$$\mathcal{L}_\mu(A) = \left\{ \lim_{n \rightarrow \infty} (A^*A\xi_n, \xi_n) : \xi_n \in H, \|\xi_n\| = 1, \lim_{n \rightarrow \infty} (A\xi_n, \xi_n) = \mu \right\}$$

if H is infinite-dimensional, while

$$DV_{ub}(A) = \{(\mu, r) \in DW(A) : r = \max \mathcal{L}_\mu(A)\},$$

where

$$\mathcal{L}_\mu(A) = \{(A^*A\xi, \xi) : \xi \in H, \|\xi\| = 1, (A\xi, \xi) = \mu\}$$

if $3 \leq \dim H < \infty$.

It remains to see what happens in 2-dimensional case. So, let $\dim H = 2$ and $A \in \mathbb{B}(H)$. Let α and β be eigenvalues of A . Then the classical numerical range

$$W(A) = \{(A\xi, \xi) : \xi \in H, \|\xi\| = 1\}$$

is an elliptical disc (possibly degenerate) centered at $\frac{1}{2}\operatorname{tr}(A)$ with foci α and β , and the Davis–Wielandt shell $DW(A)$ is an ellipsoid without the interior centered at $\left(\frac{\operatorname{tr}(A)}{2}, \frac{\operatorname{tr}(A^*A)}{2}\right)$ (see [18, 27]). (Here $\operatorname{tr}(T)$ stands for the trace of $T \in \mathbb{B}(H)$ with respect to some fixed orthonormal basis of H .)

In this case, the Roberts orthogonality of A and I can be described in terms of the classical numerical range $W(A)$ of A . Indeed, since A acts on finite-dimensional Hilbert space, it holds $V(A) = W(A)$ (see [40]). Thus, by Proposition 3.1, $A \perp_R I$ implies $W(A) = -W(A)$. Conversely, assume that $W(A) = -W(A)$. Then $W(A + \lambda I) = W(-A + \lambda I)$ for all $\lambda \in \mathbb{C}$. Then, for every $\lambda \in \mathbb{C}$, there exists a unitary $U_\lambda \in \mathbb{B}(H)$ such that $A + \lambda I = U_\lambda^*(-A + \lambda I)U_\lambda$ (see [27, Theorem 3.1]), from which it follows that $\|A + \lambda I\| = \|A - \lambda I\|$, that is, $A \perp_R I$. Let us also note that, since $W(A)$ is an elliptical disc centered at $\frac{1}{2}\operatorname{tr}(A)$, the condition $W(A) = -W(A)$ is satisfied precisely when $\operatorname{tr}(A) = 0$.

So, in the case of $\mathbb{B}(H)$ we have the following characterization.

Theorem 3.7. *Let $A \in \mathbb{B}(H)$.*

- (i) *If $\dim H = 2$, then $A \perp_R I \Leftrightarrow W(A) = -W(A) \Leftrightarrow \operatorname{tr} A = 0$.*
- (ii) *If $\dim H \geq 3$, then $A \perp_R I \Leftrightarrow DV_{ub}(A) = DV_{ub}(-A)$.*

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