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The Arithmetic Tutte polynomial of two matrices associated to Trees

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Abstract: Arithmetic matroids arising from a list $\mathcal{A}$ of integral vectors in $\mathbb{Z}^n$ are of recent interest and the arithmetic Tutte polynomial $M_{\mathcal{A}}(x, y)$ of $\mathcal{A}$ is a fundamental invariant with deep connections to several areas. In this work, we consider two lists of vectors coming from the rows of matrices associated to a tree T . Let $T = (V, E)$ be a tree with $|V| = n$ and let $\mathcal{L}_T$ be the q -analogue of its Laplacian L in the variable q . Assign $q = r$ for $r \in \mathbb{Z}$ with $r \neq 0, \pm 1$ and treat the n rows of $\mathcal{L}_T$ after this assignment as a list containing elements of $\mathbb{Z}^n$. We give a formula for the arithmetic Tutte polynomial $M_{\mathcal{L}_T}(x, y)$ of this list and show that it depends only on n, r and is independent of the structure of T . An analogous result holds for another polynomial matrix associated to T : ED_T , the $n \times n$ exponential distance matrix of T . More generally, we give formulae for the multivariate arithmetic Tutte polynomials associated to the list of row vectors of these two matrices which shows that even the multivariate arithmetic Tutte polynomial is independent of the tree T .

As a corollary, we get the Ehrhart polynomials of the following zonotopes:

- Z_{ED_T} obtained from the rows of ED_T and
- $Z_{\mathcal{L}_T}$ obtained from the rows of $\mathcal{L}_T$.

Further, we explicitly find the maximum volume ellipsoid contained in the zonotopes $Z_{\text{ED}_T}, Z_{\mathcal{L}_T}$ and show that the volume of these ellipsoids are again tree independent for fixed n, q . A similar result holds for the minimum volume ellipsoid containing these zonotopes.

MSC: 05C05, 52A40, 05B35

1 Introduction

Computing the arithmetic Tutte polynomial $M_{\mathcal{A}}(x, y)$ of a list $\mathcal{A}$ of vectors in $\mathbb{Z}^n$ is a topic of recent interest due to its connections with several areas. These include invariants of toric arrangements (see D’Adderio and Moci [13]), the dimension of the Dahmen-Micchelli space (see [13]), the dimension of the De Concini-Procesi-Vergne space (see De Concini, Procesi and Vergne [16]), arithmetic colorings and arithmetic flows in graphs (see D’Adderio and Moci [15]), the Ehrhart polynomial of the zonotope $Z_{\mathcal{A}}$ (see Stanley [25], D’Adderio and Moci [12]) as well as the Ehrhart theory of Lawrence polytopes (see Stapledon [27]).

When $\mathcal{A}$ is the list of positive roots of classical root systems, Ardila et al [1] have found its arithmetic Tutte polynomial $M_{\mathcal{A}}(x, y)$. In this work, we consider lists that arise from the rows of two matrices with entries being polynomials in the variable q , associated to a tree T on n vertices. Let $\mathcal{L}_T$ be the q -analogue of T ’s Laplacian L and let ED_T be the exponential distance matrix of T . Both are $n \times n$ matrices (see subsection 2.2 for definitions). We set $q = r$ for $r \in \mathbb{Z}$ and after this, treat the rows of both matrices as a list $\mathcal{A}$ of integral vectors. We forbid three values, namely $q = 0, \pm 1$ for minor technical issues (see subsection 2.2 for our reasons). In the literature, the column vectors of an integral matrix M are usually used to define a list. In this work, we prefer to use the

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row vectors of integral matrices to define lists. A list of such integral vectors defines an arithmetic matroid and we give a formula for the arithmetic Tutte polynomials $M_{\mathcal{L}_T}(x, y)$ and $M_{\text{ED}_T}(x, y)$. More generally, we also give formulae for the multivariate arithmetic Tutte polynomials $\text{Multi}_{\mathcal{L}_T}(t, \mathbf{v})$ and $\text{Multi}_{\text{ED}_T}(t, \mathbf{v})$. Though our results should involve n and r (the value set to q), we give our results as formulae involving n and q . It will then be straightforward to substitute $q = r$ for integral values $r \in \mathbb{Z}$ in our formulae to get the respective arithmetic Tutte polynomial when $q = r$.

Computing the arithmetic Tutte polynomial of $\mathcal{A}$, $M_{\mathcal{A}}(x, y)$ is #P-hard in general and the finite field method is often used to obtain the arithmetic coboundary polynomial of $\mathcal{A}$ which is a polynomial that is equivalent to $M_{\mathcal{A}}(x, y)$. For both arithmetic matroids that we consider, the underlying lattice is $\mathbb{Z}^n$ and we use the GCD rule to assign multiplicities $m(\mathcal{B})$ for $\mathcal{B} \subseteq \mathcal{A}$. Thus $m(\mathcal{B})$ equals the greatest common divisor of the maximal rank minors of $\mathcal{B}$. Using this multiplicity function enables us to give a bare-hands computation of $M_{\mathcal{A}}(x, y)$ without using the finite field method. Our main results are Theorems 7 and 8 for the list of rows of $\mathcal{L}_T$ and Theorems 12 and 13 for the list of rows of ED_T . These results show that both the arithmetic Tutte polynomial and the multivariate arithmetic Tutte polynomial are independent of the underlying tree. These results are proved in Section 3.

Each arithmetic matroid $\mathcal{A}$ naturally gives rise to a zonotope $Z_{\mathcal{A}}$ whose Ehrhart polynomial $\text{Ehr}_{Z_{\mathcal{A}}}(X)$ (a polynomial in the variable X) is a specialization of $M_{\mathcal{A}}(x, y)$. The polynomial $\text{Ehr}_{Z_{\mathcal{A}}}(X)$ contains information about the number of integer points in $Z_{\mathcal{A}}$ (or the discrete volume of $Z_{\mathcal{A}}$) and the number of interior integer points in $Z_{\mathcal{A}}$. When $\mathcal{A}$ is obtained from the rows of the matrices ED_T and $\mathcal{L}_T$, the independence of the respective arithmetic Tutte polynomials on the tree T naturally yields tree independent results about integer points in Z_{ED_T} and $Z_{\mathcal{L}_T}$. These corollaries are obtained in subsection 4.1.

Motivated by results on the discrete volume, we then consider the volume of extremal ellipsoids that approximate $Z_{\mathcal{A}}$ both from above and below. More precisely, we have the *minimum volume ellipsoid* containing $Z_{\mathcal{A}}$ and the *maximum volume ellipsoid* contained in $Z_{\mathcal{A}}$. We give a surprising formula showing that the volume of these ellipsoids for Z_{ED_T} and $Z_{\mathcal{L}_T}$ are again only dependent on n and (the value set to) q . Our results are Theorems 21, 22 and 23 which are proved in Section 6. To find these extremal volume ellipsoids, we will need to find the facets of the zonotope $Z_{\mathcal{A}}$. We determine the facets of Z_{ED_T} and $Z_{\mathcal{L}_T}$ in Section 5.

The Tutte polynomial of the matroid induced on the edge set of a tree T on n vertices is well known (see Chaudhury and Gordon [11]) to only be a function of n and is thus independent of the structure of T . Our results can be viewed as a generalization of this result to two arithmetic matroids defined using trees.

Analogous to the Tutte polynomial of all trees T on n vertices being identical, several results involving distances in trees T that only depend on the number n of vertices in T and not on the structure of the tree T are known. These include the determinant of the distance matrix D_T of T (see Graham and Pollak [21]), the determinants of both D_q , the q -analogue of D_T and ED_T (see Bapat, Lal and Pati [2]), the second immanant of ED_T (see Bapat and Sivasubramanian [4]) and the Smith Normal Form of ED_T (see Bapat and Sivasubramanian [5] and [2]). In this work, we add more results of this kind.

2 Preliminaries

We recall some preliminaries about arithmetic matroids, tree-related matrices, zonotopes and extremal volume ellipsoids below.

2.1 Arithmetic matroids

Given a list $\mathcal{A}$ of elements of $\mathbb{Z}^n$, an arithmetic matroid captures both the linear algebraic as well as arithmetic information contained in $\mathcal{A}$. We refer the reader to Brändén and Moci [10] or D'Adderio and Moci [14] for a formal definition of arithmetic matroids. Let U be a finite set of integral vectors in $\mathbb{R}^n$. An arithmetic matroid $\mathcal{A}$ on U assigns for every $R \subseteq U$, a rank $\text{rk}(R) \in \mathbb{Z}^+$ where $\text{rk}(R)$ is the dimension of the subspace of $\mathbb{R}^n$

spanned by the vectors in R . Hence $\text{rk}(R) \leq |R|$. Further, each $R \subseteq U$ also has a multiplicity $m(R) \in \mathbb{Z}^+$ satisfying certain axioms.

For a finite matrix M , let $\text{rows}(M)$ and $\text{cols}(M)$ be the set of its rows and columns. Given a finite matrix M , define $U = \text{rows}(M)$. For $R \subseteq \text{rows}(M)$ and $C \subseteq \text{cols}(M)$, define $M[R, C]$ to be the submatrix of M induced on the rows in R and the columns in C . By $M(R, C)$, we denote the submatrix obtained from M by omitting the rows in R and the columns in C . For $R \subseteq U$, we assign its multiplicity $m(R)$ as follows.

Define $m(R)$ to be the greatest common divisor of the maximal rank minors $M[R, C]$. As we only deal with square and invertible matrices M in this work, $m(R) = \gcd_{C \subseteq \text{cols}(M), |C|=|R|} \det M[R, C]$ with gcd being taken over choices of C . Without loss of generality, we assume that the gcd of a list of integers is non-negative. Thus for $R \subseteq U$, we have

$$m(R) = \gcd_{C \subseteq \text{cols}(M), |C|=|R|} \det M[R, C]. \quad (1)$$

The arithmetic Tutte polynomial is a bivariate polynomial while the multivariate arithmetic Tutte polynomial has one variable v_i for each $i \in U$ which we abbreviate to a vector $\mathbf{v}$ and another variable t . The definitions of these polynomials are given below.

$$M_{\mathcal{A}}(x, y) = \sum_{R \subseteq U} m(R) (x-1)^{\text{rk}(U)-\text{rk}(R)} (y-1)^{|R|-\text{rk}(R)} \quad (2)$$

$$\text{Multi}_{\mathcal{A}}(t, \mathbf{v}) = \sum_{R \subseteq U} m(R) t^{-\text{rk}(R)} \prod_{i \in R} v_i. \quad (3)$$

2.2 Two matrices associated to trees

We review two matrices associated to trees that we need in this work. Let T be a tree on n vertices. For a positive integer n , denote by $[n]$ the set $\{1, 2, \dots, n\}$. We assume that the vertex set of T is $[n]$ and hence denote vertices of T by i for $1 \leq i \leq n$.

Define $\mathcal{L}_T$, the q -analogue of T 's Laplacian to be $\mathcal{L}_T = I + q^2(D - I) - qA$, where q is a variable, A is the adjacency matrix of T and D is a diagonal matrix whose (i, i) -th entry is $\deg(i)$, the degree of vertex i . Note that setting $q = 1$ yields $\mathcal{L}_T = L_T = D - A$ which is the Laplacian of T . $\mathcal{L}_T$ and more generally $\mathcal{L}_G$ for a graph G occurs in connection with the Ihara-Selberg zeta function of G , see Bass [7] and Foata and Zeilberger [20]. $\mathcal{L}_T$ has also occurred in connection with tree-independent results shown by Bapat, Lal and Pati [2].

Next define the q -exponential distance matrix $\text{ED}_T = (e_{i,j})$ of T by $e_{i,j} = q^{d_{i,j}}$ where q is a variable with $q^0 = 1$ and $d_{i,j}$ is the length of the unique path from i to j in T . We will need the following theorem of Bapat, Lal and Pati (see [2, Propositions 3.3, 3.4]).

Theorem 1 (Bapat, Lal and Pati). *Let T be a tree with q -exponential distance matrix ED_T and q -Laplacian matrix $\mathcal{L}_T$. Then, $\text{ED}_T^{-1} = \frac{1}{1-q^2} \mathcal{L}_T$. Further, $\det \text{ED}_T = (1-q^2)^{n-1}$, $\det \mathcal{L}_T = 1-q^2$ and hence both determinants depend only on n, q and are independent of the structure of T .*

When we set integral values to q , for the sake of non-singularity of matrices (after setting values), we will not set the values $q = \pm 1$. As remarked earlier, we will assume that the gcd of a set of integers is non-negative. Since our gcd results (see Lemmas 6 and 11) involve expressions like $q^2 - 1$, these becomes negative when we plug in $q = 0$. It is simple to see that when $q = 0$, both $\mathcal{L}_T$ and ED_T become the $n \times n$ identity matrix for which results are straightforward. Though we could express results separately for the case when $q = 0$, we instead choose not to assign $q = 0$. We incur this mild loss so as to present our results in a unified manner.

2.3 Zonotopes associated to integral matrices

We refer the reader to the book by Ziegler [28] for concepts on polytopes not defined below. Let $v_1, v_2, \dots, v_k$ be k integral vectors in $\mathbb{R}^n$. Consider the closed line segment ℓ_i connecting the origin 0 to v_i . These line segments define a zonotope

$$Z(\ell_1, \ell_2, \dots, \ell_k) = \{x_1 + \dots + x_k : x_i \in \ell_i, 1 \leq i \leq k\}. \quad (4)$$

Given a list $L = (v_1, v_2, \dots, v_k)$ of vectors in $\mathbb{R}^n$, we denote by M the $k \times n$ integral matrix whose i -th row is the vector v_i for $1 \leq i \leq k$. This process can be reversed: given a finite matrix M , define a list $L = (v_i)$ of vectors where v_i is the i -th row of M . We denote the zonotope obtained from the rows of M by Z_M .

For a tree T on n vertices and any integer $r = 0, \pm 1$, set $q = r$ in ED_T . After this assignment, treat the n row vectors v_i for $1 \leq i \leq n$ of ED_T as integral points in $\mathbb{R}^n$ and consider the zonotope Z_{ED_T} , whose line segments ℓ_i are given by the lines connecting the origin 0 to v_i .

For the zonotope $Z_{\mathcal{L}_T}$, set $q = r$ for $r = 0, \pm 1$ in the matrix $\mathcal{L}_q$. After this assignment, consider the row vectors u_i for $1 \leq i \leq n$ as points in $\mathbb{R}^n$ and let $Z_{\mathcal{L}_T}$ be the zonotope whose line segments are the lines connecting the origin 0 to u_i .

For any integral polytope $P \subseteq \mathbb{R}^n$ and a positive integer m , define its m -th dilation by $mP = \{mx : x \in P\}$. Let $i(P, m) = |mP \cap \mathbb{Z}^n|$ be the number of integer points that are contained in mP . There exists a polynomial $\text{Ehr}_P(X)$, called the Ehrhart polynomial of P of degree $\dim(P)$ such that for all positive integers m , $\text{Ehr}_P(m) = i(P, m)$. Ehrhart [19], Stanley [26, Chapter 4] and Beck and Robbins [8, Theorem 3.8] are useful references for background on the Ehrhart polynomial. Since x is a variable we use with the arithmetic Tutte polynomial, we will use the variable X for the Ehrhart polynomial.

For a set S of integers, we denote by $\gcd_{x \in S} x$, the greatest common divisor of all $x \in S$. Though we write $\gcd_{x \in S} x$, we actually compute $\gcd$ over the non-zero elements of S . If S has no non-zero elements, then define $\gcd_{x \in S} = 0$. When the polytope is a zonotope, its Ehrhart polynomial can be computed using the following result of Stanley [25].

Theorem 2 (Stanley). *Let $M_{k \times n}$ be an integral matrix and let Z_M be the zonotope in $\mathbb{R}^n$ defined using $v_1, v_2, \dots, v_k$, where for $1 \leq i \leq k$, v_i is the i -th row of M . The Ehrhart polynomial of Z_M is*

$$\text{Ehr}_{Z_M}(X) = \sum_{R \subseteq \text{rows}(M)} g(R) X^{|R|}$$

where $g(R) = \gcd_{x \in S_R} x$ with $S_R = \{\det M[R, C] : C \subseteq \text{cols}(M), |C| = |R|\}$.

By convention, when $R = \emptyset$, $g(R) = 1$.

2.4 Ellipsoids contained in polytopes

We refer the reader to Barvinok's book [6] and to Henk [22] for background material on ellipsoids. Let $B_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$ be the closed ball in $\mathbb{R}^n$ and let T be an invertible linear transformation of $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ be a column vector. An ellipsoid E in $\mathbb{R}^n$ is defined as the affine image of the n -dimensional unit ball by an invertible linear transformation. Hence, E has the form $E = T(B_n) + x_0$. The vector x_0 is said to be the center of E . Thus, if $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$, we have

$$\begin{aligned} E &= \{x \in \mathbb{R}^n : \langle T^{-1}(x - x_0), T^{-1}(x - x_0) \rangle \leq 1\} \\ &= \{x \in \mathbb{R}^n : \langle Q(x - x_0), (x - x_0) \rangle \leq 1\} \end{aligned}$$

where $Q = (TT^*)^{-1}$. Sometimes, we also write $E = \{x : (x - x_0)^t Q(x - x_0) \leq 1\}$. The ellipsoid $E = (x - x_0)^t Q(x - x_0)$ has volume $\text{vol}(E) = \frac{\text{vol}(B_n)}{\sqrt{\det(Q)}}$ (see [6, page 204]).

The following result of Löwner and John shows that a maximum volume ellipsoid is well defined for polytopes (see [6, Page 207, Theorem 2.2]). A similar result holds for the minimum volume ellipsoid containing P (see [6, Page 209, problem 4]).

Theorem 3 (Löwner and John). *For any polytope P , among all ellipsoids E contained in P , there exists a unique ellipsoid $E_{\max}$ of maximum volume.*

3 Computing the arithmetic Tutte polynomial

In this section, we compute both the arithmetic Tutte polynomial and the multivariate arithmetic Tutte polynomial of the list of vectors given by the rows of $\mathcal{L}_T$ and ED_T in separate subsections.

3.1 Arithmetic Tutte polynomials of the rows of $\mathcal{L}_T$

Recall that for a tree T , its q -Laplacian is $\mathcal{L}_T = I + q^2(D - I) - qA$. We set $q = r$ for $r \neq 0, \pm 1$ and consider the rows of $\mathcal{L}_T$ as vectors in $\mathbb{Z}^n$. Recall that the vertex set of T is $[n] = \{1, 2, \dots, n\}$. We will index the rows and columns of $\mathcal{L}_T$ by the vertices of T . For $1 \leq i \leq n$, let $\mathcal{L}_T(i, i)$ be the $(n-1) \times (n-1)$ sized matrix obtained from $\mathcal{L}_T$ by deleting the row and column indexed by vertex i . We begin with the following simple corollary of Theorem 1, whose proof we omit. Recall that unimodular matrices are square matrices M with integer entries such that $\det M = \pm 1$.

Corollary 4. *Let T be a tree and $\mathcal{L}_T$ be its q -Laplacian matrix. Then for all $i \in [n]$, $\det \mathcal{L}_T(i, i) = 1$. That is, $\mathcal{L}_T(i, i)$ is unimodular.*

To the best of our knowledge, we could not find following lemma in the literature. We show that when M is unimodular, the multiplicity $m(R)$ as defined in (1) for $R \subseteq \text{rows}(M)$ is easy to compute.

Lemma 5. *Let M be a unimodular matrix and let $R \subseteq \text{rows}(M)$. Then, $m(R) = 1$.*

Proof. Recall $m(R) = \gcd_{C \subseteq \text{cols}(M)} \det M[R, C]$ with $|C| = |R|$. By Laplace's expansion of $\det M$ using the rows of R , we get

$$\begin{aligned} \det M &= \sum_{C \subseteq [n], |C|=|R|} \epsilon_{R,C} \det M[R, C] \det M(R, C) \\ \pm 1 &= \sum_{C \subseteq [n], |C|=|R|} \epsilon_{R,C} \det M[R, C] \det M(R, C) \end{aligned} \quad (5)$$

where $\epsilon_{R,C}$ is a sign depending on R, C . Since M has integer entries, $\det M(R, C)$ is an integer. By (5), varying over $C \subseteq \text{cols}(M)$, a linear combination of $\det M[R, C]$ gives us 1. Hence, the gcd of $\det M[R, C]$ over choices $C \subseteq \text{cols}(M)$ with $|C| = |R|$ equals 1, completing the proof. $\square$

The following is a key lemma of this subsection, which we believe is of independent interest.

Lemma 6. *Let T be a tree with q -Laplacian $\mathcal{L}_T$. Then $m(R) = 1$ when $R \subseteq \text{rows}(\mathcal{L}_T)$ with $R \neq \text{rows}(\mathcal{L}_T)$. If $R = \text{rows}(\mathcal{L}_T)$, then $m(R) = q^2 - 1$.*

Proof. When $R = \text{rows}(\mathcal{L}_T)$, the only choice for C with $|C| = |R|$ is $C = \text{cols}(\mathcal{L}_T)$ and by Theorem 1, $\det \mathcal{L}_T = 1 - q^2$. Since $q \in \mathbb{Z}$ with $|q| > 1$, we get $m(R) = q^2 - 1$.

Let $R \subseteq \text{rows}(\mathcal{L}_T)$ with $R \neq \text{rows}(\mathcal{L}_T)$. Choose $i \in R$. Consider the matrix $\mathcal{L}'_T = \mathcal{L}_T + q^2 E_{i,i}$ where $E_{i,i}$ is an $n \times n$ matrix with (i, i) -th entry being 1 and all other entries being 0. For $1 \leq i \leq n$, denote the i -th column of matrix M by $\text{col}_i(M)$ and the $n \times 1$ column vector with a 1 in the i -th coordinate and zero elsewhere as e_i .

We compute $\det \mathcal{L}'_T$ by using the multilinearity of the determinant and writing $\text{col}_i(\mathcal{L}'_T) = \text{col}_i(\mathcal{L}_T) + q^2 e_i$. By Theorem 1 and Corollary 4, $\det \mathcal{L}'_T = 1 - q^2 + q^2 = 1$.

As $\mathcal{L}'_T$ is unimodular, by Lemma 5, we have $\gcd_{C \subseteq \text{cols}(\mathcal{L}'_T)} \det \mathcal{L}'_T[R, C] = 1$. As sets, it is clear that $\text{cols}(\mathcal{L}'_T) = \text{cols}(\mathcal{L}_T)$ and as $i \in R$, it is easy to see that $\mathcal{L}'_T[R, C] = \mathcal{L}_T[R, C]$ for all $C \subseteq \text{cols}(\mathcal{L}_T)$. Thus $\det \mathcal{L}_T[R, C] = \det \mathcal{L}'_T[R, C]$ and so $\gcd_{C \subseteq \text{cols}(\mathcal{L}_T)} \det \mathcal{L}_T[R, C] = 1$. Hence $m(R) = 1$, completing the proof. $\square$

We end this subsection by computing the arithmetic Tutte polynomial of the vectors obtained from $\text{rows}(\mathcal{L}_T)$ below. As mentioned, we express our results by retaining the variable q so that we can plug in values for q .

Theorem 7. *Let $\mathcal{L}_T$ be the q -Laplacian of a tree T on n vertices and consider the vectors $\text{rows}(\mathcal{L}_T)$ obtained after setting $q = r$ where $r \neq 0, \pm 1$. Then, the arithmetic Tutte polynomial*

$$M_{\mathcal{L}_T}(x, y) = x^n + q^2 - 2.$$

Proof. When $q \neq \pm 1$, $\mathcal{L}_T$ is non-singular. Hence, all subsets $R \subseteq \text{rows}(\mathcal{L}_T)$ have $\text{rk}(R) = |R|$ with $\text{rk}(\text{rows}(M)) = n$. By Lemma 6, when $R = \text{rows}(\mathcal{L}_T)$, then $m(R) = q^2 - 1$ and when $R \subsetneq \text{rows}(M)$ with $R \neq \text{rows}(M)$, then $m(R) = 1$. Thus,

$$\begin{aligned} M_{\mathcal{L}_T}(x, y) &= \sum_{R \subseteq \text{rows}(\mathcal{L}_T)} m(R)(x-1)^{n-\text{rk}(R)}(y-1)^{|R|-\text{rk}(R)} \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (x-1)^{n-k} + (q^2 - 1) = x^n + q^2 - 2 \end{aligned}$$

The proof is complete. $\square$

We use Lemma 6 to find the multivariate arithmetic Tutte polynomial of the rows of $\mathcal{L}_T$.

Theorem 8. *Let $\mathcal{L}_T$ be the q -Laplacian of a tree T on n vertices and consider the vectors $\text{rows}(\mathcal{L}_T)$ obtained after setting $q = r$ where $r \neq 0, \pm 1$. Then, the arithmetic Tutte polynomial*

$$\text{Multi}_{\mathcal{L}_T}(t, v) = \prod_{i=1}^n \left(1 + \frac{v_i}{t}\right) + (q^2 - 2)t^{-n} \prod_{i=1}^n v_i.$$

Proof. From (3), we get

$$\text{Multi}_{\mathcal{L}_T}(t, v) = \sum_{R \subseteq \text{rows}(\mathcal{L}_T)} m(R)t^{-\text{rk}(R)} \prod_{i \in R} v_i \quad (6)$$

$$= \prod_{i=1}^n \left(1 + \frac{v_i}{t}\right) + (q^2 - 2)t^{-n} \prod_{i=1}^n v_i \quad (7)$$

The proof is complete. $\square$

Thus both the multivariate and the arithmetic Tutte polynomials of the list of row vectors of $\mathcal{L}_T$ are independent of the structure of T and only depend on the number n of vertices in T and the value set to q .

3.2 Arithmetic Tutte polynomials of the rows of ED_T

We compute the arithmetic Tutte polynomial of the vectors arising from the rows of ED_T in this subsection. Most of the work is already done and we just show that Lemma 6, coupled with Jacobi's identity yields results about $m(R)$ for $R \subseteq \text{rows}(\text{ED}_T)$. Since $\mathcal{L}_T$ is a symmetric matrix, we rephrase Lemma 6 with the roles of rows and columns interchanged as follows.

Lemma 9. Let $\mathcal{L}_T$ be the q -Laplacian of a tree T on n vertices. Let $C \subseteq \text{cols}(\mathcal{L}_T)$, $|C| = k$ with $k < n$. Then, $\gcd_{R \subseteq \text{rows}(\mathcal{L}_T), |R|=k} \det \mathcal{L}_T[R, C] = 1$

We need the following result of Jacobi (see [17, Sec 4.2]). For $S \subseteq [n]$, let $\alpha(S)$ denote the sum of the integers in S .

Theorem 10 (Jacobi). Let A be an invertible $n \times n$ matrix. Let $B = A^{-1}$ and let $S, K \subseteq [n]$ with $|S| = |K|$. Then,

$$\det A[S, K] = (-1)^{\alpha(S) + \alpha(K)} \frac{\det B(K, S)}{\det B}. \quad (8)$$

We can now show the following.

Lemma 11. Let ED_T be the exponential distance matrix of a tree T on n vertices. Let $R \subseteq \text{rows}(\text{ED}_T)$, $|R| = k$ with $k > 0$. Then, $m(R) = (q^2 - 1)^{k-1}$.

Proof. From Theorem 10 and Theorem 1, we get upto signs,

$$\begin{aligned} \det \text{ED}_T[R, C] &= (q^2 - 1)^{k-1} \det \mathcal{L}_T(C, R). \text{ Thus,} \\ m(R) = \gcd_{C \subseteq [n], |C|=k} \det \text{ED}_T[R, C] &= (q^2 - 1)^{k-1} \gcd_{C \subseteq [n], |C|=k} \det \mathcal{L}_T(C, R) = (q^2 - 1)^{k-1}. \end{aligned}$$

In the first equality above, we have used Jacobi's identity with $A = \text{ED}_T$ and $B = \frac{\mathcal{L}_T}{1 - q^2}$ and the fact that $\det \mathcal{L}_T = 1 - q^2$. In the last line, we have used Lemma 9 to complete the proof. $\square$

We now compute the arithmetic Tutte polynomial of the row vectors of ED_T .

Theorem 12. Let ED_T be the exponential distance matrix of a tree on n vertices and consider the vectors $\text{rows}(\text{ED}_T)$ obtained after setting $q = r$ where $r \neq 0, \pm 1$. Then, the arithmetic Tutte polynomial

$$M_{\text{ED}_T}(x, y) = (x - 1)^n \left(\frac{q^2 - 2}{q^2 - 1} \right) + \frac{1}{q^2 - 1} (x + q^2 - 2)^n$$

Proof. Setting $q = r$ for $r \neq \pm 1$ will result in ED_T being non-singular and hence for all $R \subseteq \text{rows}(\text{ED}_T)$, $\text{rk}(R) = |R|$. From Lemma 11, when $R \subseteq \text{rows}(\text{ED}_T)$, $|R| = k > 0$, $m(R) = (q^2 - 1)^{k-1}$ and that $m(\emptyset) = 1$. Thus,

$$\begin{aligned} M_{\text{ED}_T}(x, y) &= (x - 1)^n + \sum_{k=1}^n \binom{n}{k} (q^2 - 1)^{k-1} (x - 1)^{n-k} \\ &= (x - 1)^n \left[1 + \frac{1}{q^2 - 1} \sum_{k=1}^n \binom{n}{k} \left(\frac{q^2 - 1}{x - 1} \right)^k \right] \\ &= (x - 1)^n \left[\frac{q^2 - 2}{q^2 - 1} + \frac{1}{q^2 - 1} \left(\frac{x + q^2 - 2}{x - 1} \right)^n \right] \end{aligned}$$

The proof is complete. $\square$

We use Lemma 11 to find the multivariate arithmetic Tutte polynomial of the rows of $\mathcal{L}_T$.

Theorem 13. Let ED_T be the exponential distance matrix of a tree on n vertices and consider the vectors $\text{rows}(\text{ED}_T)$ obtained after setting $q = r$ where $r \neq 0, \pm 1$. Then,

$$\text{Multi}_{\text{ED}_T}(t, v) = \frac{1}{q^2 - 1} \prod_{i=1}^n \left(1 + \frac{(q^2 - 1)v_i}{t} \right) + \frac{q^2 - 2}{q^2 - 1}.$$

Proof. From (3), we get

$$\text{Multi}_{\text{ED}_T}(t, \mathbf{v}) = \sum_{R \subseteq \text{rows}(\text{ED}_T)} m(R) t^{-\text{rk}(R)} \prod_{i \in R} v_i \quad (9)$$

$$= 1 + \frac{1}{q^2 - 1} \sum_{S \subseteq [n], |S| \geq 1} (q^2 - 1)^{|S|} t^{-|S|} \prod_{i \in S} v_i \quad (10)$$

$$= \frac{1}{q^2 - 1} \prod_{i=1}^n \left(1 + \frac{(q^2 - 1)v_i}{t} \right) + \frac{q^2 - 2}{q^2 - 1} \quad (11)$$

The proof is complete. $\square$

Thus both the multivariate and the arithmetic Tutte polynomials of the list of row vectors of ED_T are independent of the structure of T and only depend on the number n of vertices in T and the value set to q .

4 Corollaries

In this section, we draw corollaries of Theorems 7 and 12. These include results about the Ehrhart polynomials of zonotopes associated with these matrices and the Smith Normal Form of these matrices.

4.1 Ehrhart polynomials of the zonotopes Z_{ED_T} and $Z_{\mathcal{L}_T}$

The following result from Stanley [25] and from D'Adderio and Moci [12] connecting the arithmetic Tutte polynomial and the Ehrhart polynomial will be needed.

Theorem 14 (D'Adderio, Moci). *Let P be an integral matrix such that the zonotope Z_P is n -dimensional. Then,*

$$\text{Ehr}_{Z_P}(X) = X^n M_P(1 + 1/X, 1).$$

Let T be a tree on n vertices. As mentioned, when $q \neq \pm 1$, both zonotopes Z_{ED_T} and $Z_{\mathcal{L}_T}$ are n -dimensional polytopes. Thus, we get the following simple corollary.

Corollary 15. *Let T be a tree on n vertices. When $q \neq 0, \pm 1$, the Ehrhart polynomial of the zonotope Z_{ED_T} equals*

$$\text{Ehr}_{Z_{\text{ED}_T}}(X) = 1 + \frac{[1 + X(q^2 - 1)]^n - 1}{q^2 - 1} \quad (12)$$

and when $q \neq 0, \pm 1$, the Ehrhart polynomial of $Z_{\mathcal{L}_T}$ equals

$$\text{Ehr}_{Z_{\mathcal{L}_T}}(X) = X^n(q^2 - 2) + (1 + X)^n. \quad (13)$$

Thus, the number of integer points in Z_{ED_T} and $Z_{\mathcal{L}_T}$ are independent of T and only depend on n , the number of vertices of T and the parameter q . It is well known that if P is an n -dimensional polytope in $\mathbb{R}^n$, then its volume is the leading coefficient of its Ehrhart polynomial $\text{Ehr}_P(X)$ (see [8, Corollary 3.20]). Thus if T is a tree on n vertices, the volumes of the zonotopes Z_{ED_T} and $Z_{\mathcal{L}_T}$ are only dependent on n and independent of the structure of the tree T .

When P is an n -dimensional integral polytope, by the Ehrhart-Macdonald reciprocity Theorem (see [8, Theorem 4.1]), the number of interior integer points in the m -th dilation mP equals $(-1)^n \text{Ehr}_P(-m)$. Hence the number of integral interior points in the zonotopes Z_{ED_T} and $Z_{\mathcal{L}_T}$ are independent of the tree T and only depend on n and the parameter q .

4.1.1 The q^2 -analogue of n

We mention an interesting special case of Corollary 15. Recall the q -analogue of a positive integer n , is the polynomial in one variable q given by $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and that $[0]_q = 0$. Similarly, for a positive integer n , define its q^2 -analogue by $[n]_{q^2} = 1 + q^2 + \cdots + q^{2n-2}$. The q^2 -analogue of n has appeared for instance in the work of Bapat and Sivasubramanian (see [3, Corollary 16]) as a minor of $\mathcal{L}_T$ when the rows and columns corresponding to two distinct vertices i, j are omitted. By setting $X = 1$ in Corollary 15, we get another instance where the q^2 -analogue of n appears.

Corollary 16. *Let T be a tree on n vertices with exponential distance matrix ED_T . For $q \in \mathbb{Z}$ with $q \neq 0, \pm 1$, the number of integer points in the zonotope Z_{ED_T} equals $1 + [n]_{q^2}$.*

4.2 Smith Normal Form of ED_T and $\mathcal{L}_T$

Lemmas 6 and 11 imply the Smith Normal Form (SNF henceforth) of the matrices ED_T and $\mathcal{L}_T$ for a tree T on n vertices. We recall some preliminaries about the SNF of a matrix with entries from a principal ideal domain (PID henceforth). We refer the reader to Newman's book [24, Chapter II] for a nice introduction to this topic.

Let A be an $n \times n$ matrix with entries from a PID. For $1 \leq k \leq n$, consider submatrices $A[R, C]$ where $R, C \subseteq [n]$, $|R| = |C| = k$. Define $d_k(A)$ to be the gcd of $\det(A[R, C])$ over the $\binom{n}{k}^2$ choices of R, C . If all the $\binom{n}{k}^2$ matrices $A[R, C]$ are singular, we set $d_k(A) = 0$.

It can be shown (see [24, Section 16]) that $d_k(A)$ divides $d_{k+1}(A)$ for $1 \leq k \leq n-1$. Set $d_0(A) = 1$ and define for $1 \leq k \leq n$

$$s_k(A) = \frac{d_k(A)}{d_{k-1}(A)}.$$

$s_k(A)$ is defined to be zero when both $d_k(A) = d_{k-1}(A) = 0$. The quantities $s_k(A)$ are called the *invariant factors* of A which we denote by $s_1, s_2, \dots, s_r$, when the matrix A is clear. The main result in the theory of SNF is the following (see [24, Section 15]).

Theorem 17 (Smith Normal Form). *Let A be an $n \times n$ matrix with entries from a PID. Then, there exists $n \times n$ matrices U, V with $\det U$ and $\det V$ being units in the PID such that $S = UAV = \text{Diag}(s_1, s_2, \dots, s_r, 0, 0, \dots, 0)$, where r is the rank of A . The matrix S is unique up to multiplication by units of the PID.*

The matrix S in the Theorem 17 is called the SNF of the matrix A . Since $\mathbb{Q}[q]$ and $\mathbb{Z}$ are PIDs, the theory of SNF is applicable to the matrices ED_T and $\mathcal{L}_T$ (both before and after plugging in values for q). The SNF of the matrix ED_T is already known (see [2] and [5]) and the SNF of $\mathcal{L}_T$ can be inferred from it. This work gives a simple reproof of the SNF of both ED_T and $\mathcal{L}_T$.

Corollary 18. *Let T be a tree on n vertices. The SNF of its exponential distance matrix ED_T is $S = \text{Diag}(1, q^2 - 1, q^2 - 1, \dots, q^2 - 1)$. The SNF of its q -Laplacian $\mathcal{L}_T$ is $R = \text{Diag}(1, 1, \dots, q^2 - 1)$. Thus, the SNF of these matrices are also tree independent.*

Proof. By Lemma 6, when $k < n$, for $\mathcal{L}_q$, we get the gcd of $\det(\mathcal{L}_q[R, C])$ over the $\binom{n}{k}^2$ choices of R, C with $|R| = |C| = k$, is 1, and for $k = n$, $\det(\mathcal{L}_q) = 1 - q^2$. Thus $d_k(\mathcal{L}_q) = 1$ when $k < n$ and $d_n(\mathcal{L}_q) = q^2 - 1$.

Similarly by Lemma 11, for a fixed $R \subseteq [n]$, with $|R| = k$ when $1 \leq k \leq n$, the gcd of $\det(\text{ED}_T[R, C])$ over all $C \subseteq [n]$, $|C| = k$ is $(q^2 - 1)^{k-1}$. Thus $d_k(\text{ED}_T) = (q^2 - 1)^{k-1}$, completing the proof. $\square$

5 Facets of the zonotopes

In this section, we begin by finding the facets of these zonotopes. We will need this for finding the equation and the volume of the extremal ellipsoids associated to both the zonotopes.

5.1 Facets of Z_{ED_T}

We first find the facets of the zonotope Z_{ED_T} . For this, we introduce a seemingly new polytope. Let T be a tree on n vertices with vertex set V and with edge set $E(T)$. Let $q \in \mathbb{Z}$ with $q \neq 0, \pm 1$. Define a new polyhedron $Z_n(T, q)$ to be the solution set to the following set of $2n$ inequalities

$$- \left[q^2(\deg(v) - 1) + 1 \right] x_v + q \left(\sum_{w: \{v, w\} \in E(T)} x_w \right) \leq q^2 - 1; \quad \forall v \in V \quad (14)$$

$$\left[q^2(\deg(v) - 1) + 1 \right] x_v - q \left(\sum_{w: \{v, w\} \in E(T)} x_w \right) \leq 0; \quad \forall v \in V. \quad (15)$$

Clearly $Z_n(T, q)$ is a polyhedron. Let $\mathbf{0}$ and $\mathbf{1}$ be the all zero and the all ones vector respectively in $\mathbb{R}^n$. Clearly, we could alternatively write $Z_n(T, q) = \{x \in \mathbb{R}^n : \mathbf{0} \leq \mathcal{L}_q \cdot (-x) \leq (q^2 - 1)\mathbf{1}\}$, where we interpret the above inequalities as being componentwise.

Theorem 19. *Let T be a tree on n vertices. When $q \neq 0, \pm 1$, $Z_{ED_T} = Z_n(T, q)$. Thus, the facets of Z_{ED_T} are those given in inequalities (14) and (15).*

Proof.

$$\begin{aligned} Z_{ED_T} &= \{x \in \mathbb{R}^n : x = ED_T \cdot \alpha \text{ for some } \alpha \in \mathbb{R}^n, \mathbf{0} \leq \alpha \leq \mathbf{1}\} \text{ (by definition)} \\ &= \{x \in \mathbb{R}^n : \frac{\mathcal{L}_T}{1 - q^2} \cdot x = \alpha \text{ for some } \alpha \in \mathbb{R}^n, \mathbf{0} \leq \alpha \leq \mathbf{1}\} \text{ by Theorem 1} \\ &= \{x \in \mathbb{R}^n : \mathbf{0} \leq \frac{\mathcal{L}_T}{1 - q^2} \cdot x \leq \mathbf{1}\} \\ &= \{x \in \mathbb{R}^n : \mathbf{0} \leq \mathcal{L}_T \cdot (-x) \leq (q^2 - 1)\mathbf{1}\} \\ &= Z_n(T, q). \end{aligned}$$

Again, we interpret the above inequalities as being componentwise. We omit the value $q = 0$ so as to keep $q^2 - 1$ positive (required in the penultimate line). $\square$

5.2 Facets of $Z_{\mathcal{L}_T}$

We again introduce a new polytope. Define a polyhedron $P_n(T, q)$ as the solution set to the following set of $2n$ inequalities

$$- \left(\sum_{u \in V} q^{d_{u,v}} x_u \right) \leq q^2 - 1; \quad \forall v \in V \quad (16)$$

$$\sum_{u \in V} q^{d_{u,v}} x_u \leq 0; \quad \forall v \in V. \quad (17)$$

As the proof of the next result is identical to the proof of Theorem 19, we merely state it without giving a proof.

Theorem 20. *Let T be a tree on n vertices. When $q \neq 0, \pm 1$, $Z_{\mathcal{L}_T} = P_n(T, q)$. Thus, the facets of $Z_{\mathcal{L}_T}$ are those given in inequalities (16) and (17).*

6 Extremal volume ellipsoids associated to Z_{ED_T} and $Z_{\mathcal{L}_T}$

In this section, we give an explicit formula for the extremal volume ellipsoids associated to the polytopes Z_{ED_T} and $Z_{\mathcal{L}_T}$. In general, computing extremal ellipsoids is NP hard, if the dimension is part of the input (see Matoušek [23]). Surprisingly, for Z_{ED_T} and $Z_{\mathcal{L}_T}$, the volume of these ellipsoids also turn out to be tree independent.

6.1 Extremal volume ellipsoids associated to Z_{ED_T}

Let $E_{\max}^T$ be the maximum volume ellipsoid contained in Z_{ED_T} and let $E_{\min}^T$ be the minimum volume ellipsoid containing Z_{ED_T} .

Theorem 21. *Let T be a tree on n vertices. For $q \in \mathbb{Z}$ with $q \neq 0, \pm 1$, the maximum volume ellipsoid $E_{\max}$ contained in the zonotope Z_{ED_T} has center $x_0 = \frac{1-q^2}{2} \mathcal{L}_T^{-1} \mathbb{1}$ and $Q = \left(\frac{2}{q^2-1} \mathcal{L}_T \right)^2$.*

Proof. We will use the affine invariance property of the maximum volume ellipsoid (see Boyd and Vandenberghe [9, Section 8.4.3]). If E is the maximum volume ellipsoid contained in a polytope P and R is a non-singular matrix then this property asserts that $R(E)$ is the maximum volume ellipsoid contained in the polytope $R(P)$. Let $\mathcal{L}_T x = y$, so that $x = \mathcal{L}_T^{-1} y$. Then, $(x - x_0)^t Q (x - x_0) \leq 1$ is equivalent to $(y - \frac{1-q^2}{2} \mathbb{1})^t (y - \frac{1-q^2}{2} \mathbb{1}) \leq (\frac{1-q^2}{2})^2$, which is the maximum volume ellipsoid contained in the n -dimensional cuboid $\{y : (1-q^2)\mathbb{1} \leq y \leq 0\}$ (see [9, Page 413]). Consequently, the maximum volume ellipsoid $E_{\max}^T$ contained in Z_{ED_T} has

$$\text{vol}(E_{\max}^T) = \frac{\text{vol}(B_n)}{\det((\frac{2}{q^2-1}) \cdot \mathcal{L}_T)} = \frac{\text{vol}(B_n) \cdot (q^2-1)^{n-1}}{2^n}.$$

We have used $|\det(\mathcal{L}_T)| = q^2 - 1$ in the last step (from Theorem 1). Thus $\text{vol}(E_{\max}^T)$ is independent of the tree T and only depends on n and q . The proof is complete. $\square$

Theorem 22. *Let T be a tree on n vertices. For $q \in \mathbb{Z}$ with $q \neq 0, \pm 1$, the minimum volume ellipsoid containing Z_{ED_T} has center $x_0 = \frac{1-q^2}{2} \mathcal{L}_T^{-1} \mathbb{1}$ and $Q = \frac{1}{n} \left(\frac{2}{q^2-1} \mathcal{L}_T \right)^2$.*

Proof. (Sketch). An identical proof works, just that this time, the minimum volume ellipsoid containing the n -dimensional cuboid $\{y : 0 \leq y \leq (q^2-1)\mathbb{1}\}$ (see [9, Section 8.4]) is:

$$(y - \frac{q^2-1}{2} \mathbb{1})^t (y - \frac{q^2-1}{2} \mathbb{1}) \leq n \cdot \left(\frac{q^2-1}{2} \right)^2. \text{ Thus, we have } Q = \frac{4}{n \cdot (q^2-1)^2} \mathcal{L}_T^2 \text{ and}$$

$$\text{vol}(E_{\min}^T) = \frac{\text{vol}(B_n) \cdot n^{n/2} \cdot (q^2-1)^{n-1}}{2^n}.$$

Thus again $\text{vol}(E_{\min}^T)$ is independent of the tree T , completing the proof. $\square$

6.2 Extremal volume ellipsoids associated to $Z_{\mathcal{L}_T}$

We prove similar results for the zonotope $Z_{\mathcal{L}_T}$. The proofs are identical to that of the previous subsection. Let $F_{\max}^T$ be the maximum volume ellipsoid contained in $Z_{\mathcal{L}_T}$ and let $F_{\min}^T$ be the minimum volume ellipsoid containing $Z_{\mathcal{L}_T}$.

Theorem 23. *Let T be a tree on n vertices. For $q \in \mathbb{Z}$ with $q \neq 0, \pm 1$, the maximum volume ellipsoid $F_{\max}$ contained in the zonotope $Z_{\mathcal{L}_T}$ has center $x_0 = \frac{1-q^2}{2} \text{ED}_T^{-1} \mathbb{1}$ and $Q = \left(\frac{2}{q^2-1} \text{ED}_T \right)^2$.*

The minimum volume ellipsoid containing $Z_{\mathcal{L}_T}$ has the same center $x_0 = \frac{1-q^2}{2} \text{ED}_T^{-1} \mathbb{1}$ and $Q = \frac{1}{n} \left(\frac{2}{q^2-1} \text{ED}_T \right)^2$. Hence, both $\text{vol}(F_{\max}^T)$ and $\text{vol}(F_{\min}^T)$ are independent of the tree T and only depend on n and q .

Proof. (Sketch) As this proof is identical to the proofs of Theorems 21 and 22, we just mention the volume of the ellipsoids. It is easy to see that

$$\text{vol}(F_{\max}^T) = \frac{\text{vol}(B_n)}{\det(\frac{2}{q^2-1} \cdot \text{ED}_T)} = \frac{\text{vol}(B_n)(q^2-1)}{2^n}.$$

For the minimum volume ellipsoid, it is easy to see that

$$\text{vol}(F_{\min}^T) = \frac{\text{vol}(B_n)}{\det(\frac{2}{\sqrt{n}(q^2-1)} \cdot \text{ED}_T)} = \frac{\text{vol}(B_n) \cdot n^{n/2} \cdot (q^2-1)}{2^n}.$$

Thus, $\text{vol}(F_{\max}^T)$ and $\text{vol}(F_{\min}^T)$ are independent of the tree T and only depend on n, q . $\square$

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