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Commuting decomposition of $K_{n_1, n_2, \dots, n_k}$ through realization of the product $A(G)A(G_k^P)$

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Abstract: In this paper, we introduce the notion of perfect matching property for a k -partition of vertex set of given graph. We consider nontrivial graphs G and G_k^P , the k -complement of graph G with respect to a k -partition of $V(G)$, to prove that $A(G)A(G_k^P)$ is realizable as a graph if and only if P satisfies perfect matching property. For $A(G)A(G_k^P) = A(\Gamma)$ for some graph Γ , we obtain graph parameters such as chromatic number, domination number etc., for those graphs and characterization of P is given for which G_k^P and Γ are isomorphic. Given a 1-factor graph G with $2n$ vertices, we propose a partition P for which G_k^P is a graph of rank r and $A(G)A(G_k^P)$ is graphical, where $n \leq r \leq 2n$. Motivated by the result of characterizing decomposable $K_{n,n}$ into commuting perfect matchings [2], we characterize complete k -partite graph $K_{n_1, n_2, \dots, n_k}$ which has a commuting decomposition into a perfect matching and its k -complement.

Keywords: Adjacency matrix, k -complement, Graphical, Matrix product, Commutativity, Commuting Decomposition

MSC: 05C50; 15A24

1 Introduction

Graphs considered in this paper are simple, undirected, and without self-loops. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. For any two vertices v_i and v_j , $i \neq j$ in the graph G , $v_i \sim_G v_j$ denotes that the vertices are adjacent in the graph G , and $v_i \not\sim_G v_j$ denotes that the vertices are not adjacent in the graph G . The suffix G in the notations $\sim_G$ and $\not\sim_G$ are conveniently ignored if the graph under discussion is clearly understood. The notation $A(G)$ denotes the adjacency matrix of the graph G .

A symmetric $(0, 1)$ matrix is said to be *graphical* [2], if its diagonal entries are all equal to zero. In this case, a graph G such that $A(G) = B$ is called the realization of the matrix B , and B is said to be realizable as graph G .

In the following we provide some useful definitions.

Definition 1.1. [6] A graph Γ is said to be the matrix product of graphs G and H if $A(\Gamma) = A(G)A(H)$.

Definition 1.2. [6] (*GH path*) Given graphs G and H on the same set of vertices $\{v_1, v_2, \dots, v_n\}$, two vertices v_i and v_j ($i \neq j$) are said to have a *GH path* from v_i to v_j , if there exists a vertex v_k , different from v_i and v_j , such that $v_i \sim_G v_k$ and $v_k \sim_H v_j$.

The following theorem characterizes graphs G and H with the property that $A(G)A(H)$ is graphical.

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Theorem 1.3. [6, Theorem 1] Let G and H be graphs on the same set of vertices $\{v_1, v_2, \dots, v_n\}$. Then $C = A(G)A(H)$ is graphical if and only if the following three statements hold.

- (i) G and H are edge disjoint.
- (ii) For every two vertices v_i and v_j , there exists at most one GH path from the vertex v_i to the vertex v_j .
- (iii) For every two vertices v_i and v_j , if there exists a GH path from the vertex v_i to the vertex v_j , then there exists a HG path from v_i to v_j .

The following two properties regarding degree of vertices of graphs G , H and Γ satisfying $A(G)A(H) = A(\Gamma)$ are useful.

Lemma 1.4. [6, Corollary 2] Let Γ be the matrix product of G and H . If the vertices u and v are connected by a path in G , then $\deg_H u = \deg_H v$.

Theorem 1.5. [6, Theorem 7] Let G , H and Γ be the graphs such that Γ is the matrix product of G and H . Then $\deg_\Gamma u = \deg_G u \cdot \deg_H u$ for all vertices u .

The notion of k -complement of graph G with respect to a partition P of $V(G)$ with k partite sets is a generalization of complement of a graph G introduced by E. Sampath Kumar and L. Pushpalatha [3].

Definition 1.6. [3] Given a graph G and a k -partition $P = \{V_1, V_2, \dots, V_k\}$ of $V(G)$, the k -complement of G with respect to the partition P is a graph H with $V(H) = V(G)$ in which the adjacency between $v \in V_i$ and $w \in V_j$ is defined by

$$v \sim_H w \text{ if } \begin{cases} v \sim_G w, & \text{for } i \neq j \\ v \sim_G w, & \text{for } i = j. \end{cases}$$

The above graph H is usually denoted by G_k^P and is also known as generalized complement of graph G , where it reduces to the well known complement of G when k is equal to n , the number of vertices in the graph G . From the definition it is clear that the subgraph induced by V_i , $1 \leq i \leq k$, in G and G_k^P are same. We refer [3], [4], [5], [7] and [8] for the further properties and results associated with generalized complement of a graph.

Definition 1.7. A collection of subgraphs $H_1, H_2, \dots, H_k$ of G is said to be a decomposition of G if $\bigcup_{i=1}^k H_i = G$ and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$.

Definition 1.8. A decomposition $H_1, H_2, \dots, H_k$ of G is said to be a commuting decomposition of G if $H_1, H_2, \dots, H_k$ are spanning subgraphs and $A(H_i)A(H_j) = A(H_j)A(H_i)$ for all i and j with $1 \leq i, j \leq k$.

Readers are referred to [1] for the results on commuting decomposition of K_n and [2] for the results on the commuting decomposition of $K_{n,n}$ into perfect matchings and Hamiltonian cycles.

In this paper, we focus our study on characterization of the partition P of $V(G)$ for which the matrix product $A(G)A(G_k^P)$ is graphical. We discuss about some parameters of graphs G , G_k^P and Γ , when $A(G)A(G_k^P) = A(\Gamma)$. We comment on rank of G , G_k^P and Γ , whenever $A(G)A(G_k^P) = A(\Gamma)$. We also derive a result on commuting decomposition of the complete k -partite graph $K_{n_1, n_2, \dots, n_k}$ into a perfect matching G and its k -complement G_k^P with respect to k -partition $P = \{V_1, V_2, \dots, V_k\}$ of $V(K_{n_1, n_2, \dots, n_k})$.

Readers are referred to [9] for all the elementary notations and definitions not described but used in this paper.

2 Realization of $A(G)A(G_k^P)$ into graph

In this section we explore the characterization of graph G and its k -partition $P = \{V_1, V_2, \dots, V_k\}$ so that $A(G)A(G_k^P)$ is graphical.

Theorem 2.1. *Let G be a graph, $P = \{V_1, V_2, \dots, V_k\}$ be a k -partition of $V(G)$. The matrix product $A(G)A(G_k^P)$ is graphical if and only if the following statements are true.*

- (i) *The subgraph $\langle V_i \rangle$ is totally disconnected for every i , $1 \leq i \leq k$.*
- (ii) *For every pair of vertices $u \in V_i$ and $v \in V_j$, there exists at most one vertex $w \in V_r$, $r \neq i, j$ and $1 \leq i, j, r \leq k$, such that $u \sim_G w$ and $w \approx_G v$.*
- (iii) *For a pair of vertices u and v , if there exists w satisfying (ii), then there exists a unique $w' \in V_l$, $l \neq i, j$, such that $v \sim_G w'$ and $w' \approx_G u$.*

Proof. Let $A(G)A(G_k^P)$ be graphical. If $u \sim_G v$ for any $u, v \in V_i$, then from the definition of G_k^P we have that $u \sim_{G_k^P} v$. It contradicts the fact that G_k^P is subgraph of $\bar{G}$. This proves (i).

From (ii) of Theorem 1.3 we have that for every pair of vertices u and v , there exists w different from u and v such that $u \sim_G w$ and $w \sim_{G_k^P} v$. Since G_k^P is subgraph of $\bar{G}$, it is trivial that $w \approx_G v$. Further, if $w \in V_r$ for some r , from (i), we get that r is neither equal to i nor equal to j . Thus (ii) is proved.

Proof of (iii) follows immediately from (iii) of Theorem 1.3.

Conversely, let (i), (ii) and (iii) of the theorem are satisfied. Consider any two vertices $u \in V_i$ and $v \in V_j$. Since $\langle V_i \rangle$ and $\langle V_j \rangle$ are totally disconnected in G , from the definition of G_k^P and from (ii), we get that there is at most one GG_k^P path between u and v . For the same reason, (iii) implies that there exists at most one $G_k^P G$ path between u and v when ever there exists one GG_k^P path between u and v . Now, referring to Theorem 1.3, we get that $A(G)A(G_k^P)$ is graphical. $\square$

Note 2.2. *Let $A(G)A(G_k^P)$ be graphical, and let $u \in V_i$ and $v \in V_j$, $1 \leq i, j \leq k$, $i \neq j$. Let $w \in V_r$, r different from i and j such that $u \sim_G w$ and $w \approx_G v$. In other words, there exists a GG_k^P path between u and v . Now, from the definition of G_k^P and uniqueness of GG_k^P path between u and v , we have that for $w' \neq v$ in the neighbourhood of u , either w' belongs to V_j or $w' \sim_G v$.*

We describe the above theorem with the help of the following example.

Example 2.3. *Consider a partition $P_1 = \{V_1, V_2, V_3, V_4\}$ of $V(G)$ as shown in the Figure 1. $G_4^{P_1}$ is constructed and is shown in the same Figure 1. From the figure, it is clear that $\langle V_1 \rangle, \langle V_2 \rangle, \langle V_3 \rangle$ and $\langle V_4 \rangle$ are totally disconnected and also satisfy conditions (ii) and (iii) given in the Theorem 2.1. Further, by direct computation of $A(G)A(G_4^{P_1})$, which are also shown below, we observe that the product is graphical and the product Γ realizing $A(G)A(G_4^{P_1})$ is also shown in the same figure.*

The partition $P_2 = \{U_1, U_2, U_3, U_4\}$ of $V(G)$ is such that from the vertex v_3 to the vertex v_4 , there is a $GG_4^{P_2}$ path but there is no $G_4^{P_2} G$ path. That is, P_2 does not satisfy the condition (iii) of the Theorem 2.1. Figure 2 shows graph G and its 4-complement $G_4^{P_2}$. In $A(G)A(G_4^{P_2})$, the (i, j) entry is different from (j, i) entry for $i = 3$ and $j = 4$. Therefore $A(G)A(G_4^{P_2})$ is not symmetric and hence is not graphical.

Figure 1: Graphs G , $G_4^{P_1}$ and Γ with respect to partition P_1 of $V(G)$

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A(G_4^{P_1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$$A(G)A(G_4^{P_1}) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Figure 2: Graphs G , $G_4^{P_2}$ with respect to partition P_2 of $V(G)$

$$A(G_4^{P_2}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad A(G)A(G_4^{P_2}) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & 1 \\ 1 & 0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & 1 \\ 1 & 1 & 0 & \mathbf{1} & \mathbf{1} & 0 & 1 & 1 \\ 0 & 0 & \mathbf{0} & 0 & 1 & 1 & 1 & \mathbf{1} \\ 0 & 0 & \mathbf{0} & 1 & 0 & 1 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & 0 & 1 & 1 & 0 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \end{pmatrix}.$$

Let G be a graph on n vertices and $P = \{V_1, V_2, \dots, V_n\}$ be a partition of $V(G)$ of size n . Then every partite set is a singleton set and the n -complement G_n^P is same as the actual complement $\bar{G}$ of G . Then, $A(G)A(G_n^P) = A(G)A(\bar{G})$ is graphical if and only if G is either a 1-factor graph or $(n-2)$ regular graph or else C_5 , the cycle on five vertices ([6, Theorem 10]). So, while characterizing graph G and partition P of $V(G)$ satisfying the property that $A(G)A(G_k^P)$ is graphical, we consider $k \neq n$.

Definition 2.4. Given a graph G , a k -partition $P = \{V_1, V_2, \dots, V_k\}$ of $V(G)$ is said to have perfect matching property in G if V_i is an independent set for every i , $1 \leq i \leq k$, and for every V_i there exists exactly one V_j , $1 \leq j \leq k$, and $i \neq j$, such that the graph induced by the set $V_i \cup V_j$ is a perfect matching. In such a case, V_i and V_j are said to be the matching partite sets.

Lemma 2.5. *Let G be a 1-factor graph and let $P = \{V_1, V_2, \dots, V_k\}$ be a k -partition of $V(G)$. Then $A(G)A(G_k^P)$ is graphical if and only if the partition P has the perfect matching property.*

Proof. Given a 1-factor graph G , let $A(G)A(G_k^P)$ be graphical. Since $k \neq n$, there exists at least one V_i with two or more vertices. Consider any two vertices u and v from V_i , $1 \leq i \leq k$. Since $A(G)A(G_k^P)$ is graphical, from (i) of Theorem 2.1, we have that $\langle V_i \rangle$ is totally disconnected and therefore u and v are not adjacent with each other. Since G is a 1-factor graph, there exists distinct vertices w_1 and w_2 such that $u \sim_G w_1$ and $v \sim_G w_2$ and further w_1 and w_2 are not adjacent in G .

Now to prove that P has perfect matching property, it is enough if we prove that both w_1 and w_2 are in the same partite set in P . To the contrary, suppose w_1 and w_2 are in different partite sets, say $w_1 \in V_r$ and $w_2 \in V_s$, where $1 \leq r, s \leq k$, $r \neq s$. From the definition of G_k^P , we observe that there exists a GG_k^P path ($u \sim_G w_1 \sim_{G_k^P} w_2$). But there is no $G_k^P G$ path between u and w_2 , as v is the only vertex adjacent with w_2 in G and u and v remains to be non adjacent in G_k^P . This would imply that $A(G)A(G_k^P)$ is not graphical, which is a contradiction. Therefore w_1 and w_2 must be in the same partite set, say V_r . Thus $\langle V_i \cup V_r \rangle$ is a perfect matching and therefore P has perfect matching property.

Conversely, let the partition $P = \{V_1, V_2, \dots, V_k\}$ of $V(G)$ be with perfect matching property. So, each of partite set V_i is an independent set in the graph G and therefore, from the definition of G_k^P , we get that G_k^P is a subgraph of $\bar{G}$. Since G is a 1-factor graph, there exists at most one GG_k^P path between any two vertices. Now let there be a GG_k^P path between the vertices u and v given by $u \sim_G x \sim_{G_k^P} v$. Let $u \in V_i$, $v \in V_j$ and $x \in V_r$ and $1 \leq i, j, r \leq k$. Since P has perfect matching property, we have that $r \neq i$, and V_i and V_r are matching partite sets. Also, $r \neq j$ as V_j is an independent set in the graph G as well as in the graph G_k^P . Now for $y \sim_G v$, it is clear that y is not in V_i , as V_i and V_j are not matching partite sets. So, y is not adjacent with u in G and therefore we obtain $G_k^P G$ path between u and v , given by $u \sim_{G_k^P} y \sim_G v$. So, from Theorem 1.3, we get that $A(G)A(G_k^P)$ is graphical. $\square$

Theorem 2.6. *Let G be a graph on n vertices and $P = \{V_1, V_2, \dots, V_k\}$ be a k -partition of $V(G)$ with $k \neq n$. Then $A(G)A(G_k^P)$ is graphical if and only if one of the following holds.*

- (i) *Either G or G_k^P is a null graph, in which case the other one is a complete k -partite graph.*
- (ii) *The partition P has perfect matching property either in G or in G_k^P .*

Proof. Let $A(G)A(G_k^P)$ be graphical, in which case each of partite set is an independent set in G . Therefore from the definition of k -complement of G , it is trivial that G is a null graph (complete k -partite graph) if and only if G_k^P is a complete k -partite graph (null graph). In this case (i) holds.

Suppose G is neither null graph nor a complete k -partite graph. Consider a pair of vertices u and v from same partite set in the partition, say V_1 , which is possible as $k \neq n$. Since $A(G)A(G_k^P)$ is graphical, by Theorem 2.1, we have one of the following cases.

- (a) *For all $w \notin V_1$, $u \sim_G w$ and $w \sim_G v$ (equivalently, $u \sim_{G_k^P} w$ and $w \sim_{G_k^P} v$).*
- (b) *There exist unique $r \notin V_1$ such that $u \sim_G r$ and $r \sim_{G_k^P} v$ in which case there exists a vertex s such that, $u \sim_{G_k^P} s$ and $s \sim_G v$.*

Case (a): For any $x \notin V_1$, it is not possible that $u \sim_{G_k^P} x$ and $x \sim_{G_k^P} v$, otherwise there would be two GG_k^P paths between w and x . So, for every $x \notin V_1$, x is adjacent with both of u and v in G . This implies $\deg_{G_k^P} u = 0$. By Lemma 1.4, $\deg_{G_k^P} u = \deg_{G_k^P} x = 0$ for all $x \notin V_1$. Further, from the definition of k -complement of G and V_1 is independent set in G , it follows that $\deg_{G_k^P} v = 0$ as $\deg_{G_k^P} x = 0$ for all $x \notin V_1$. Hence G_k^P is a null graph, which is not the case in discussion.

Case (b): Without loss of generality, let there exist unique $r \notin V_1$ such that $u \sim_G r$ and $r \sim_{G_k^P} v$, in which case there exists a vertex s such that, $u \sim_{G_k^P} s$ and $s \sim_G v$. For any $t \neq r, s$ and $t \notin V_1$, both u and v are simultaneously adjacent or not adjacent with t in G . Also, note that there is no pair of vertices p and q such

that p is adjacent with both u and v in G , and q is not adjacent with both u and v in G . Otherwise, there would be two GG_k^P paths ($p \sim_G u \sim_{G_k^P} q$ and $p \sim_G v \sim_{G_k^P} q$) between p and q .

In case both the vertices u and v are not adjacent with $t \notin V_1$ and $t \neq r, s$, then $\deg_G u = \deg_G v = 1$. As u and v are adjacent with t in G_k^P , from Lemma 1.4, we have that $\deg_G t = \deg_G u = \deg_G v = 1$ for all $t \notin V_1$. Since r and v are adjacent with each other in G_k^P , $\deg_G r = 1$. For all $z \in V_1$, and $z \neq u$ we have that $r \sim_{G_k^P} z$, and by Lemma 1.4, it implies that $\deg_G z = 1$.

Similarly, in the case of both the vertices u and v are adjacent with $t \notin V_1$ and $t \neq r, s$, we prove that G_k^P is 1-factor graph. So in the case (ii), we have that either G or G_k^P is 1-factor graph. Therefore, by Lemma 2.5, it follows that the partition P satisfies perfect matching property.

Converse part is trivial if G and G_k^P satisfy the condition (i) given in the theorem. If G and G_k^P satisfy the condition (ii), the converse part follows from Lemma 2.5, as graph G with partition P satisfying perfect matching property is trivially a 1-factor graph. $\square$

We note the following whenever $A(G)A(G_k^P)$ is graphical.

- (i) If $|V_i| = n_i$, $1 \leq i \leq k$, then $A(G) + A(G_k^P) = A(K_{n_1, n_2, \dots, n_k})$, where $K_{n_1, n_2, \dots, n_k}$ is the complete k -partite graph with respect to the same partition P .
- (ii) If P is with perfect matching property in G , then G_k^P is connected except for the case when $k = 2$ and $n = 4$, in which case G and G_k^P are isomorphic.
- (iii) If P is with perfect matching property in G , then by Theorem 1.5, degree sequences of graphs G_k^P and Γ are the same, where $A(\Gamma) = A(G)A(G_k^P)$. In fact, complement of graph Γ is the disjoint union of complete bipartite graph, where each component is of the form $\langle V_i \cup V_j \rangle$ in which V_i and V_j are the partite sets in this component and are matching partite sets in G .

3 Some more results on the graphs G , G_k^P and Γ satisfying $A(G)A(G_k^P) = A(\Gamma)$

In this section, we investigate the graph parameters like chromatic number, domination number etc., of G_k^P and Γ where, $A(G)A(G_k^P) = A(\Gamma)$ and G is a 1-factor graph with perfect matching property.

We recall that, a set $S \subseteq V(G)$ is a dominating set if every vertex not in S has a neighbour in S . The domination number $\gamma(G)$ is the minimum size of a dominating set in G . A dominating set S in G is a connected dominating set if $\langle S \rangle$ is connected, an independent dominating set if $\langle S \rangle$ is independent.

A clique in a graph is a set of pairwise adjacent vertices. The clique number of G , denoted by $\omega(G)$, is the maximum size of a clique in G .

The chromatic number of a graph G , denoted by $\chi(G)$, is the minimum number of colours needed to colour the vertices so that adjacent vertices receive different colours.

In all the discussions that follow, we consider a graph G and $2k = K$ -partition $P = \{V_1, V_2, \dots, V_k, V'_1, V'_2, \dots, V'_k\}$ of $V(G)$. Let V_i and V'_i , $1 \leq i \leq k$ be the matching partite sets. The following remarks hold when $A(G)A(G_k^P) = A(\Gamma)$.

Remark 3.1. By the definition of Γ , for a vertex $u \in V_i$, $1 \leq i \leq k$, $u \sim_\Gamma v$ for every vertex v in V'_i and $u \sim_\Gamma w$ for every vertex $w \notin V'_i$. Hence, for a vertex u in V_i and a vertex v in V'_i , $1 \leq i \leq k$, $\{u, v\}$ is an independent dominating set of the graph Γ . The set $\{x, y\}$, where $x \in V_i$ and $y \in V'_j$, $1 \leq i, j \leq k$, $i \neq j$, is a connected dominating set of Γ . Since no single vertex can dominate all other vertices, both connected domination number and independent domination number are equal to two for the graph Γ .

Remark 3.2. Similarly, for a vertex $u \in V_i$, $u \sim_{G_k^P} v$ for every vertex v in V_i and also $u \sim_{G_k^P} w$ for that vertex w in V'_i for which $u \sim_G w$, $1 \leq i \leq k$. Hence the set $\{u, v\}$ where, $u \in V_i$ and $v \in V'_i$ and $u \sim_G v$ is an independent

dominating set of G_K^P , $1 \leq i \leq k$. Also, the set $\{x, y\}$ forms a connected dominating set of G_K^P in all the following cases.

- (i) $x \in V_i$ and $y \in V_j$,
- (ii) $x \in V_i$ and $y \in V'_j$,
- (iii) $x \in V'_i$ and $y \in V'_j$, $1 \leq i, j \leq k$, $i \neq j$.

As no single vertex dominates all other vertices in G_K^P , both connected domination number and independent domination number of G_K^P are equal to two.

Remark 3.3. By definition of Γ , the graphs $G_1 = \langle \bigcup_{i=1}^k V_i \rangle$ and $G_2 = \langle \bigcup_{i=1}^k V'_i \rangle$ are complete subgraphs of Γ . Since no vertex of G_1 is adjacent to all the vertices of G_2 and no vertex of G_2 is adjacent to all the vertices of G_1 , both G_1 and G_2 are maximal complete subgraphs of Γ , with order $\frac{n}{2}$, where n is the order of the graph Γ . Hence, $\omega(\Gamma) \geq \frac{n}{2}$.

Now we show that the complete subgraph of Γ with maximum size is of order $\frac{n}{2}$. Let H be a complete subgraph of Γ . For any vertex $x \in V(H)$, $x \in V_i$, $1 \leq i \leq k$, we have that every $y \in V_i$ is also in $V(H)$ and every $z \in V'_i$ is not in $V(H)$. Otherwise, by the structure of Γ , for $y \in V_i$ and not in $V(H)$, note that for every vertex v with $v \sim_\Gamma x$, we have $v \sim_\Gamma y$ also. Therefore, the graph $\langle y \cup V(H) \rangle$ is a complete graph with cardinality more than that of $V(H)$, which contradicts the maximality. Similarly, for $z \in V'_i$, if $z \in V(H)$ then by the structure of Γ , we have that $z \sim_\Gamma x$, which contradicts the completeness of Γ .

Also note that, for every i , either V_i is a subset of $V(H)$ or else V'_i is a subset of $V(H)$. Suppose that for an i , $1 \leq i \leq k$, V'_i is not a subset of $V(H)$. If V_i is also not a subset of $V(H)$, then for any $x \in V_i$, we have that $x \sim_\Gamma y$ for every $y \in \langle V_j \cup V'_j \rangle$ for $j \neq i$. Therefore, $\langle x \cup V(H) \rangle$ is complete, which contradicts the maximality of $V(H)$. Therefore V_i must be a subset of $V(H)$. This proves that the cardinality of $V(H)$ is $\frac{n}{2}$.

Remark 3.4. In G_K^P , since every partite set is independent, no clique can have more than one element from same partite set. Hence, $\omega(G_K^P) \leq K$. Consider the case when every partite set has at least two vertices. Let H be a maximum clique in G_K^P . Suppose $V(H)$ does not contain any vertex from some partite set V_i , $1 \leq i \leq k$. If $V(H)$ contains no vertex of V'_i also, then by definition of G_K^P , for every vertex x in V'_i , $x \sim_{G_K^P} y$ for every y in H . Hence $\langle x \cup V(H) \rangle$ is a clique with cardinality more than that of H , which is a contradiction to the maximality of H . If $V(H)$ contains a vertex z of V'_i , then for any vertex $x \in V_i$, $x \sim_{G_K^P} z$, by definition of G_K^P , $x \sim_{G_K^P} y$ for every $y \in V(H)$. This implies that $\langle x \cup V(H) \rangle$ is complete, contradicting maximality of H . Thus $V(H)$ contains exactly one vertex from every partite set and hence $|V(H)| = K$.

Let V_i and V'_i be two matching partite sets which are singletons. Then, maximum complete graph H cannot contain both the vertices u and v where $u \in V_i$ and $v \in V'_i$, since $u \sim_{G_K^P} v$. If H neither contains u nor contains v then both $\langle u \cup V(H) \rangle$ and $\langle v \cup V(H) \rangle$ are complete graphs of order greater than that of $V(H)$, a contradiction to the maximality of H . Therefore, $V(H)$ contains exactly one of the two vertices u and v . Let there be k_1 partite sets, which are singletons. Then any maximum complete graph H can contain exactly one of two vertices of $\langle V_i \cup V'_i \rangle$. Thus, when the partition P has k_1 singleton partite sets, $\omega(G_K^P) = K - \frac{k_1}{2}$.

Remark 3.5. By Theorem 2.1, every partite set V_i and V'_i are independent. Hence $\chi(G_K^P) \leq K$. Suppose that $|V_i| \geq 2$ for every i , $1 \leq i \leq k$. By definition of G_K^P , it follows that, in any proper colouring of vertices of G_K^P , two vertices u and v may receive the same colour if and only if either they are in the same partite set or one is in V_i and the other is in V'_i and are adjacent to each other in G , $1 \leq i \leq k$. Suppose that $u \in V_i$ and $v \in V'_i$ with $u \sim_G v$ receive same colour say c , in a proper colouring of G_K^P . Then the colour c cannot be given to any vertex in V_i other than u (since they are all adjacent to v) and any vertex in V'_i , other than v (since they are all adjacent to u). At least two colours (2 colours when $|V_i| = 2$ and 3 colours when $|V_i| \geq 3$) are needed to colour vertices in $V_i \cup V'_i$, $1 \leq i \leq k$. Thus, optimal colouring results when all the vertices of any partite set receive the same colour and no two vertices of two different partite sets receive the same colour. Thus $\chi(G_K^P) = K$.

Now consider the case that there are k_1 partite sets which are singletons. Let $u \in V_i$ and $v \in V'_i$ where $|V_i| = |V'_i| = 1$ for some i , $1 \leq i \leq k$. Then u and v can share the same colour in any proper colouring of G_K^P . But, the colour shared by u and v , cannot be given to any other vertex because every other vertex is adjacent to both u and v in G_K^P . A proper colouring of G_K^P with minimum number of colours can be obtained as follows.

All the vertices of any partite set with cardinality greater than or equal to two receive the same colour so that no two vertices from two different partite sets with cardinality greater than or equal to two share the same colour. This requires $K - k_1$ colours. For the remaining k_1 partite sets which are singletons, by sharing the colour between the vertices in the matching partite sets, a proper minimal colouring of G_K^P can be obtained with $(K - k_1) + \frac{k_1}{2} = K - \frac{k_1}{2}$ colours. This proves that $\chi(G_K^P) = K - \frac{k_1}{2}$, where k_1 is the number of singleton partite sets in the partition P .

Remark 3.6. From the Remark 3.3, we have $\omega(\Gamma) = \frac{n}{2}$. By sharing the colour between the vertices u and v such that $u \sim_G v$, we get a proper minimal colouring of Γ with $\frac{n}{2}$ colours. Therefore $\chi(\Gamma) = \frac{n}{2}$.

Remark 3.7. For any $u \in V_i$, $1 \leq i \leq k$, $\deg_{G_K^P} u = \deg_\Gamma u = n - n_i - 1$, where $n_i = |V_i| = |V'_i|$. When $|V_i| = |V_j|$ for every i and j , $1 \leq i, j \leq k$, both G_K^P and Γ are regular with regularity $(K - 1)\frac{n}{K} - 1$. If $K = n$ then $G_K^P = G_n^P = \overline{G}$, is regular with regularity $(n - 2)$ and has maximum number of edges, $\frac{n(n - 2)}{2}$. If $K = 2$, then $n_1 = |V_1| = |V'_1| = \frac{n}{2}$ and both G_K^P and Γ are regular with regularity $\frac{n - 2}{2}$. In this case, they have minimum number of edges, given by $\frac{n(n - 2)}{4}$.

Using the clique size of the graphs G_K^P and Γ , we characterize the partition P such that G_K^P is isomorphic to Γ .

Theorem 3.8. Let G be a graph on n vertices and let $P = \{V_1, V_2, \dots, V_k, V'_1, V'_2, \dots, V'_k\}$ be a $2k$ -partition of $V(G)$ satisfying perfect matching property in G . Let V_i and V'_i be the matching partite sets, $1 \leq i \leq k$, and let $A(G)A(G_K^P) = A(\Gamma)$ where $K = 2k$. Then the graphs G_K^P and Γ are isomorphic to each other if and only if the partition P satisfies the property that $|V_i| = |V'_i| \leq 2$ for every i , $1 \leq i \leq k$.

Proof. Let $|V_i| = |V'_i| \leq 2$ for every i , $1 \leq i \leq k$. Define a mapping σ from $V(G_K^P)$ to $V(\Gamma)$ as follows. For every vertex u in V_i or V'_i with $|V_i| = |V'_i| = 1$, $\sigma(u) = u$. And, for every $u_1, u_2 \in V_j$ and $u'_1, u'_2 \in V'_j$ with $|V_j| = |V'_j| = 2$, and $u_1 \sim_G u'_1, u_2 \sim_G u'_2$, $\sigma(u_1) = u_1, \sigma(u_2) = u'_2, \sigma(u'_1) = u'_1$ and $\sigma(u'_2) = u_2$, i.e., σ is given by

$$\sigma = \begin{pmatrix} u_1 & u_2 & u'_1 & u'_2 \\ u_1 & u'_2 & u'_1 & u_2 \end{pmatrix}.$$

The mapping σ is a permutation from $V(G_K^P)$ to $V(\Gamma)$ which fixes u_1, u'_1 and swaps u_2 and u'_2 . To prove that σ is an isomorphism, we show the following.

$$\text{For every } u \in V(G_K^P), \sigma(u) \in V(\Gamma), \text{ an element } v \in N_{G_K^P}(u) \text{ if and only if } \sigma(v) \in N_\Gamma(\sigma(u)). \quad (1)$$

Observe that, when $|V_i| = |V'_i| = 1$ for every i , $1 \leq i \leq k$, both G_K^P and Γ are same as $\overline{G}$, the complement of G . Hence they are same.

Now consider the case that the partition P has partite sets of cardinality two also. We show that σ is an isomorphism from G_K^P to Γ using the property in 1.

Consider V_i and V'_i with $u_1, u_2 \in V_i$ and $u'_1, u'_2 \in V'_i$ with $u_1 \sim_G u'_1$ and $u_2 \sim_G u'_2$ (Refer Figure 3).

We consider the following two cases..

Case (i): Consider $u_1 \in V_i$ such that $\sigma(u_1) = u_1$. Then, by definition of G_K^P ,

$$N_{G_K^P}(u_1) = u'_2 \cup \{x | x \notin V_i \cup V'_i\},$$

Figure 3: Adjacencies of u_1, u_2, u'_1 and u'_2 in the graphs G, G_K^P and Γ

and

$$\begin{aligned} N_{\Gamma}(\sigma(u_1)) = N_{\Gamma}(u_1) &= \{u_2\} \cup \{y | y \notin V_i \cup V'_i\} \\ &= \{\sigma(u'_2)\} \cup \{y | y \notin V_i \cup V'_i\}. \end{aligned}$$

Case (ii): Consider $u_2 \in V_i$ with $\sigma(u_2) = u'_2$. Now,

$$N_{G_K^P}(u_2) = u'_1 \cup \{x | x \notin V_i \cup V'_i\},$$

and

$$\begin{aligned} N_{\Gamma}(\sigma(u_2)) = N_{\Gamma}(u'_2) &= \{u'_1\} \cup \{y | y \notin V_i \cup V'_i\} \\ &= \{\sigma(u'_1)\} \cup \{y | y \notin V_i \cup V'_i\}. \end{aligned}$$

Thus, we observe that Equation 1 is satisfied which implies that σ is an isomorphism from G_K^P to Γ .

Conversely, let $G_K^P \cong \Gamma$. Then $\omega(G_K^P) = \omega(\Gamma)$

$\Rightarrow K - \frac{k_1}{2} = \frac{n}{2}$ where k_1 is the number of singleton sets in P i.e., $2K = n + k_1$.

If every partite sets with cardinality greater than one contains exactly two elements, then there are $\frac{n - k_1}{2}$ partite sets which are not singletons. Now suppose that there is a partite set with three or more vertices in it, then K satisfies

$$K < \frac{n - k_1}{2} + k_1 = \frac{n + k_1}{2}.$$

$\Rightarrow 2K < n + k_1$, which is a contradiction, since $2K = n + k_1$. Therefore there is no partite set V_i with $|V_i| \geq 3$. $\square$

4 Rank of the graphs G, G_K^P and Γ when $A(G)A(G_K^P) = A(\Gamma)$

Throughout this section we consider the graph G and G_K^P are such that $A(G)A(G_K^P)$ is realizable and let Γ be the realization of the matrix product. Since the rank of null graph and complete k -partite graph are trivial, we consider the nontrivial case in which the partition P has the perfect matching property in G . Also note that the rank of 1-factor graph is same as the number of vertices in the graph. Therefore, we restrict ourselves to find rank of G_K^P , which is same as that of rank of Γ .

Given a 1-factor graph G with $2n$ vertices, we propose a partition P for which G_K^P is a graph of rank r and $A(G)A(G_K^P)$ is graphical, where $n \leq r \leq 2n$.

In the following theorem we obtain relation between rank of G_K^P , the number of vertices and the number of partite sets with cardinality one.

Theorem 4.1. Let G be a graph and let P be a partition of size $2k = K$ of $V(G)$ satisfying perfect matching property in G . Suppose that $A(G)A(G_K^P) = A(\Gamma)$. Then the rank of G_K^P is $n - l$ where n is the number of vertices in G and $2l$ is the number of partite sets of cardinality one in the partition P .

Proof. Let $P = \{V_1, V_2, \dots, V_m, V_{m+1}, \dots, V_k, V'_1, V'_2, \dots, V'_m, V'_{m+1}, \dots, V'_k\}$ be a partition of $V(G)$ with $|V_i| = |V'_i| = n_i$, satisfying the following properties.

(i) $n_i \geq 2$ for $1 \leq i \leq m$,

(ii) $n_i = 1$ for $m + 1 \leq i \leq k$, with $k - m = l$.

Further, let $V_i = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$ and $V'_i = \{v'_{i1}, v'_{i2}, \dots, v'_{in_i}\}$ be the matching partite set with v_{ij} adjacent with v'_{ij} in G for $1 \leq i \leq k$, $1 \leq j \leq n_i$.

We prove that the rank of G_K^P is $n - l$ by showing that the dimension of the right null space of $A(G_K^P)$ is equal to l .

We index the rows of $A(G_K^P)$ as follows. The first n_1 rows correspond to the n_1 vertices in V_1 , and the next n_1 rows correspond to n_1 vertices in V'_1 . Further, the next n_2 rows correspond to vertices in V_2 and so on. The last row correspond to the vertex in V'_k .

Then $A(G_K^P)$ can be viewed as follows;

$$A(G_K^P) = \begin{pmatrix} 0_{n_1 \times n_1} & (J - I)_{n_1 \times n_1} & J_{n_1 \times n_2} & \dots & \dots & \dots & J_{n_1 \times 1} \\ (J - I)_{n_1 \times n_1} & 0_{n_1 \times n_1} & J_{n_1 \times n_2} & \dots & \dots & \dots & J_{n_1 \times 1} \\ J_{n_2 \times n_1} & J_{n_2 \times n_1} & \dots & \dots & \dots & \dots & J_{n_2 \times 1} \\ J_{n_2 \times n_1} & J_{n_2 \times n_1} & \dots & \dots & \dots & \dots & J_{n_2 \times 1} \\ \vdots & \vdots & & & & & \\ J_{n_m \times n_1} & J_{n_m \times n_1} & \dots & 0_{n_m \times n_m} & (J - I)_{n_m \times n_m} & \dots & J_{n_m \times 1} \\ J_{n_m \times n_1} & J_{n_m \times n_1} & \dots & (J - I)_{n_m \times n_m} & 0_{n_m \times n_m} & \dots & J_{n_m \times 1} \\ \vdots & \ddots & & \ddots & & & \\ J_{1 \times n_1} & \dots & & & & & 0_{1 \times 1} \end{pmatrix}.$$

Observe that the row in $A(G_K^P)$ corresponding to the vertex v_{ij} of V_i has zero entries at column positions corresponding to vertices in V_i and at one more column position corresponding to the vertex v'_{ij} . All other entries in the row except these $(n_i + 1)$ entries are equal to one.

Let X be a vector of order $(n \times 1)$ given by

$X = (x_{11}, \dots, x_{1n_1}, x'_{11}, \dots, x'_{1n_1}, \dots, x_{m1}, \dots, x_{mn_m}, x'_{m1}, \dots, x'_{mn_m}, x_{m+11}, x'_{m+11}, \dots, x_k, x'_k)$ which is in the right null space of $A(G_K^P)$. Then $A(G_K^P)X = 0$.

For our convenience, we say that component x_{ij} of the vector X corresponds to the vertex v_{ij} , $1 \leq i \leq k$, $1 \leq j \leq n_i$.

Consider two vertices v_{ij} and v_{ir} in the same partite set V_i , $1 \leq i \leq m$, $1 \leq j, r \leq n_i$, $j \neq r$. Then v'_{ij} and v'_{ir} are their corresponding vertices in V'_i , such that $v_{ij} \sim_G v'_{ij}$ and $v_{ir} \sim_G v'_{ir}$. The rows of $A(G_K^P)$ corresponding to the vertices v_{ij} and v_{ir} differ only at the two columns corresponding to the vertices v'_{ij} and v'_{ir} .

Taking difference of the two equations in $A(G_K^P)X = 0$ corresponding to the above two rows, we get, $x'_{ij} = x'_{ir}$. Since i, j and r are arbitrary, the above says that, for each given i , all the entries in X corresponding to elements of V'_i are same. Similarly, by taking V'_i instead of V_i , one can show that, all the entries in X corresponding to vertices of V_i for each i are same. Hence X is as follows.

$X = (\alpha_1, \alpha_1, \dots, \alpha_1, \alpha'_1, \alpha'_1, \dots, \alpha'_1, \dots, \alpha_m, \alpha_m, \dots, \alpha_m, \alpha'_m, \alpha'_m, \dots, \alpha'_m, \alpha_{m+1}, \alpha'_{m+1}, \dots, \alpha_k, \alpha'_k)$.

Observe that α_i and α'_i repeat n_i times for every i , $1 \leq i \leq m$.

Now consider the vertices, v_{ij} from V_i and v'_{ir} from V'_i , $1 \leq i \leq m$, $1 \leq j, r \leq n_i$. In G_K^P , v_{ij} and v'_{ir} are commonly adjacent with every vertex outside $V_i \cup V'_i$ and v_{ij} is adjacent with every vertex in V'_i , except v'_{ij} and v'_{ir} is adjacent with every vertex in V_i , except v_{ir} . Thus the rows corresponding to v_{ij} and v'_{ir} differ at $(n_i - 1)$ column positions. Taking the difference of the corresponding equations in $A(G_K^P)X = 0$, we get,

$$(n_i - 1)\alpha_i = (n_i - 1)\alpha'_i.$$

The above implies that $\alpha_i = \alpha'_i$, provided $n_i \neq 1$.

Then X is as follows,

$X = (\alpha_1, \alpha_1, \dots, \alpha_1, \alpha_2, \dots, \alpha_2, \dots, \alpha_m, \alpha_m, \dots, \alpha_m, \alpha_{m+1}, \alpha'_{m+1}, \dots, \alpha_k, \alpha'_k)$ where α_i repeats $2n_i$ times, $1 \leq i \leq m$.

Suppose that, for every i , $1 \leq i \leq k$, $n_i \neq 1$. Then all the entries in X corresponding to vertices of V_i and V'_i are same for each i , $1 \leq i \leq k$. Then X becomes,

$X = (\alpha_1, \alpha_1, \dots, \alpha_1, \alpha_2, \alpha_2, \dots, \alpha_2, \dots, \alpha_k, \alpha_k, \dots, \alpha_k)$ where each α_i repeats $2n_i$ times, $1 \leq i \leq k$.

Consider two vertices, $v_{ir} \in V_i$ and $v_{jl} \in V_j$, where $1 \leq i, j \leq k$, $i \neq j$, $1 \leq l \leq n_j$, $1 \leq r \leq n_i$. The rows of $A(G_K^P)$ corresponding to the above two vertices differ at $(n_i + 1) + (n_j + 1)$ column positions. Taking the difference of the equations in $A(G_K^P)X = 0$ corresponding to the above two rows, we get,

$$(n_i + 1)x_{jl} = (n_j + 1)x_{ir}, \text{ i.e.,}$$

$$\alpha_i = \frac{n_i + 1}{n_j + 1} \alpha_j, \quad 1 \leq i, j \leq k.$$

Since the above is true for every i and j , $1 \leq i, j \leq k$, $i \neq j$, every α_i can be written as a scalar multiple of α where $\alpha_1 = \alpha$. Then X becomes,

$$X = (\alpha, \alpha, \dots, \alpha, r_2\alpha, r_2\alpha, \dots, r_2\alpha, \dots, r_k\alpha, r_k\alpha, \dots, r_k\alpha) \text{ where } r_t = \frac{n_t + 1}{n_1 + 1}, \quad 2 \leq t \leq k.$$

Now, $A(G_k^P)X = 0$ implies $\alpha = 0$. Hence X is a zero vector and the right null space is a zero space with dimension zero. This implies that rank of G_k^P is n .

Now suppose that there are $2l$ partite sets which are singletons, in addition to $2m$ partite sets with cardinality greater than or equal to two. Then X is of the form

$$X = (\alpha, \alpha, \dots, \alpha, r_2\alpha, r_2\alpha, \dots, r_2\alpha, \dots, r_m\alpha, r_m\alpha, \dots, r_m\alpha, \alpha_{m+1}, \alpha'_{m+1}, \dots, \alpha_k, \alpha'_k)$$

$$\text{where } r_t = \frac{n_t + 1}{n_1 + 1}, \quad 2 \leq t \leq m.$$

To resolve the above case, we consider two vertices, $v_{1r} \in V_1$ and $v_{j1} \in V_j$ where $m + 1 \leq j \leq k$, $1 \leq r \leq n_1$. The rows of $A(G_k^P)$ corresponding to these vertices differ at $(n_1 + 1) + (1 + 1)$ column positions. Taking the difference of the equations in $A(G_k^P)X = 0$ corresponding to the above two rows, we get,

$$(n_1 + 1)x_{1r} = x_{j1} + x'_{j1} \text{ which implies that}$$

$$\alpha_j + \alpha'_j = (n_1 + 1)\alpha \text{ for } m + 1 \leq j \leq k.$$

Observe that the equations in $A(G_k^P)X = 0$ corresponding to the vertices v_{j1} and v'_{j1} are same for $m + 1 \leq j \leq k$.

Now consider two vertices v_{i1} and v_{j1} in V_i and V_j respectively, with $m + 1 \leq i, j \leq k$, $i \neq j$.

Taking the difference of the equations corresponding to the above two rows, we get,

$$\alpha_j + \alpha'_j = \alpha_i + \alpha'_i, \text{ which is true for arbitrary } i \text{ and } j.$$

Finally, consider the l equations corresponding to the l vertices v_{j1} , $m + 1 \leq j \leq k$. Since $\alpha_j + \alpha'_j$ is a scalar multiple of α for every j , substituting in any of the above l equations we get $\alpha = 0$. But, $\alpha_j + \alpha'_j = (n_1 + 1)\alpha$,

$$\Rightarrow \alpha_j + \alpha'_j = 0 \text{ for every } j, m + 1 \leq j \leq k,$$

$$\Rightarrow \alpha_j = -\alpha'_j.$$

Thus, there are l linearly independent vectors, say $\beta_{m+1}, \beta_{m+2}, \dots, \beta_k$, in the right null space of $A(G_k^P)$, which are given as follows.

The vertex β_{m+i} has all the components equal to zero except at the positions corresponding to the vertices v_{m+i} and v'_{m+i} , $1 \leq i \leq (k - m)$, and the entries at these positions are c and $-c$ where c is any non zero constant. Since these vectors form a basis for the right null space of $A(G_k^P)$, rank of G_k^P is $n - l$. $\square$

Since G is a graph with partition P having perfect matching property, $A(G)$ is a permutation matrix. Therefore, when $A(G)A(G_k^P) = A(\Gamma)$, rank of Γ is same as rank of G_k^P . Hence, the following theorem follows.

Theorem 4.2. Let G be a graph and let P be a partition of size $2k = K$ of $V(G)$ satisfying perfect matching property in G . Let $A(G)A(G_k^P) = A(\Gamma)$. Then the rank of Γ is $n - l$ where n is the number of vertices in G and $2l$ is the number of partite sets of cardinality one in the partition P .

Remark 4.3. Given a 1-factor graph, using the Theorems 4.1 and 4.2, we propose a partition P for which G_k^P is a graph of rank r and $A(G)A(G_k^P)$ is graphical, where $n \leq r \leq 2n$.

Example 4.4. In the following example, we illustrate a procedure to construct a nontrivial graph H with the given rank r , $n \leq r \leq 2n$. We consider a properly chosen 1-factor graph G of suitable order n and use the Theorems 4.1 and 4.2, to partition the vertex set of G so that $n - l = r$ where, $2l$ is the number of singleton sets in the partition and $H = G_k^P$ is a graph of rank r .

Let $r = 7$. Consider the 1-factor graph G with 10 vertices and a partition P of size 8 as shown in the Figure 4. The adjacent graph shown in the Figure 4, is G_8^P with respect to the same partition P . The graph G_8^P is of rank $n - l = 10 - 3 = 7$. In this case $G_8^P \cong \Gamma_1$, where $A(\Gamma_1) = A(G)A(G_8^P)$.

For the 1-factor graph H_1 with 8 vertices and a partition P of size 4 as shown in the Figure 5, the graph $(H_1)_4^P$ is of rank $n - l = 8 - 1 = 7$. In this case, $(H_1)_4^P \not\cong \Gamma_2$, where $A(\Gamma_2) = A(H_1)A((H_1)_4^P)$. Therefore Γ_2 is also of rank 7.

For the 1-factor graph H_2 with 14 vertices and a partition P of size 14 as shown in the Figure 5, and the graph $(H_2)_{14}^P = \bar{G}$ is of rank $n - l = 14 - 7 = 7$. In this case, $(H_2)_{14}^P \cong \Gamma_3$, where $A(\Gamma_3) = A(H_2)A((H_2)_{14}^P)$.

Figure 4: Graphs G and G_8^P

Figure 5: Graph H_1 ($n = 8, k = 4$) and graph H_2 ($n = 14, k = 14$)

5 A Commuting decomposition of $K_{n_1, n_2, \dots, n_k}$ into a 1-factor subgraph and its k -complement

In this section we obtain a commuting decomposition of a complete k -partite graph G into a perfect matching and its k -complement with respect to the k -partition of G .

Note 5.1. From the result of Theorem 1 of [6], one can conclude that the (i, j) entry of $A(G)A(H)$ represents number of GH paths from v_i to v_j . Hence G and H commute with each other if and only if for every two vertices v_i and v_j , $i \neq j$, $1 \leq i, j \leq n$, the number of GH paths from v_i to v_j is same as number of HG paths from v_i to v_j .

The Theorem 5.2 explains the commuting decomposition of $K_{n_1, n_2, \dots, n_k}$ into a perfect matching and its k -complement. We consider the partition P to be the k -partition of the complete k -partite graph $K_{n_1, n_2, \dots, n_k}$.

Theorem 5.2. Let G be a complete k -partite graph $K_{n_1, n_2, \dots, n_k}$ on n vertices with the k -partition of $V(G)$ given by $P = \{V_1, V_2, \dots, V_k\}$ and let $|V_i| = n_i$, $1 \leq i \leq k$. Then G is decomposable into two commuting subgraphs, one of which is perfect matching of G , say H_1 , and the other one is the k -complement of H_1 with respect to the

same partition $P = \{V_1, V_2, \dots, V_k\}$, if and only if the number of partite sets in P with same cardinality n_i is even for every i , $1 \leq i \leq k$.

Proof. Consider the complete k -partite graph $G = K_{n_1, n_2, \dots, n_k}$. Suppose that G has a commuting decomposition into two subgraphs H_1 and H_2 , one of which, say H_1 , is a perfect matching of G and the other one is given by $H_2 = (H_1)_k^P$, the k -complement of H_1 with respect to the partition $P = \{V_1, V_2, \dots, V_k\}$.

Since G has a perfect matching H_1 , the number of vertices in G is even. If all the partite sets are singletons, then $k = n$ is also even and assertion of the theorem is obviously true.

Suppose that there is a partite set, say V_i , $1 \leq i \leq k$, which has two or more vertices. Let $u \in V_i$ and $v \in V_j$ with $u \sim_{H_1} v$, $1 \leq i, j \leq k$, $i \neq j$. Let $w \in V_i$ and let $w \sim_{H_1} x$. If $x \in V_t$, $t \neq j$, then there is a $H_1 H_2$ path from w to v but no $H_2 H_1$ path from w to v which by Note 5.1, contradicts the assumption that H_1 and H_2 commute with each other. Hence if $u \in V_i$ and $v \in V_j$, $1 \leq i, j \leq k$, $i \neq j$, if $u \sim_{H_1} v$, then for each vertex in V_i the only adjacent vertex of H_1 is also in V_j . Same is true for V_j which implies that $|V_i| = |V_j|$. Similarly, partite sets with same cardinality are paired and since H_1 is a perfect matching, the number of partite sets with cardinality n_i is even for every i , $1 \leq i \leq k$.

Conversely, assume that the number of partite sets with same cardinality is even. Then we can pair the partite sets having the same cardinality. Without loss of generality, assume that $|V_1| = |V_2| = n_1$, $|V_3| = |V_4| = n_2, \dots, |V_{2r-1}| = |V_{2r}| = n_r$, where $2r = k$. Observe that n_i may be equal to n_j for some i and j , $1 \leq i, j \leq r$, $i \neq j$. Since for every i , $1 \leq i \leq r$, $\langle V_{2i-1} \cup V_{2i} \rangle$ is a complete bipartite graph with $|V_{2i-1}| = |V_{2i}|$, there is a perfect matching in the induced graph. Union of all these perfect matching gives a perfect matching in G , denoted by H_1 . In the graph $G \setminus H_1$, obtained by deleting the edges of H_1 from G , two vertices are adjacent if and only if they are in two different partite sets and they are not adjacent in H_1 . Hence $G \setminus H_1$ is same as $(H_1)_k^P$ which implies that $G = H_1 \cup (H_1)_k^P$ and $E(H_1) \cap E((H_1)_k^P) = \emptyset$. Thus H_1 and $H_2 = (H_1)_k^P$ form a decomposition of G . To show that H_1 and $(H_1)_k^P$ form a commuting decomposition of G , we show that for any two vertices u and v in G , the number of $H_1 H_2$ path from u to v is same as the number of $H_2 H_1$ path from u to v .

Consider u, v in the same partite set V_i . Let $u \sim_{H_1} u'$ and $v \sim_{H_1} v'$. Then, $u \sim_{H_1} u' \sim_{H_2} v$ is the only $H_1 H_2$ path and $u \sim_{H_2} v' \sim_{H_1} v$ is the only $H_2 H_1$ path from u to v .

Consider $u \in V_i$ and let $u' \in V_j$ with $u \sim_{H_1} u'$. For any other vertex x in V_j , there neither exists a $H_1 H_2$ path nor exists a $H_2 H_1$ path from u to x . Consider a vertex $x \in V_l$, $l \neq i, j$, with $x \sim_{H_1} x'$. Then $u \sim_{H_1} u' \sim_{H_2} x$ is the only $H_1 H_2$ path and $u \sim_{H_2} x' \sim_{H_1} x$ is the only $H_2 H_1$ path from u to x . Hence the decomposition $\{H_1, H_2 = (H_1)_k^P\}$ of G is a commuting decomposition. $\square$

Note 5.3. Let G be a complete k -partite graph $K_{n_1, n_2, \dots, n_k}$ with respect to a partition $P = \{V_1, V_2, \dots, V_k\}$ of $V(G)$. Then the graph G is decomposable into two commuting perfect matchings i.e., both H_1 and $H_2 = (H_1)_k^P$ are perfect matchings only when $n = 4$ and $k = 2$. Equivalently, $K_{2,2}$ is the only complete k -partite graph, with $k = 2$ which is decomposable into two commuting perfect matchings as above.

Note 5.4. Let G be a complete k -partite graph $K_{n_1, n_2, \dots, n_k}$ with respect to a partition $P = \{V_1, V_2, \dots, V_k\}$ of $V(G)$. Then the graph G is decomposable into two commuting subgraphs, one of them is a perfect matching and the other one is a Hamiltonian cycle i.e., H_1 is a perfect matching and $H_2 = (H_1)_k^P$ is a Hamiltonian cycle when either $n = 6$ and $k = 2$ or $n = 4$ and $k = 4$. Equivalently, $K_{3,3}$ and $K_{1,1,1,1}$ are the only complete k -partite graphs, with $k = 2$ and $k = 4$ respectively, which are decomposable into two commuting subgraphs, one of them is a perfect matching and the other one is a Hamiltonian cycle.

Conclusion: This paper contains characterization of a graph G and the k -partition P of $V(G)$ satisfying the property that $A(G)A(G_k^P)$ is graphical. It also contains characterization of the partition P such that $G_k^P \cong \Gamma$, when $A(G)A(G_k^P) = A(\Gamma)$. Given a 1-factor graph G with $2n$ vertices, we propose a partition P for which G_k^P is a graph of rank r and $A(G)A(G_k^P)$ is graphical, where $n \leq r \leq 2n$. Finally, a characterization of the existence of commuting decomposition of the complete k -partite graph $K_{n_1, n_1, \dots, n_k}$ into a perfect matching and its k -complement is obtained.

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