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Self-dual Leonard pairs

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Abstract: Let \mathbb{F} denote a field and let V denote a vector space over \mathbb{F} with finite positive dimension. Consider a pair A, A^* of diagonalizable \mathbb{F} -linear maps on V , each of which acts on an eigenbasis for the other one in an irreducible tridiagonal fashion. Such a pair is called a Leonard pair. We consider the self-dual case in which there exists an automorphism of the endomorphism algebra of V that swaps A and A^* . Such an automorphism is unique, and called the duality $A \leftrightarrow A^*$. In the present paper we give a comprehensive description of this duality. In particular, we display an invertible \mathbb{F} -linear map T on V such that the map $X \mapsto TXT^{-1}$ is the duality $A \leftrightarrow A^*$. We express T as a polynomial in A and A^* . We describe how T acts on 4 flags, 12 decompositions, and 24 bases for V .

Keywords: Leonard pair, tridiagonal matrix, self-dual

Classification: 17B37, 15A21

1 Introduction

Let \mathbb{F} denote a field and let V denote a vector space over \mathbb{F} with finite positive dimension. We consider a pair A, A^* of diagonalizable \mathbb{F} -linear maps on V , each of which acts on an eigenbasis for the other one in an irreducible tridiagonal fashion. Such a pair is called a Leonard pair (see [13, Definition 1.1]). The Leonard pair A, A^* is said to be self-dual whenever there exists an automorphism of the endomorphism algebra of V that swaps A and A^* . In this case such an automorphism is unique, and called the duality $A \leftrightarrow A^*$.

The literature contains many examples of self-dual Leonard pairs. For instance (i) the Leonard pair associated with an irreducible module for the Terwilliger algebra of the hypercube (see [4, Corollaries 6.8, 8.5]); (ii) a Leonard pair of Krawtchouk type (see [10, Definition 6.1]); (iii) the Leonard pair associated with an irreducible module for the Terwilliger algebra of a distance-regular graph that has a spin model in the Bose-Mesner algebra (see [1, Theorem], [3, Theorems 4.1, 5.5]); (iv) an appropriately normalized totally bipartite Leonard pair (see [11, Lemma 14.8]); (v) the Leonard pair consisting of any two of a modular Leonard triple A, B, C (see [2, Definition 1.4]); (vi) the Leonard pair consisting of a pair of opposite generators for the q -tetrahedron algebra, acting on an evaluation module (see [5, Proposition 9.2]). The example (i) is a special case of (ii), and the examples (iii), (iv) are special cases of (v).

Let A, A^* denote a Leonard pair on V . We can determine whether A, A^* is self-dual in the following way. By [13, Lemma 1.3] each eigenspace of A, A^* has dimension one. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A . For $0 \leq i \leq d$ let v_i denote a θ_i -eigenvector for A . The ordering $\{\theta_i\}_{i=0}^d$ is said to be standard whenever A^* acts on the basis $\{v_i\}_{i=0}^d$ in an irreducible tridiagonal fashion. If the ordering $\{\theta_i\}_{i=0}^d$ is standard then the ordering $\{\theta_{d-i}\}_{i=0}^d$ is also standard, and no further ordering is standard. Similar comments apply to A^* . Let $\{\theta_i\}_{i=0}^d$ denote a standard ordering of the eigenvalues of A . Then A, A^* is self-dual if and only if $\{\theta_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A^* (see [7, Proposition 8.7]).

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For a given self-dual Leonard pair, it is not obvious what is the corresponding duality. The purpose of this paper is to describe this duality. Our description is summarized as follows. Let A, A^* denote a self-dual Leonard pair on V , and let $\{\theta_i\}_{i=0}^d$ denote a standard ordering of the eigenvalues of A . By construction $\{\theta_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A^* . For $0 \leq i \leq d$ let $E_i : V \rightarrow V$ (resp. $E_i^* : V \rightarrow V$) denote the projection onto the eigenspace of A (resp. A^*) corresponding to θ_i . Using the projections $\{E_i\}_{i=0}^d$ and $\{E_i^*\}_{i=0}^d$ we define a certain \mathbb{F} -linear map $T : V \rightarrow V$. We show that T is invertible, and the map $X \mapsto TXT^{-1}$ is the duality $A \leftrightarrow A^*$. In order to illuminate the nature of T , we show how T acts on 4 flags, 12 decompositions, and 24 bases attached to A, A^* . Here are some details. By a flag on V we mean a sequence $\{H_i\}_{i=0}^d$ of subspaces of V such that H_i has dimension $i + 1$ for $0 \leq i \leq d$ and $H_{i-1} \subseteq H_i$ for $1 \leq i \leq d$. By a decomposition of V we mean a sequence $\{V_i\}_{i=0}^d$ of one dimensional subspaces whose direct sum is V . For a decomposition $\{V_i\}_{i=0}^d$ of V , define $H_i = V_0 + V_1 + \cdots + V_i$ for $0 \leq i \leq d$. The sequence $\{H_i\}_{i=0}^d$ is a flag on V , said to be induced by $\{V_i\}_{i=0}^d$. Two flags $\{H_i\}_{i=0}^d$ and $\{H'_i\}_{i=0}^d$ on V are called opposite whenever there exists a decomposition $\{V_i\}_{i=0}^d$ of V that induces $\{H_i\}_{i=0}^d$ and $\{V_{d-i}\}_{i=0}^d$ induces $\{H'_i\}_{i=0}^d$. In this case $V_i = H_i \cap H'_{d-i}$ for $0 \leq i \leq d$. In particular the decomposition $\{V_i\}_{i=0}^d$ is uniquely determined by the ordered pair $\{H_i\}_{i=0}^d, \{H'_i\}_{i=0}^d$; we say that this ordered pair induces $\{V_i\}_{i=0}^d$. For each symbol z among the symbols $0, D, 0^*, D^*$ we define a flag $[z]$ on V as follows. The flag $[0]$ is induced by $\{E_i V\}_{i=0}^d$ and the flag $[D]$ is induced by $\{E_{d-i} V\}_{i=0}^d$. The flag $[0^*]$ is induced by $\{E_i^* V\}_{i=0}^d$ and the flag $[D^*]$ is induced by $\{E_{d-i}^* V\}_{i=0}^d$. By [14, Theorem 7.3] the flags $[0], [D], [0^*], [D^*]$ are mutually opposite. For distinct z, w among the symbols $0, D, 0^*, D^*$, let $[zw]$ denote the decomposition of V induced by $[z]$ and $[w]$. There are 12 choices for the ordered pair z, w and this gives 12 decompositions of V . For each decomposition, pick a nonzero vector in each component of the decomposition. The resulting sequence of vectors is a basis for V . We normalize the basis in two ways that seem attractive; this yields two bases for each decomposition. By this procedure we obtain 24 bases for V . We obtain the action of T on each of these bases. As we will see, with respect to four of the bases among the 24, the matrix representing T is independent of the four bases and its entries take a very attractive form.

The paper is organized as follows. In Sections 2–6 we review some background and establish some basic results for general Leonard pairs. Starting in Section 7 we consider a self-dual Leonard pair A, A^* . In Section 8 we introduce the map T and discuss its basic properties. In Sections 9, 10 we show that T is invertible, and the map $X \mapsto TXT^{-1}$ is the duality $A \leftrightarrow A^*$. In Section 11 we use A, A^* to define 4 flags and 12 decompositions. In Section 12 we obtain the action of T on these flags and decompositions. In Sections 13, 14 we obtain two bases from each of the 12 decompositions, and describe how these two bases are related. In Section 15 we obtain the action of T on the 24 bases. We also display four bases among the 24, with respect to which the matrix representing T is independent of the bases and its entries take an attractive form.

2 Leonard pairs

We now begin our formal argument. In this section we recall the notion of a Leonard pair. We use the following terms. A square matrix is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. Let \mathbb{F} denote a field.

Definition 2.1. [13, Definition 1.1] Let V denote a vector space over \mathbb{F} with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair of \mathbb{F} -linear maps $A : V \rightarrow V$ and $A^* : V \rightarrow V$ that satisfy the following (i), (ii).

- (i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
- (ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We say that A, A^* is *over* \mathbb{F} .

Note 2.2. According to a common notational convention, for a matrix A its conjugate-transpose is denoted by A^* . We are not using this convention. In a Leonard pair A, A^* the linear maps A, A^* are arbitrary subject to (i) and (ii) above.

We refer the reader to [16] for background on Leonard pairs

Note 2.3. Assume that A, A^* is a Leonard pair on V . Then A^*, A is a Leonard pair on V .

For the rest of this paper, let V denote a vector space over \mathbb{F} with finite positive dimension. Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the \mathbb{F} -linear maps from V to V . The algebra $\text{End}(V)$ is called the *endomorphism algebra* of V .

Lemma 2.4. [16, Corollary 5.6] Let A, A^* denote a Leonard pair on V . Then A, A^* together generate the algebra $\text{End}(V)$.

We recall the notion of an isomorphism for Leonard pairs. Let A, A^* denote a Leonard pair on V . Let V' denote a vector space over \mathbb{F} with finite positive dimension, and let A', A'^* denote a Leonard pair on V' . By an *isomorphism of Leonard pairs* from A, A^* to A', A'^* we mean an isomorphism of \mathbb{F} -algebras from $\text{End}(V)$ to $\text{End}(V')$ that sends $A \mapsto A'$ and $A^* \mapsto A'^*$. The Leonard pairs A, A^* and A', A'^* are said to be *isomorphic* whenever there exists an isomorphism of Leonard pairs from A, A^* to A', A'^* . In this case, the isomorphism involved is unique by Lemma 2.4. An isomorphism of Leonard pairs can be seen from the following point of view. By the Skolem-Noether theorem (see [12, Corollary 7.125]), a map $\sigma : \text{End}(V) \rightarrow \text{End}(V')$ is an \mathbb{F} -algebra isomorphism if and only if there exists an \mathbb{F} -linear bijection $K : V \rightarrow V'$ such that $X^\sigma = KXK^{-1}$ for all $X \in \text{End}(V)$. In this case, we say that K gives σ . Assume that K gives σ . Then an \mathbb{F} -linear map $\tilde{K} : V \rightarrow V'$ gives σ if and only if there exists $0 \neq \alpha \in \mathbb{F}$ such that $\tilde{K} = \alpha K$.

Definition 2.5. A Leonard pair A, A^* is said to be *self-dual* whenever A, A^* is isomorphic to A^*, A .

Let A, A^* denote a self-dual Leonard pair on V . For an automorphism σ of $\text{End}(V)$ the following are equivalent:

- (i) σ is an isomorphism of Leonard pairs from A, A^* to A^*, A ;
- (ii) σ is an isomorphism of Leonard pairs from A^*, A to A, A^* .

There exists a unique automorphism σ of $\text{End}(V)$ that satisfies (i), (ii).

Definition 2.6. Let A, A^* denote a self-dual Leonard pair on V . By the *duality* $A \leftrightarrow A^*$ we mean the automorphism σ of $\text{End}(V)$ that satisfies (i), (ii) above.

3 Leonard systems

When working with a Leonard pair, it is convenient to consider a closely related object called a Leonard system [13]. Before we define a Leonard system, we recall a few concepts from linear algebra.

We denote by I the identity element of $\text{End}(V)$. For $A \in \text{End}(V)$ let $\langle A \rangle$ denote the subalgebra of $\text{End}(V)$ generated by A . For an integer $d \geq 0$ let $\text{Mat}_{d+1}(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, d$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . For $X \in \text{End}(V)$ and $Y \in \text{Mat}_{d+1}(\mathbb{F})$, we say that Y represents X with respect to $\{v_i\}_{i=0}^d$ whenever $Xv_j = \sum_{i=0}^d Y_{i,j}v_i$ for $0 \leq j \leq d$. Let $A \in \text{End}(V)$. For $\theta \in \mathbb{F}$ define $V(\theta) = \{v \in V \mid Av = \theta v\}$. Observe that $V(\theta)$ is a subspace of V . The scalar θ is called an *eigenvalue* of A whenever $V(\theta) \neq 0$. In this case, $V(\theta)$ is called the *eigenspace* of A corresponding to θ . We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A . We say that A is *multiplicity-free* whenever A is diagonalizable, and each eigenspace of A has dimension one.

Assume that A is multiplicity-free, and let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A . Then $\{V_i\}_{i=0}^d$ is a decomposition of V . For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A corresponding to V_i . For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ if $j \neq i$ ($0 \leq j \leq d$). Thus E_i is the projection onto V_i . Observe that (i) $V_i = E_i V$ ($0 \leq i \leq d$); (ii) $E_i E_j = \delta_{i,j} E_i$ ($0 \leq i, j \leq d$); (iii) $I = \sum_{i=0}^d E_i$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Also

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d). \quad (1)$$

We call E_i the *primitive idempotent* of A for θ_i ($0 \leq i \leq d$). Observe that $\{A^i\}_{i=0}^d$ is a basis for the \mathbb{F} -vector space $\langle A \rangle$, and $\prod_{i=0}^d (A - \theta_i I) = 0$. Also observe that $\{E_i\}_{i=0}^d$ is a basis for the \mathbb{F} -vector space $\langle A \rangle$.

Let A, A^* denote a Leonard pair on V . By [13, Lemma 1.3] each of A, A^* is multiplicity-free. Let $\{E_i\}_{i=0}^d$ denote an ordering of the primitive idempotents of A . For $0 \leq i \leq d$ pick $0 \neq v_i \in E_i V$. Then $\{v_i\}_{i=0}^d$ is a basis for V . The ordering $\{E_i\}_{i=0}^d$ is said to be *standard* whenever the basis $\{v_i\}_{i=0}^d$ satisfies Definition 2.1(ii). A standard ordering of the primitive idempotents of A^* is similarly defined.

Definition 3.1. By a *Leonard system* on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \quad (2)$$

of elements in $\text{End}(V)$ that satisfy the following (i)–(iii):

- (i) A, A^* is a Leonard pair on V ;
- (ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A ;
- (iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^* .

We say that Φ is *over* \mathbb{F} .

Referring to Definition 3.1, the Leonard pair A, A^* from part (i) is said to be *associated with* Φ .

We recall the notion of an isomorphism for Leonard systems. Consider the Leonard system (2). Let V' denote a vector space over \mathbb{F} with dimension $d + 1$. For an \mathbb{F} -algebra isomorphism $\sigma : \text{End}(V) \rightarrow \text{End}(V')$ define

$$\Phi^\sigma = (A^\sigma; \{E_i^\sigma\}_{i=0}^d; (A^*)^\sigma; \{(E_i^*)^\sigma\}_{i=0}^d).$$

Then Φ^σ is a Leonard system on V' . Let Φ' denote a Leonard system on V' . By an *isomorphism of Leonard systems* from Φ to Φ' we mean an \mathbb{F} -algebra isomorphism $\sigma : \text{End}(V) \rightarrow \text{End}(V')$ such that $\Phi' = \Phi^\sigma$. The Leonard systems Φ and Φ' are said to be *isomorphic* whenever there exists an isomorphism of Leonard systems from Φ to Φ' . In this case, the isomorphism involved is unique.

Consider a Leonard system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ over \mathbb{F} . For $0 \leq i \leq d$ let θ_i (resp. $\{\theta_i^*\}_{i=0}^d$) denote the eigenvalue of A (resp. A^*) corresponding to E_i (resp. E_i^*). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ . Note that $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in \mathbb{F} . Similarly $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and contained in \mathbb{F} .

Consider a Leonard system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ over \mathbb{F} . Then each of the following is a Leonard system over \mathbb{F} :

$$\begin{aligned} \Phi^* &= (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d), \\ \Phi^\downarrow &= (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d), \\ \Phi^\updownarrow &= (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d). \end{aligned}$$

Viewing $*, \downarrow, \updownarrow$ as permutations on the set of all Leonard systems over \mathbb{F} ,

$$*^2 = \downarrow^2 = \updownarrow^2 = 1, \quad (3)$$

$$\downarrow * = * \downarrow, \quad \downarrow * = * \downarrow, \quad \downarrow \downarrow = \downarrow \downarrow. \quad (4)$$

The group generated by the symbols $*, \downarrow, \updownarrow$ subject to the relations (3), (4) is the dihedral group D_4 . Recall that D_4 is the group of symmetries of a square, and has 8 elements. The elements $*, \downarrow, \updownarrow$ induce an action

of D_4 on the set of all Leonard systems over \mathbb{F} . Two Leonard systems over \mathbb{F} will be called *relatives* whenever they are in the same orbit of this D_4 action.

Definition 3.2. Let Φ denote a Leonard system, and let $g \in D_4$. For any object f attached to Φ , let f^g denote the corresponding object attached to $\Phi^{g^{-1}}$.

Lemma 3.3. Let A, A^* denote a Leonard pair on V , and let Φ denote an associated Leonard system. Then the Leonard systems associated with A, A^* are $\Phi, \Phi^\downarrow, \Phi^\uparrow, \Phi^{\downarrow\downarrow}$.

Proof. By the comments above Definition 3.1. □

Definition 3.4. A Leonard system Φ is said to be *self-dual* whenever Φ is isomorphic to Φ^* .

Let Φ denote a self-dual Leonard system on V . For an automorphism σ of $\text{End}(V)$ the following are equivalent:

- (i) σ is an isomorphism of Leonard systems from Φ to Φ^* ;
- (ii) σ is an isomorphism of Leonard systems from Φ^* to Φ .

There exists a unique automorphism σ of $\text{End}(V)$ that satisfies (i), (ii).

Definition 3.5. Let Φ denote a self-dual Leonard system on V . By the *duality* $\Phi \leftrightarrow \Phi^*$ we mean the automorphism of $\text{End}(V)$ that satisfies (i), (ii) above.

4 Antiautomorphisms and bilinear forms

In this section we recall a few notions from the theory of Leonard pairs. Let \mathcal{A} denote an \mathbb{F} -algebra. By an *antiautomorphism* of \mathcal{A} we mean an \mathbb{F} -linear bijection $\xi : \mathcal{A} \rightarrow \mathcal{A}$ such that $(XY)^\xi = Y^\xi X^\xi$ for all $X, Y \in \mathcal{A}$.

Lemma 4.1. [16, Theorem 6.1] Let A, A^* denote a Leonard pair on V . Then there exists a unique antiautomorphism \dagger of $\text{End}(V)$ such that $A^\dagger = A$ and $(A^*)^\dagger = A^*$. Moreover $(X^\dagger)^\dagger = X$ for all $X \in \text{End}(V)$.

Lemma 4.2. [16, Lemma 6.3] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V . Then the following hold.

- (i) We have $X^\dagger = X$ for all $X \in \langle A \rangle$. In particular, $E_i^\dagger = E_i$ for $0 \leq i \leq d$.
- (ii) We have $X^\dagger = X$ for all $X \in \langle A^* \rangle$. In particular, $(E_i^*)^\dagger = E_i^*$ for $0 \leq i \leq d$.

By a *bilinear form* on V we mean a map $(,) : V \times V \rightarrow \mathbb{F}$ that satisfies the following four conditions for all $u, v, w \in V$ and for all $\alpha \in \mathbb{F}$: (i) $(u+v, w) = (u, w) + (v, w)$; (ii) $(\alpha u, v) = \alpha(u, v)$; (iii) $(u, v+w) = (u, v) + (u, w)$; (iv) $(u, \alpha v) = \alpha(u, v)$. Let $(,)$ denote a bilinear form on V . This form is said to be *symmetric* whenever $(u, v) = (v, u)$ for all $u, v \in V$. Let $(,)$ denote a bilinear form on V . Then the following are equivalent: (i) there exists a nonzero $u \in V$ such that $(u, v) = 0$ for all $v \in V$; (ii) there exists a nonzero $v \in V$ such that $(u, v) = 0$ for all $u \in V$. The form $(,)$ is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise. Let ξ denote an antiautomorphism of $\text{End}(V)$. Then there exists a nonzero bilinear form $(,)$ on V such that $(Xu, v) = (u, X^\xi v)$ for all $u, v \in V$ and for all $X \in \text{End}(V)$. The form is unique up to multiplication by a nonzero scalar in \mathbb{F} . The form is nondegenerate. We refer to this form as the *bilinear form on V associated with ξ* . This form is not symmetric in general.

Let A, A^* denote a Leonard pair on V . Recall the antiautomorphism \dagger of $\text{End}(V)$ from Lemma 4.1. Let $(,)$ denote the bilinear form on V associated with \dagger . By [15, Corollary 15.4] the bilinear form $(,)$ is symmetric. By construction, for $X \in \text{End}(V)$ we have

$$(Xu, v) = (u, X^\dagger v) \quad (u, v \in V).$$

In particular,

$$(Au, v) = (u, Av), \quad (A^*u, v) = (u, A^*v) \quad (u, v \in V).$$

5 The split decomposition and the parameter array

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V . In this section we recall the Φ -split decomposition of V and the parameter array of Φ . Recall the eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and the dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$ of Φ . Let x denote an indeterminate, and let $\mathbb{F}[x]$ denote the \mathbb{F} -algebra consisting of the polynomials in x that have all coefficients in \mathbb{F} .

Definition 5.1. [13, Definition 4.3] For $0 \leq i \leq d$ we define some polynomials in $\mathbb{F}[x]$:

$$\begin{aligned} \tau_i &= (x - \theta_0)(x - \theta_1) \cdots (x - \theta_{i-1}), \\ \eta_i &= (x - \theta_d)(x - \theta_{d-1}) \cdots (x - \theta_{d-i+1}), \\ \tau_i^* &= (x - \theta_0^*)(x - \theta_1^*) \cdots (x - \theta_{i-1}^*), \\ \eta_i^* &= (x - \theta_d^*)(x - \theta_{d-1}^*) \cdots (x - \theta_{d-i+1}^*). \end{aligned}$$

For $0 \leq i \leq d$ define

$$U_i = (E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV). \quad (5)$$

By [16, Theorem 20.7] the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V . This decomposition is called the Φ -split decomposition of V . By [16, Lemma 20.9],

$$\begin{aligned} (A - \theta_i I)U_i &= U_{i+1} & (0 \leq i \leq d-1), & & (A - \theta_d I)U_d &= 0, \\ (A^* - \theta_i^* I)U_i &= U_{i-1} & (1 \leq i \leq d), & & (A^* - \theta_0^* I)U_0 &= 0. \end{aligned}$$

For $0 \leq i \leq d$,

$$\tau_i(A)U_0 = U_i, \quad \eta_i^*(A^*)U_d = U_{d-i}.$$

Pick a nonzero $v \in E_0^*V$. For $0 \leq i \leq d$ define $u_i = \tau_i(A)v$. Then $0 \neq u_i \in U_i$ for $0 \leq i \leq d$. Moreover, the vectors $\{u_i\}_{i=0}^d$ form a basis for V . We call $\{u_i\}_{i=0}^d$ a Φ -split basis for V . With respect to a Φ -split basis, the matrices representing A and A^* are

$$A : \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & \cdot & \varphi_d \\ & & & & & \theta_d^* \end{pmatrix},$$

where $\{\varphi_i\}_{i=1}^d$ are nonzero scalars in \mathbb{F} . The sequence $\{\varphi_i\}_{i=1}^d$ is uniquely determined by Φ , and called the *first split sequence* of Φ . Let $\{\phi_i\}_{i=1}^d$ denote the first split sequence of Φ^\downarrow . We call $\{\phi_i\}_{i=1}^d$ the *second split sequence* of Φ . By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$. By [13, Theorem 1.9] the Leonard system Φ is determined up to isomorphism by its parameter array.

For the rest of this section let

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$$

denote the parameter array of Φ .

Lemma 5.2. [13, Theorem 1.11] *The following (i)–(iii) hold.*

(i) *The parameter array of Φ^* is*

$$(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d).$$

(ii) *The parameter array of Φ^\downarrow is*

$$(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d).$$

(iii) *The parameter array of Φ^\uparrow is*

$$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d).$$

We mention some results for later use.

Lemma 5.3. *We have*

$$E_0 = \frac{\eta_d(A)}{\eta_d(\theta_0)}, \quad E_d = \frac{\tau_d(A)}{\tau_d(\theta_d)}, \quad E_0^* = \frac{\eta_d^*(A^*)}{\eta_d^*(\theta_0^*)}, \quad E_d^* = \frac{\tau_d^*(A^*)}{\tau_d^*(\theta_d^*)}. \quad (6)$$

Proof. By (1) and Definition 5.1. □

Lemma 5.4. *For the \mathbb{F} -vector spaces $\langle A \rangle$ and $\langle A^* \rangle$, we give three bases:*

vector space U	three bases for U		
$\langle A \rangle$	$\{E_i\}_{i=0}^d$	$\{\tau_i(A)\}_{i=0}^d$	$\{\eta_i(A)\}_{i=0}^d$
$\langle A^* \rangle$	$\{E_i^*\}_{i=0}^d$	$\{\tau_i^*(A^*)\}_{i=0}^d$	$\{\eta_i^*(A^*)\}_{i=0}^d$

Proof. By the comments below (1) along with Definition 5.1. □

6 Some traces

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

Later in the paper we will need some facts about Φ that involve the trace function tr . Consider the scalars

$$\text{tr}(E_r E_0^*), \quad \text{tr}(E_r E_d^*), \quad \text{tr}(E_r^* E_0), \quad \text{tr}(E_r^* E_d) \quad (7)$$

for $0 \leq r \leq d$. By [15, Theorem 17.12] we find that for $0 \leq r \leq d$,

$$\text{tr}(E_r E_0^*) = \frac{\varphi_1 \varphi_2 \cdots \varphi_r \phi_1 \phi_2 \cdots \phi_{d-r}}{\eta_d^*(\theta_0^*) \tau_r(\theta_r) \eta_{d-r}(\theta_r)}, \quad (8)$$

$$\text{tr}(E_r E_d^*) = \frac{\phi_d \phi_{d-1} \cdots \phi_{d-r+1} \varphi_d \varphi_{d-1} \cdots \varphi_{r+1}}{\tau_d^*(\theta_d^*) \tau_r(\theta_r) \eta_{d-r}(\theta_r)}, \quad (9)$$

$$\text{tr}(E_r^* E_0) = \frac{\varphi_1 \varphi_2 \cdots \varphi_r \phi_d \phi_{d-1} \cdots \phi_{r+1}}{\eta_d(\theta_0) \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*)}, \quad (10)$$

$$\text{tr}(E_r^* E_d) = \frac{\phi_1 \phi_2 \cdots \phi_r \varphi_d \varphi_{d-1} \cdots \varphi_{r+1}}{\tau_d(\theta_d) \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*)}. \quad (11)$$

Note that the scalars in (8)–(11) are nonzero. In particular $\text{tr}(E_0 E_0^*)$ is nonzero. Define $v \in \mathbb{F}$ by

$$v = \text{tr}(E_0 E_0^*)^{-1}. \quad (12)$$

By [16, Lemma 9.4],

$$v E_0 E_0^* E_0 = E_0, \quad v E_0^* E_0 E_0^* = E_0^*. \quad (13)$$

By (8)–(12),

$$v = \frac{\eta_d(\theta_0)\eta_d^*(\theta_0^*)}{\phi_1 \cdots \phi_d}, \quad v^\downarrow = \frac{\eta_d(\theta_0)\tau_d^*(\theta_d^*)}{\varphi_1 \cdots \varphi_d}, \quad (14)$$

$$v^\downarrow\downarrow = \frac{\tau_d(\theta_d)\eta_d^*(\theta_0^*)}{\phi_1 \cdots \phi_d}, \quad v^\downarrow\downarrow\downarrow = \frac{\tau_d(\theta_d)\tau_d^*(\theta_d^*)}{\phi_1 \cdots \phi_d}. \quad (15)$$

We mention a result for later use. Let $\{U_i\}_{i=0}^d$ denote the Φ -split decomposition of V . For $0 \leq i \leq d$ define $F_i \in \text{End}(V)$ such that $(F_i - I)U_i = 0$ and $F_i U_j = 0$ if $j \neq i$ ($0 \leq j \leq d$). Thus F_i is the projection onto U_i . Observe that (i) $U_i = F_i V$ ($0 \leq i \leq d$); (ii) $F_i F_j = \delta_{i,j} F_i$ ($0 \leq i, j \leq d$); (iii) $I = \sum_{i=0}^d F_i$.

Lemma 6.1. [6, Corollary 7.4] For $0 \leq i \leq d$,

$$F_i = \frac{v\tau_i(A)E_0^*E_0\tau_i^*(A^*)}{\varphi_1 \cdots \varphi_i}. \quad (16)$$

7 Self-dual Leonard pairs and systems

Earlier we defined the concept of a self-dual Leonard pair and system. In this section we make some observations about this concept.

Lemma 7.1. Let A, A^* denote a self-dual Leonard pair on V , and let σ denote the duality $A \leftrightarrow A^*$. Then $\sigma^2 = 1$.

Proof. By construction, σ^2 fixes each of A, A^* . By this and Lemma 2.4, σ^2 fixes every element of $\text{End}(V)$. So $\sigma^2 = 1$. \square

Lemma 7.2. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a self-dual Leonard system, and let σ denote the duality $\Phi \leftrightarrow \Phi^*$. Then the Leonard pair A, A^* is self-dual. Moreover σ is the duality $A \leftrightarrow A^*$.

Proof. By construction. \square

Lemma 7.3. Let A, A^* denote a self-dual Leonard pair, and let σ denote the duality $A \leftrightarrow A^*$. Let $\{E_i\}_{i=0}^d$ denote a standard ordering of the primitive idempotents of A . Then the following (i)–(iii) hold:

- (i) $\{E_i^\sigma\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^* ;
- (ii) the sequence $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^\sigma\}_{i=0}^d)$ is a self-dual Leonard system;
- (iii) σ is the duality $\Phi \leftrightarrow \Phi^*$.

Proof. Note that $A^\sigma = A^*$ and $(A^*)^\sigma = A$.

(i) Let $\{E_i^*\}_{i=0}^d$ denote a standard ordering of the primitive idempotents for A^* , and consider the Leonard system

$$\Phi' = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

We have $(\Phi')^\sigma = (A^*; \{E_i^\sigma\}_{i=0}^d; A; \{(E_i^*)^\sigma\}_{i=0}^d)$. The result follows since $(\Phi')^\sigma$ is a Leonard system.

(ii), (iii) By (i) above and the construction, Φ is a Leonard system. Applying σ to Φ and using Lemma 7.1, we obtain

$$\Phi^\sigma = (A^*; \{E_i^\sigma\}_{i=0}^d; A; \{E_i\}_{i=0}^d) = \Phi^*.$$

The result follows. \square

The self-dual Leonard systems are characterized as follows.

Lemma 7.4. [7, Proposition 8.7] Let Φ denote a Leonard system over \mathbb{F} with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$. Then Φ is self-dual if and only if

$$\theta_i = \theta_i^* \quad (0 \leq i \leq d). \quad (17)$$

In this case

$$\phi_i = \phi_{d-i+1} \quad (1 \leq i \leq d). \quad (18)$$

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a self-dual Leonard system on V , and let σ denote the duality $\Phi \leftrightarrow \Phi^*$. Our next general goal is to describe σ . To do this we will display an invertible $T \in \text{End}(V)$ that gives σ .

8 The element T

For the rest of the paper, fix a Leonard system on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d). \quad (19)$$

In this section we introduce an element $T \in \text{End}(V)$; this element will be used to describe the duality $\Phi \leftrightarrow \Phi^*$ in the self-dual case. Let

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

denote the parameter array of Φ . Let \dagger denote the antiautomorphism of $\text{End}(V)$ that fixes each of A, A^* . Let $(,)$ denote the bilinear form on V associated with \dagger , as discussed at the end of Section 4.

Definition 8.1. Define $T \in \text{End}(V)$ by

$$T = \sum_{i=0}^d \eta_{d-i}(A) E_0^* E_d \tau_i^*(A^*). \quad (20)$$

Note 8.2. Sometimes it is convenient to express T as a polynomial in A, A^* . Evaluating (20) using (6) we get

$$T = \sum_{i=0}^d \frac{\eta_{d-i}(A) \eta_d^*(A^*) \tau_d(A) \tau_i^*(A^*)}{\tau_d(\theta_d) \eta_d^*(\theta_0^*)}.$$

We have

$$\begin{aligned} T^* &= \sum_{i=0}^d \eta_{d-i}^*(A^*) E_0 E_d^* \tau_i(A), \\ T^\dagger &= \sum_{i=0}^d \tau_i^*(A^*) E_d E_0^* \eta_{d-i}(A), \\ (T^*)^\dagger &= \sum_{i=0}^d \tau_i(A) E_d^* E_0 \eta_{d-i}^*(A^*). \end{aligned}$$

We now state our first main result.

Theorem 8.3. Assume that Φ is self-dual. Then the elements $T, T^*, T^\dagger, (T^*)^\dagger$ are equal and this common element gives the duality $\Phi \leftrightarrow \Phi^*$.

Our proof of Theorem 8.3 is contained in Section 10.

9 Some products

We continue to discuss the Leonard system Φ from (19). Recall the element T from Definition 8.1. In this section we consider the elements $T, T^*, T^\dagger, (T^*)^\dagger$. We obtain formulas for the products of these elements with the elements E_0, E_0^* . These formulas are used to show that $T = T^* = T^\dagger$ in our proof of Theorem 8.3.

Lemma 9.1. *We have*

$$TE_0^* = \frac{\eta_d(\theta_0)\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)\eta_d^*(\theta_0^*)} E_0 E_0^*, \quad (21)$$

$$T^* E_0 = \frac{\eta_d^*(\theta_0^*)\varphi_1 \cdots \varphi_d}{\tau_d^*(\theta_d^*)\eta_d(\theta_0)} E_0^* E_0, \quad (22)$$

$$E_0^* T^\dagger = \frac{\eta_d(\theta_0)\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)\eta_d^*(\theta_0^*)} E_0^* E_0, \quad (23)$$

$$E_0(T^*)^\dagger = \frac{\eta_d^*(\theta_0^*)\varphi_1 \cdots \varphi_d}{\tau_d^*(\theta_d^*)\eta_d(\theta_0)} E_0 E_0^*. \quad (24)$$

Proof. We first show (21). In (20), multiply each side on the right by E_0^* . Simplify the result using $\tau_i^*(A^*)E_0^* = \tau_i^*(\theta_0^*)E_0^*$ and $\tau_i^*(\theta_0^*) = \delta_{i,0}$ ($0 \leq i \leq d$) to get

$$TE_0^* = \eta_d(A)E_0^* E_d E_0^*.$$

By (6) we have $\eta_d(A) = \eta_d(\theta_0)E_0$. By (13) applied to Φ^\downarrow , $E_0^* E_d E_0^* = (v^\downarrow)^{-1} E_0^*$. By these comments and (15) we obtain (21). The line (22) is obtained by applying (21) to Φ^* . The lines (23) and (24) are obtained by applying \dagger to (21) and (22), respectively. \square

Lemma 9.2. [9, Lemma 7.1] *For $0 \leq i, j \leq d$,*

$$E_0^* \tau_i(A) \tau_j^*(A^*) E_0 = \delta_{i,j} \varphi_1 \varphi_2 \cdots \varphi_i E_0^* E_0. \quad (25)$$

Lemma 9.3. *We have*

$$TE_0 = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)} E_0^* E_0, \quad (26)$$

$$T^* E_0^* = \frac{\varphi_1 \cdots \varphi_d}{\tau_d^*(\theta_d^*)} E_0 E_0^*, \quad (27)$$

$$E_0 T^\dagger = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)} E_0 E_0^*, \quad (28)$$

$$E_0^*(T^*)^\dagger = \frac{\varphi_1 \cdots \varphi_d}{\tau_d^*(\theta_d^*)} E_0^* E_0. \quad (29)$$

Proof. In (20), multiply each side on the right by E_0 . Simplify the result using $E_d = \tau_d(A)/\tau_d(\theta_d)$ and (25) to get (26). The line (27) is obtained by applying (26) to Φ^* . The lines (28) and (29) are obtained by applying \dagger to (26) and (27), respectively. \square

10 The proof of Theorem 8.3

In this section we prove Theorem 8.3. Recall the Leonard system Φ from (19) and the element T from Definition 8.1.

Lemma 10.1. *We have*

$$T^2 = (v^\downarrow)^{-1} \varphi_1 \cdots \varphi_d \sum_{j=0}^d \frac{\eta_j(A) E_0^* E_d \tau_j^*(A^*)}{\varphi_d \cdots \varphi_{d-j+1}}. \quad (30)$$

Proof. By (20),

$$T^2 = \sum_{i=0}^d \sum_{j=0}^d \eta_{d-i}(A) E_0^* E_d \tau_i^*(A^*) \eta_{d-j}(A) E_0^* E_d \tau_j^*(A^*). \quad (31)$$

Applying (25) to $\Phi^{\downarrow*}$,

$$E_d \tau_i^*(A^*) \eta_j(A) E_0^* = \delta_{i,j} \phi_1 \cdots \phi_i E_d E_0^* \quad (0 \leq i, j \leq d).$$

In this line, replace j with $d - j$ to get

$$E_d \tau_i^*(A^*) \eta_{d-j}(A) E_0^* = \delta_{i,d-j} \phi_1 \cdots \phi_{d-j} E_d E_0^* \quad (0 \leq i, j \leq d).$$

By this and (31),

$$T^2 = \sum_{j=0}^d \phi_1 \cdots \phi_{d-j} \eta_j(A) E_0^* E_d E_0^* E_d \tau_j^*(A^*). \quad (32)$$

Applying (13) to Φ^{\downarrow} ,

$$E_0^* E_d E_0^* = (v^{\downarrow})^{-1} E_0^*.$$

By this and (32) we get (30). \square

Proposition 10.2. *Assume that Φ is self-dual, Then T is invertible. Moreover, $T^2 = \lambda I$, where*

$$\lambda = (v^{\downarrow})^{-2} \phi_1 \cdots \phi_d.$$

Proof. By Lemma 7.4 the sum in (30) is equal to

$$\sum_{j=0}^d \frac{\eta_j(A) E_0^* E_d \tau_j^*(A^*)}{\phi_1 \cdots \phi_j}.$$

Applying (16) to Φ^{\downarrow} and using $I = \sum_{j=0}^d F_j^{\downarrow}$, we find that the above sum is equal to $(v^{\downarrow})^{-1} I$. Thus $T^2 = \lambda I$. By construction $\lambda \neq 0$ so T is invertible. \square

Lemma 10.3. *Assume that Φ is self-dual. Then*

$$AT = TA^*, \quad A^* T = TA. \quad (33)$$

Proof. We first show $AT = TA^*$. For $0 \leq i \leq d$ define

$$T_i = \eta_{d-i}(A) E_0^* E_d \tau_i^*(A^*).$$

The element T_i is the i -summand in (20), so $T = \sum_{i=0}^d T_i$. By Definition 5.1 along with $\prod_{\ell=0}^d (A - \theta_\ell I) = 0$ and $\prod_{\ell=0}^d (A^* - \theta_\ell^* I) = 0$,

$$\begin{aligned} AT_0 - \theta_0 T_0 &= 0, \\ AT_i - \theta_i T_i &= T_{i-1} A^* - \theta_{i-1}^* T_{i-1} \quad (1 \leq i \leq d), \\ 0 &= T_d A^* - \theta_d^* T_d. \end{aligned}$$

By these comments

$$AT - \sum_{i=0}^d \theta_i T_i = TA^* - \sum_{i=0}^d \theta_i^* T_i.$$

By this and (17) we see that $AT = TA^*$. In this equation, multiply each side on the left and right by T . Simplify the result using Proposition 10.2 to get $A^* T = TA$. \square

Corollary 10.4. *Assume that Φ is self-dual. Then*

$$E_i T = T E_i^*, \quad E_i^* T = T E_i \quad (0 \leq i \leq d). \quad (34)$$

Proof. By (1) and (33). □

Proof of Theorem 8.3. By Proposition 10.2, T is invertible. By (33) and (34), T gives the duality $\Phi \leftrightarrow \Phi^*$.

Next we show that $T = T^*$. In the above statement, we replace T by T^* and swap the roles of Φ , Φ^* to see that T^* gives the duality $\Phi \leftrightarrow \Phi^*$. Thus each of T and T^* gives the duality $\Phi \leftrightarrow \Phi^*$. By this and the comment above Definition 2.5, there exists $0 \neq \zeta \in \mathbb{F}$ such that $T^* = \zeta T$. We show that $\zeta = 1$. By (22), (26) together with (17) we find that $T^* E_0$ and $T E_0$ have the same trace. This trace is nonzero by the comments above (12). Thus $\zeta = 1$ and so $T = T^*$.

Next we show that $T = T^\dagger$. In (33) and (34), apply \dagger to each side and use Lemma 4.2 to find that T^\dagger gives the duality $\Phi \leftrightarrow \Phi^*$. Thus each of T and T^\dagger gives the duality $\Phi \leftrightarrow \Phi^*$. By this and the comment above Definition 2.5, there exists $0 \neq \zeta' \in \mathbb{F}$ such that $T^\dagger = \zeta' T$. We show that $\zeta' = 1$. By (23), (27) together with (17) and $T = T^*$, we find that $E_0^* T^\dagger$ and $E_0^* T$ have the same trace. This trace is nonzero by the comments above (12). Thus $\zeta' = 1$ and so $T = T^\dagger$.

In the equation $T = T^*$, apply \dagger to each side to get $T^\dagger = (T^*)^\dagger$. We have shown that the elements T , T^* , T^\dagger , $(T^*)^\dagger$ are equal. □

11 Some decompositions and flags associated with a Leonard system

Consider the Leonard system Φ from (19). Recall from Section 1 the notion of a flag on V , and what it means for two flags on V to be opposite. In this section we use Φ to obtain four mutually opposite flags on V ; these are induced by the eigenspace decompositions of A and A^* , as well as the split decomposition for Φ and its relatives. In the next section, we will describe how T acts on these flags and decompositions.

Definition 11.1. For notational convenience let Ω denote the set consisting of four symbols $0, D, 0^*, D^*$.

Definition 11.2. For $z \in \Omega$ we define a flag on V which we denote by $[z]$. To define this flag we display the i^{th} component for $0 \leq i \leq d$.

z	i^{th} component of $[z]$
0	$E_0 V + E_1 V + \cdots + E_i V$
D	$E_d V + E_{d-1} V + \cdots + E_{d-i} V$
0^*	$E_0^* V + E_1^* V + \cdots + E_i^* V$
D^*	$E_d^* V + E_{d-1}^* V + \cdots + E_{d-i}^* V$

Lemma 11.3. [14, Theorem 7.3] *The four flags in Definition 11.2 are mutually opposite.*

Definition 11.4. Let z, w denote an ordered pair of distinct elements of Ω . By Lemma 11.3 the flags $[z]$, $[w]$ are opposite. Let $[zw]$ denote the decomposition of V induced by $[z]$, $[w]$.

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . By the *inversion* of this decomposition we mean the decomposition $\{V_{d-i}\}_{i=0}^d$. By [8, Lemma 8.6] the decompositions in Definition 11.4 have the following features. For distinct $z, w \in \Omega$ we have (i) the decomposition $[zw]$ is the inversion of $[wz]$; (ii) for $0 \leq i \leq d$ the i^{th} component of $[zw]$ is the intersection of the i^{th} component of $[z]$ and the $(d-i)^{\text{th}}$ component of $[w]$; (iii) the decomposition $[zw]$ induces $[z]$ and the inversion of $[zw]$ induces $[w]$.

Example 11.5. We display some of the decompositions from Definition 11.4. For each decomposition in the table below we give the i^{th} component for $0 \leq i \leq d$.

decomposition	i^{th} component
$[0^*D]$	$(E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV)$
$[D^*D]$	$(E_d^*V + \cdots + E_{d-i}^*V) \cap (E_iV + \cdots + E_dV)$
$[0^*0]$	$(E_0^*V + \cdots + E_i^*V) \cap (E_{d-i}V + \cdots + E_0V)$
$[D^*0]$	$(E_d^*V + \cdots + E_{d-i}^*V) \cap (E_{d-i}V + \cdots + E_0V)$
$[0D]$	E_iV
$[0^*D^*]$	E_i^*V

12 The action of T on the flags and decompositions

Recall the Leonard system Φ from (19) and the element T from Definition 8.1. In this section we describe how T acts on the flags from Definition 11.2 and the decompositions from Definition 11.4.

Lemma 12.1. *Assume that Φ is self-dual. Then*

$$TE_iV = E_i^*V, \quad TE_i^*V = E_iV \quad (0 \leq i \leq d).$$

Proof. By (34), $TE_iV = E_i^*TV$. We have $TV = V$ since T is invertible. By these comments $TE_iV = E_i^*V$. Similarly $TE_i^*V = E_iV$. \square

For a sequence $H = \{H_i\}_{i=0}^d$ of subspaces of V , let TH denote the sequence $\{TH_i\}_{i=0}^d$.

Proposition 12.2. *Assume that Φ is self-dual. Then*

$$T[0] = [0^*], \quad T[0^*] = [0], \quad T[D] = [D^*], \quad T[D^*] = [D].$$

Proof. By Definition 11.2 and Lemma 12.1. \square

Proposition 12.3. *Assume that Φ is self-dual. In the table below we give some decompositions u of V . For each decomposition u we give Tu .*

u	$[0^*D]$	$[D^*D]$	$[0^*0]$	$[D^*0]$	$[0D]$	$[0^*D^*]$
Tu	$[0D^*]$	$[DD^*]$	$[00^*]$	$[D0^*]$	$[0^*D^*]$	$[0D]$

Proof. First consider the case $u = [0^*D]$. By Definition 11.4 the decomposition u is induced by the ordered pair of flags $[0^*], [D]$. By this and since T is invertible, the decomposition Tu is induced by the ordered pair of flags $T[0^*], T[D]$. By Proposition 12.2 we have $T[0^*] = [0]$ and $T[D] = [D^*]$. By these comments and Definition 11.4, $Tu = [0D^*]$. We have shown the result for the case $u = [0^*D]$. For the other cases the proof is similar. \square

13 The 24 bases

Recall the Leonard system Φ from (19) and the element T from Definition 8.1. In [14] the second author introduced 24 bases for V on which A, A^* act in an attractive manner. Our next goal is to describe how T acts on these bases. In this section we define the 24 bases and give their basic properties.

Let v_0, v_d, v_0^*, v_d^* denote nonzero vectors in V such that

$$v_0 \in E_0V, \quad v_d \in E_dV, \quad v_0^* \in E_0^*V, \quad v_d^* \in E_d^*V. \quad (35)$$

We consider the decompositions from Definition 11.4.

Lemma 13.1. For each row in the table below, consider the decomposition $\{U_i\}_{i=0}^d$ of V in the first column. For $0 \leq i \leq d$ the vector in the second column and third column is a basis for U_i .

decomposition $\{U_i\}_{i=0}^d$	basis for U_i	basis for U_i
$[0^*D]$	$\tau_i(A)v_0^*$	$\eta_{d-i}^*(A^*)v_d$
$[D^*D]$	$\tau_i(A)v_d^*$	$\tau_{d-i}^*(A^*)v_d$
$[0^*0]$	$\eta_i(A)v_0^*$	$\eta_{d-i}^*(A^*)v_0$
$[D^*0]$	$\eta_i(A)v_d^*$	$\tau_{d-i}^*(A^*)v_0$

Proof. By [8, Lemma 8.8]. □

Corollary 13.2. Each of the following 8 sequences is a basis for V :

$$\{\tau_i(A)v_0^*\}_{i=0}^d, \quad \{\tau_i(A)v_d^*\}_{i=0}^d, \quad \{\eta_i(A)v_0^*\}_{i=0}^d, \quad \{\eta_i(A)v_d^*\}_{i=0}^d, \quad (36)$$

$$\{\tau_{d-i}^*(A^*)v_0\}_{i=0}^d, \quad \{\tau_{d-i}^*(A^*)v_d\}_{i=0}^d, \quad \{\eta_{d-i}^*(A^*)v_0\}_{i=0}^d, \quad \{\eta_{d-i}^*(A^*)v_d\}_{i=0}^d. \quad (37)$$

Proof. By Lemma 13.1. □

Lemma 13.3. For each row in the table below, consider the decomposition $\{U_i\}_{i=0}^d$ of V in the first column. For $0 \leq i \leq d$ the vector in the second column and third column is a basis for U_i .

decomposition $\{U_i\}_{i=0}^d$	basis for U_i	basis for U_i
$[D0^*]$	$\eta_i^*(A^*)v_d$	$\tau_{d-i}(A)v_0^*$
$[DD^*]$	$\tau_i^*(A^*)v_d$	$\tau_{d-i}(A)v_d^*$
$[00^*]$	$\eta_i^*(A^*)v_0$	$\eta_{d-i}(A)v_0^*$
$[0D^*]$	$\tau_i^*(A^*)v_0$	$\eta_{d-i}(A)v_d^*$

Proof. These are the inversions of the decompositions in Lemma 13.1. □

Corollary 13.4. Each of the following 8 sequences is a basis for V :

$$\{\tau_i^*(A^*)v_0\}_{i=0}^d, \quad \{\tau_i^*(A^*)v_d\}_{i=0}^d, \quad \{\eta_i^*(A^*)v_0\}_{i=0}^d, \quad \{\eta_i^*(A^*)v_d\}_{i=0}^d, \quad (38)$$

$$\{\tau_{d-i}(A)v_0^*\}_{i=0}^d, \quad \{\tau_{d-i}(A)v_d^*\}_{i=0}^d, \quad \{\eta_{d-i}(A)v_0^*\}_{i=0}^d, \quad \{\eta_{d-i}(A)v_d^*\}_{i=0}^d. \quad (39)$$

Proof. By Lemma 13.3. □

Lemma 13.5. For each row in the table below, consider the decomposition $\{U_i\}_{i=0}^d$ of V in the first column. For $0 \leq i \leq d$ the vector in the second column and third column is a basis for U_i .

decomposition $\{U_i\}_{i=0}^d$	basis for U_i	basis for U_i
$[0D]$	$E_i v_0^*$	$E_i v_d^*$
$[0^*D^*]$	$E_i^* v_0$	$E_i^* v_d$
$[D0]$	$E_{d-i} v_0^*$	$E_{d-i} v_d^*$
$[D^*0^*]$	$E_{d-i}^* v_0$	$E_{d-i}^* v_d$

Proof. First consider the decomposition $[0D]$. By Example 11.5, $E_i v_0^* \in U_i$. By [16, Lemma 10.2], $E_i v_0^* \neq 0$. Thus $E_i v_0^*$ is a basis for U_i . Similarly $E_i v_d^*$ is a basis for U_i . The proof is similar for the remaining decompositions. □

Corollary 13.6. *Each of the following 8 sequences is a basis for V :*

$$\{E_i v_0^*\}_{i=0}^d, \quad \{E_i v_d^*\}_{i=0}^d, \quad \{E_{d-i} v_0^*\}_{i=0}^d, \quad \{E_{d-i} v_d^*\}_{i=0}^d, \quad (40)$$

$$\{E_i^* v_0\}_{i=0}^d, \quad \{E_i^* v_d\}_{i=0}^d, \quad \{E_{d-i}^* v_0\}_{i=0}^d, \quad \{E_{d-i}^* v_d\}_{i=0}^d. \quad (41)$$

Proof. By Lemma 13.5. □

Note 13.7. The 24 bases (36)–(41) are investigated by the second author in [14]. In [14, Theorem 11.2] the matrices representing A and A^* with respect to these 24 bases are given. In [14, Section 15] the transition matrices between these 24 bases are given.

Let $\{u_i\}_{i=0}^d$ denote a basis for V . Then $\{u_{d-i}\}_{i=0}^d$ is a basis for V , called the *inversion* of $\{u_i\}_{i=0}^d$. For each of the 24 bases listed in (36)–(41), its inversion is listed in (36)–(41).

14 Some relationship among the 24 bases

Recall the Leonard system Φ from (19). In Lemmas 13.1, 13.3, 13.5 we gave some decompositions of V . For each decomposition and $0 \leq i \leq d$ we gave two bases for its i^{th} component. In this section we show how these bases are related. To do this, we consider the following inner products:

$$(v_0, v_0), \quad (v_d, v_d), \quad (v_0^*, v_0^*), \quad (v_d^*, v_d^*), \quad (42)$$

$$(v_0, v_0^*), \quad (v_0, v_d^*), \quad (v_d, v_0^*), \quad (v_d, v_d^*). \quad (43)$$

The above scalars are all nonzero by [15, Lemma 15.5] applied to the relatives of Φ .

Lemma 14.1. [9, Lemma 9.5] *We have*

$$E_0 v_0^* = \frac{(v_0, v_0^*)}{(v_0, v_0)} v_0, \quad E_d v_0^* = \frac{(v_d, v_0^*)}{(v_d, v_d)} v_d, \quad (44)$$

$$E_0 v_d^* = \frac{(v_0, v_d^*)}{(v_0, v_0)} v_0, \quad E_d v_d^* = \frac{(v_d, v_d^*)}{(v_d, v_d)} v_d, \quad (45)$$

$$E_0^* v_0 = \frac{(v_0, v_0^*)}{(v_0^*, v_0^*)} v_0^*, \quad E_d^* v_0 = \frac{(v_0, v_d^*)}{(v_d^*, v_d^*)} v_d^*, \quad (46)$$

$$E_0^* v_d = \frac{(v_d, v_0^*)}{(v_0^*, v_0^*)} v_0^*, \quad E_d^* v_d = \frac{(v_d, v_d^*)}{(v_d^*, v_d^*)} v_d^*. \quad (47)$$

The scalars (42), (43) satisfy the following relations.

Lemma 14.2. [9, Lemma 9.7] *We have*

$$\frac{(v_0, v_d^*)(v_d, v_0^*)}{(v_0, v_0^*)(v_d, v_d^*)} = \frac{\varphi_1 \cdots \varphi_d}{\phi_1 \cdots \phi_d}. \quad (48)$$

Lemma 14.3. [9, Corollary 8.3, Lemma 9.6] *We have*

$$\frac{(v_0, v_0)(v_0^*, v_0^*)}{(v_0, v_0^*)^2} = \frac{\eta_d(\theta_0)\eta_d^*(\theta_0^*)}{\phi_1 \cdots \phi_d}, \quad (49)$$

$$\frac{(v_0, v_0)(v_d^*, v_d^*)}{(v_0, v_d^*)^2} = \frac{\eta_d(\theta_0)\tau_d^*(\theta_d^*)}{\varphi_1 \cdots \varphi_d}, \quad (50)$$

$$\frac{(v_d, v_d)(v_0^*, v_0^*)}{(v_d, v_0^*)^2} = \frac{\tau_d(\theta_d)\eta_d^*(\theta_0^*)}{\varphi_1 \cdots \varphi_d}, \quad (51)$$

$$\frac{(v_d, v_d)(v_d^*, v_d^*)}{(v_d, v_d^*)^2} = \frac{\tau_d(\theta_d)\tau_d^*(\theta_d^*)}{\phi_1 \cdots \phi_d}. \quad (52)$$

Note 14.4. By (49)–(52) the scalars (43) are determined up to sign by the scalars (42) and the parameter array.

Our next goal is to describe how the bases in Lemmas 13.1, 13.3, 13.5 are related. The bases in Lemma 13.1 are related as follows.

Lemma 14.5. For $0 \leq i \leq d$,

$$\tau_{d-i}^*(A^*)v_0 = \frac{\tau_d^*(\theta_d^*)}{\varphi_d \cdots \varphi_{d-i+1}} \frac{(v_0, v_d^*)}{(v_d^*, v_d^*)} \eta_i(A)v_d^*, \quad (53)$$

$$\eta_{d-i}^*(A^*)v_0 = \frac{\eta_d^*(\theta_0^*)}{\phi_1 \cdots \phi_i} \frac{(v_0, v_0^*)}{(v_0^*, v_0^*)} \eta_i(A)v_0^*, \quad (54)$$

$$\tau_{d-i}^*(A^*)v_d = \frac{\tau_d^*(\theta_d^*)}{\phi_d \cdots \phi_{d-i+1}} \frac{(v_d, v_d^*)}{(v_d^*, v_d^*)} \tau_i(A)v_d^*, \quad (55)$$

$$\eta_{d-i}^*(A^*)v_d = \frac{\eta_d^*(\theta_0^*)}{\varphi_1 \cdots \varphi_i} \frac{(v_d, v_0^*)}{(v_0^*, v_0^*)} \tau_i(A)v_0^*. \quad (56)$$

Proof. We first show (53). By [6, Theorem 5.2],

$$\tau_{d-i}^*(A^*)E_0 = \frac{\tau_d^*(\theta_d^*)}{\varphi_d \cdots \varphi_{d-i+1}} \eta_i(A)E_d^*E_0. \quad (57)$$

In this line, apply each side to v_0 and use $E_0v_0 = v_0$. Simplify the result using the equation on the right in (46). This gives (53). To get the remaining equations, apply (53) to Φ^\downarrow , Φ^\updownarrow , $\Phi^{\downarrow\downarrow}$, and use Lemma 5.2. \square

The bases in Lemma 13.3 are related as follows.

Lemma 14.6. For $0 \leq i \leq d$,

$$\tau_{d-i}(A)v_0^* = \frac{\tau_d(\theta_d)}{\varphi_d \cdots \varphi_{d-i+1}} \frac{(v_d, v_0^*)}{(v_d, v_d)} \eta_i^*(A^*)v_d, \quad (58)$$

$$\eta_{d-i}(A)v_0^* = \frac{\eta_d(\theta_0)}{\phi_d \cdots \phi_{d-i+1}} \frac{(v_0, v_0^*)}{(v_0, v_0)} \eta_i^*(A^*)v_0, \quad (59)$$

$$\tau_{d-i}(A)v_d^* = \frac{\tau_d(\theta_d)}{\phi_1 \cdots \phi_i} \frac{(v_d, v_d^*)}{(v_d, v_d)} \tau_i^*(A^*)v_d, \quad (60)$$

$$\eta_{d-i}(A)v_d^* = \frac{\eta_d(\theta_0)}{\varphi_1 \cdots \varphi_i} \frac{(v_0, v_d^*)}{(v_0, v_0)} \tau_i^*(A^*)v_0. \quad (61)$$

Proof. Apply Lemma 14.5 to Φ^* , and use Lemma 5.2. \square

The bases in Lemma 13.5 are related as follows.

Lemma 14.7. For $0 \leq i \leq d$,

$$E_i^*v_d = \frac{\phi_1 \cdots \phi_i}{\varphi_1 \cdots \varphi_i} \frac{(v_d, v_0^*)}{(v_0, v_0^*)} E_i^*v_0, \quad (62)$$

$$E_{d-i}^*v_d = \frac{\varphi_d \cdots \varphi_{d-i+1}}{\phi_d \cdots \phi_{d-i+1}} \frac{(v_d, v_d^*)}{(v_0, v_d^*)} E_{d-i}^*v_0, \quad (63)$$

$$E_iv_d^* = \frac{\phi_d \cdots \phi_{d-i+1}}{\varphi_1 \cdots \varphi_i} \frac{(v_0, v_d^*)}{(v_0, v_0^*)} E_iv_0^*, \quad (64)$$

$$E_{d-i}v_d^* = \frac{\varphi_d \cdots \varphi_{d-i+1}}{\phi_1 \cdots \phi_i} \frac{(v_d, v_d^*)}{(v_d, v_0^*)} E_{d-i}v_0^*. \quad (65)$$

Proof. We first show (62). In the equation on the right in (46), multiply each side on the left by $E_i^*E_d$. Simplify the result using the equation on the right in (45). This gives

$$E_i^*E_dE_d^*v_0 = \frac{(v_0, v_d^*)(v_d, v_d^*)}{(v_d, v_d)(v_d^*, v_d^*)} E_i^*v_d. \quad (66)$$

By [9, Lemma 7.2],

$$E_0 E_d^* E_d E_i^* = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d) \tau_d^*(\theta_d^*)} \frac{\phi_1 \cdots \phi_i}{\varphi_1 \cdots \varphi_i} E_0 E_i^*. \quad (67)$$

Applying \dagger to (67) we obtain

$$E_i^* E_d E_d^* E_0 = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d) \tau_d^*(\theta_d^*)} \frac{\phi_1 \cdots \phi_i}{\varphi_1 \cdots \varphi_i} E_i^* E_0. \quad (68)$$

In this line, apply each side to v_0 , and use $E_0 v_0 = v_0$. Comparing the result with (66) we find that $E_i^* v_d$ is equal to

$$\frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d) \tau_d^*(\theta_d^*)} \frac{(v_d, v_d)(v_d^*, v_d^*)}{(v_0, v_d^*)(v_d, v_d^*)} \quad (69)$$

times

$$\frac{\phi_1 \cdots \phi_i}{\varphi_1 \cdots \varphi_i} E_i^* v_0.$$

By (48) and (52), the line (69) is equal to

$$\frac{(v_d, v_0^*)}{(v_0, v_0^*)}.$$

By these comments we obtain (62). To get (63), replace i with $d - i$ in (62) and use (48). To get (64), apply (62) to Φ^* , and use Lemma 5.2. The line (65) is similarly obtained by applying (63) to Φ^* . \square

15 The action of T on the 24 bases

Recall the Leonard system Φ from (19) and the element T from Definition 8.1. Consider the 24 bases from (36)–(41). In this section we describe how T acts on these bases, under the assumption that Φ is self-dual.

Lemma 15.1. *Assume that Φ is self-dual. Then*

$$\begin{aligned} T v_0 &= \alpha v_0^*, & T v_d &= \beta v_d^*, \\ T v_0^* &= \alpha^* v_0, & T v_d^* &= \beta^* v_d, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)} \frac{(v_0, v_0^*)}{(v_0^*, v_0^*)}, & \beta &= \frac{\varphi_1 \cdots \varphi_d}{\eta_d(\theta_0)} \frac{(v_d, v_d^*)}{(v_d^*, v_d^*)}, \\ \alpha^* &= \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)} \frac{(v_0, v_0^*)}{(v_0, v_0)}, & \beta^* &= \frac{\varphi_1 \cdots \varphi_d}{\eta_d(\theta_0)} \frac{(v_d, v_d^*)}{(v_d, v_d)}. \end{aligned}$$

Proof. By Theorem 8.3 we have $T = T^*$. By this and (22),

$$T E_0 = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)} E_0^* E_0.$$

In this line, apply each side to v_0 . Simplify the result using $E_0 v_0 = v_0$ to get

$$T v_0 = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)} E_0^* v_0.$$

In this line, eliminate $E_0^* v_0$ using the equation on the left in (46) to get $T v_0 = \alpha v_0^*$. The remaining equations are obtained in a similar way. \square

Proposition 15.2. *Assume that Φ is self-dual. Then for $0 \leq i \leq d$,*

$$\begin{aligned} T E_i^* v_0 &= \alpha E_i v_0^*, & T \tau_i^*(A^*) v_0 &= \alpha \tau_i(A) v_0^*, & T \eta_i^*(A^*) v_0 &= \alpha \eta_i(A) v_0^*, \\ T E_i^* v_d &= \beta E_i v_d^*, & T \tau_i^*(A^*) v_d &= \beta \tau_i(A) v_d^*, & T \eta_i^*(A^*) v_d &= \beta \eta_i(A) v_d^*, \\ T E_i v_0^* &= \alpha^* E_i^* v_0, & T \tau_i(A) v_0^* &= \alpha^* \tau_i^*(A^*) v_0, & T \eta_i(A) v_0^* &= \alpha^* \eta_i^*(A^*) v_0, \\ T E_i v_d^* &= \beta^* E_i^* v_d, & T \tau_i(A) v_d^* &= \beta^* \tau_i^*(A^*) v_d, & T \eta_i(A) v_d^* &= \beta^* \eta_i^*(A^*) v_d, \end{aligned}$$

where $\alpha, \beta, \alpha^*, \beta^*$ are from Lemma 15.1.

Proof. By (34) we have $TE_i^*T^{-1} = E_i$. By Lemma 15.1, $Tv_0 = \alpha v_0^*$. By these comments

$$TE_i^*v_0 = TE_i^*T^{-1}Tv_0 = \alpha E_i v_0^*.$$

We have shown that $TE_i^*v_0 = \alpha E_i v_0^*$. The remaining equations are obtained in a similar way. \square

To motivate the next result we make some comments. Consider the following bases for V :

$$\{\eta_i^*(A^*)v_0\}_{i=0}^d, \quad \{\eta_i(A)v_0^*\}_{i=0}^d, \quad \{\tau_i^*(A^*)v_d\}_{i=0}^d, \quad \{\tau_i(A)v_d^*\}_{i=0}^d. \quad (70)$$

By [14, Theorem 11.2], with respect to these bases the matrices representing A and A^* are as follows.

basis	matrix representing A	matrix representing A^*
$\{\eta_i^*(A^*)v_0\}_{i=0}^d$	$\begin{pmatrix} \theta_0 & \phi_d & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \ddots & & \\ & & & \phi_2 & \\ & & & \theta_{d-1} & \phi_1 \\ \mathbf{0} & & & & \theta_d \end{pmatrix}$	$\begin{pmatrix} \theta_d^* & & & & \mathbf{0} \\ 1 & \theta_{d-1}^* & & & \\ & 1 & \cdot & & \\ & & \ddots & \cdot & \\ & & & \cdot & \theta_1^* \\ \mathbf{0} & & & & 1 & \theta_0^* \end{pmatrix}$
$\{\eta_i(A)v_0^*\}_{i=0}^d$	$\begin{pmatrix} \theta_d & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & \\ & 1 & \cdot & & \\ & & \ddots & \cdot & \\ & & & \cdot & \theta_1 \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix}$	$\begin{pmatrix} \theta_0^* & \phi_1 & & & \mathbf{0} \\ & \theta_1^* & \phi_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \\ & & & & \theta_{d-1}^* & \phi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}$
$\{\tau_i^*(A^*)v_d\}_{i=0}^d$	$\begin{pmatrix} \theta_d & \phi_1 & & & \mathbf{0} \\ & \theta_{d-1} & \phi_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \\ & & & & \theta_1 & \phi_d \\ \mathbf{0} & & & & & \theta_0 \end{pmatrix}$	$\begin{pmatrix} \theta_0^* & & & & \mathbf{0} \\ 1 & \theta_1^* & & & \\ & 1 & \cdot & & \\ & & \ddots & \cdot & \\ & & & \cdot & \theta_{d-1}^* \\ \mathbf{0} & & & & 1 & \theta_d^* \end{pmatrix}$
$\{\tau_i(A)v_d^*\}_{i=0}^d$	$\begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \cdot & & \\ & & \ddots & \cdot & \\ & & & \cdot & \theta_{d-1} \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}$	$\begin{pmatrix} \theta_d^* & \phi_d & & & \mathbf{0} \\ & \theta_{d-1}^* & \cdot & & \\ & & \ddots & \cdot & \\ & & & \cdot & \phi_2 \\ & & & & \theta_1^* & \phi_1 \\ \mathbf{0} & & & & & \theta_0^* \end{pmatrix}$

Theorem 15.3. *Assume that Φ is self-dual. Then with respect to each basis (70) the matrix representing T is*

$$\frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)\eta_d(\theta_0)} \begin{pmatrix} \mathbf{0} & & & & \phi_1 \cdots \phi_d \\ & \cdot & & & \\ & & \cdot & & \\ & & & \phi_1 \phi_2 & \\ & & & \phi_1 & \\ 1 & & & & \mathbf{0} \end{pmatrix}. \quad (71)$$

Proof. First consider the basis $\{\eta_i^*(A^*)v_0\}_{i=0}^d$. By Proposition 15.2,

$$T\eta_i^*(A^*)v_0 = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)} \frac{(v_0, v_0^*)}{(v_0^*, v_0^*)} \eta_i(A)v_0^* \quad (0 \leq i \leq d).$$

By (17) and (54),

$$\eta_i(A)v_0^* = \frac{\phi_1 \cdots \phi_i}{\eta_d(\theta_0)} \frac{(v_0^*, v_0^*)}{(v_0, v_0^*)} \eta_{d-i}^*(A^*)v_0 \quad (0 \leq i \leq d).$$

By these comments,

$$T\eta_i^*(A^*)v_0 = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)\eta_d(\theta_0)} \phi_1 \cdots \phi_i \eta_{d-i}^*(A^*)v_0 \quad (0 \leq i \leq d). \quad (72)$$

Thus the matrix (71) represents T with respect to $\{\eta_i^*(A^*)v_0\}_{i=0}^d$.

Next consider the basis $\{\tau_i^*(A^*)v_d\}_{i=0}^d$. In a similar way as above using (18) and (55), we obtain

$$T\tau_i^*(A^*)v_d = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)\eta_d(\theta_0)} \phi_1 \cdots \phi_i \tau_{d-i}^*(A^*)v_d \quad (0 \leq i \leq d). \quad (73)$$

Thus the matrix (71) represents T with respect to $\{\tau_i^*(A^*)v_d\}_{i=0}^d$.

Next consider the basis $\{\eta_i(A)v_0^*\}_{i=0}^d$. Apply (72) to Φ^* , and use Lemma 5.2. This gives

$$T^*\eta_i(A)v_0^* = \frac{\varphi_1 \cdots \varphi_d}{\tau_d^*(\theta_d^*)\eta_d^*(\theta_0^*)} \phi_d \cdots \phi_{d-i+1} \eta_{d-i}(A)v_0^* \quad (0 \leq i \leq d).$$

We have $T^* = T$ by Theorem 8.3. By this and (17), (18), the above line becomes

$$T\eta_i(A)v_0^* = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)\eta_d(\theta_0)} \phi_1 \cdots \phi_i \eta_{d-i}(A)v_0^* \quad (0 \leq i \leq d).$$

Thus the matrix (71) represents T with respect to $\{\eta_i(A)v_0^*\}_{i=0}^d$.

Next consider the basis $\{\tau_i(A)v_d^*\}_{i=0}^d$. Apply (73) to Φ^* , and use Lemma 5.2. Simplify the result in a similar way as above to get

$$T\tau_i(A)v_d^* = \frac{\varphi_1 \cdots \varphi_d}{\tau_d(\theta_d)\eta_d(\theta_0)} \phi_1 \cdots \phi_i \tau_{d-i}(A)v_d^* \quad (0 \leq i \leq d).$$

Thus the matrix (71) represents T with respect to $\{\tau_i(A)v_d^*\}_{i=0}^d$. □

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