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Explicit determinants, inverses and eigenvalues of four band Toeplitz matrices with perturbed rows

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Abstract: In this paper, four-band Toeplitz matrices and four-band Hankel matrices of type I and type II with perturbed rows are introduced. Explicit determinants, inverses and eigenvalues for these matrices are derived by using a nice inverse formula of block bidiagonal Toeplitz matrices.

Keywords: perturbed Toeplitz four band matrix, perturbed Hankel four band matrix, determinant, inverse, eigenvalues

MSC: Primary 15B05; 15A15, Secondary 11B39

1 Introduction

Throughout this paper, we consider explicit determinants, inverses and eigenvalues for two classes of four-band Toeplitz and Hankel matrices with perturbed rows, respectively.

An $n \times n$ matrix $\mathcal{M} = (d_{ij})_{n\times n}$ is said to be a perturbed Toeplitz four-band matrix of type I if its entries are defined as

$$d_{i,j} = \begin{cases} a & i = j = 1, \\ b & i = 1, j = n, \\ c & i = n, j = 1, \\ d & i = j = n, \\ s_{n-j} & i = 1,\ 2 \le j \le n-1, \\ -f & 0 \le j - i \le 2,\ 2 \le i,\ j \le n-1, \\ \frac{f+(-1)^{i-1}f}{2} & i - j = 1,\ 2 \le i,\ j \le n-1, \\ t_{n-j} & i = n,\ 2 \le j \le n-1, \\ 0 & \text{otherwise}, \end{cases}$$

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that is,

$$\mathcal{M} = \begin{pmatrix} a & s_{n-2} & s_{n-3} & s_{n-4} & s_{n-5} & \cdots & s_3 & s_2 & s_1 & b \\ 0 & -f & -f & -f & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & f & -f & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & f & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & f & -f & -f & -f & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & -f & -f & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & f & -f & 0 \\ c & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & \cdots & t_3 & t_2 & t_1 & d \end{pmatrix}_{n\times n}, \tag{1.1}$$

where a, b, c, d $(ad - bc \neq 0)$, $f \neq 0$, s_i and t_i $(i = 1, 2, \cdots, n-2)$ are arbitrary complex numbers.

We introduce some closely related matrices. An $n \times n$ matrix $\mathcal{N} = (g_{ij})_{n\times n}$ is said to be a perturbed Hankel four-band matrix of type I if $\mathcal{N} = \hat{\mathbf{I}}_n\mathcal{M}$. An $n \times n$ matrix $\mathcal{P} = (h_{ij})_{n\times n}$ is said to be a perturbed Hankel four-band matrix of type II if $\mathcal{P} = \mathcal{M}\hat{\mathbf{I}}_n$. Finally, an $n \times n$ matrix $\mathcal{Q} = (s_{ij})_{n\times n}$ is said to be a perturbed Toeplitz four-band matrix of type II if $\mathcal{Q} = \hat{\mathbf{I}}_n\mathcal{M}\hat{\mathbf{I}}_n$.

Let $\hat{\mathbf{I}}_n$ be the "reverse unit matrix", which has ones along the secondary diagonal and zeros elsewhere. Then one observes that $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{Q}$ can be generated by $\mathcal{M}$ and $\hat{\mathbf{I}}_n$ through matrix multiplications in the following way:

$$\mathcal{N} = \hat{\mathbf{I}}_n\mathcal{M}, \tag{1.2}$$

$$\mathcal{P} = \mathcal{M}\hat{\mathbf{I}}_n, \tag{1.3}$$

$$\mathcal{Q} = \hat{\mathbf{I}}_n\mathcal{M}\hat{\mathbf{I}}_n. \tag{1.4}$$

Many special banded matrices such as Toeplitz matrices [14–17, 21, 22, 43–45], symmetric Toeplitz matrices, especially tridiagonal matrices, etc., have been studied extensively; see [3–7, 9–12, 18–20, 24, 25, 35, 36, 39, 40], to name only a few. These matrices arise in many areas of pure and applied mathematics. For example, tridiagonal matrices have been used in telecommunication system analysis, finite difference methods for solving PDEs [8, 32, 33, 37, 38], linear recurrence systems with non-constant coefficients, etc. Unlike the tridiagonal matrices which have received much attention, little is known about the 4-diagonal or general banded matrices. Indeed, in this paper we are going to explore some special kinds 4-band Toeplitz matrices with purturbed entries. Though this is a topic of interest in its own right, it could have impact on other field of studies. A recent example is a problem from lattices associated with finite Abelian groups; see [2].

The rest of the paper is organized as follows: Section 2 is devoted to some auxiliary results. Section 3 consists of the main results of the paper, we derive the determinant, inverse and eigenvalues of perturbed Toeplitz four-band matrices of type I. In Section 4-6, the determinants and inverses of perturbed Hankel four-band matrices of type I and II and perturbed Toeplitz four-band matrices of type II are presented, respectively. They all base on the results in Section 3. In Section 7, algorithm and numerical examples which related to the determinant, inverse and eigenvalues of perturbed Toeplitz four-band matrix of type I are showed.

2 The inverses of some special structured matrices

The inverses of some special matrices are discussed in this section. These formulas are used in our derivation of main results in Section 3. The first two lemmas are known, while the third one seems to be new.

Lemma 2.1. *(see, e.g.[42, p. 19]) Let R and $\begin{pmatrix} J & K \\ S & R \end{pmatrix}$ be invertible matrices. Then*

$$\begin{pmatrix} J & K \\ S & R \end{pmatrix}^{-1} = \begin{pmatrix} (J - KR^{-1}S)^{-1} & -(J - KR^{-1}S)^{-1}KR^{-1} \\ -R^{-1}S(J - KR^{-1}S)^{-1} & R^{-1} + R^{-1}S(J - KR^{-1}S)^{-1}KR^{-1} \end{pmatrix}.$$

Lemma 2.2. *(see, e.g.[23, p. 126]) An $n \times n$ matrix $\mathcal{F}_n = (l_{ij})_{n\times n}$ is said to be a bidiagonal Toeplitz matrix if*

$$l_{ij} = \begin{cases} \alpha & 1 \le i = j \le n, \\ \beta & 1 \le j = i + 1 \le n, \\ 0 & \text{otherwise}. \end{cases}$$

If $\alpha \neq 0$, then

$$\mathcal{F}_n^{-1} = \begin{cases} \Delta_{j-i+1} & i \le j, \\ 0 & i > j, \end{cases}$$

where

$$\Delta_i = \frac{(-\beta)^{i-1}}{\alpha^i}, \; 1 \le i \le n.$$

Lemma 2.3. *A $2n \times 2n$ matrix $\mathcal{T}_{2n}(\Phi_{2\times2}, \Psi_{2\times2})$ is called a block bidiagonal Toeplitz matrix if*

$$\mathcal{T}_{2n}(\Phi_{2\times2}, \Psi_{2\times2}) = \begin{pmatrix} \Phi_{2\times2} & \Psi_{2\times2} & O_{2\times2} & \cdots & \cdots & O_{2\times2} \\ O_{2\times2} & \Phi_{2\times2} & \Psi_{2\times2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & O_{2\times2} \\ \vdots & & & \ddots & \Phi_{2\times2} & \Psi_{2\times2} \\ O_{2\times2} & \cdots & \cdots & \cdots & O_{2\times2} & \Phi_{2\times2} \end{pmatrix}_{n\times n},$$

where $\Phi_{2\times2}$ and $\Psi_{2\times2}$ are 2×2 matrices with $\Phi_{2\times2}$ nonsingular, $O_{2\times2}$ is a 2×2 zero matrix. The inverse of $\mathcal{T}_{2n}(\Phi_{2\times2}, \Psi_{2\times2})$ can be expressed as

$$\mathcal{T}_{2n}^{-1}(\Phi_{2\times2}, \Psi_{2\times2}) = \begin{pmatrix} (Y_1)_{2\times2} & (Y_2)_{2\times2} & (Y_3)_{2\times2} & \cdots & \cdots & (Y_n)_{2\times2} \\ O_{2\times2} & (Y_1)_{2\times2} & (Y_2)_{2\times2} & (Y_3)_{2\times2} & \cdots & (Y_n)_{2\times2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & (Y_3)_{2\times2} \\ \vdots & & & \ddots & (Y_1)_{2\times2} & (Y_2)_{2\times2} \\ O_{2\times2} & \cdots & \cdots & \cdots & O_{2\times2} & (Y_1)_{2\times2} \end{pmatrix}_{n\times n},$$

where

$$(Y_i)_{2\times2} = (-\Phi_{2\times2}^{-1}\Psi_{2\times2})^{i-1}\Phi_{2\times2}^{-1}, \quad 1 \le i \le n.$$

Proof. We first decompose

$$\mathcal{T}_{2n}(\Phi_{2\times2}, \Psi_{2\times2}) = \mathcal{T}_1\mathcal{T}_2, \tag{2.1}$$

where

$$\mathcal{T}_1 = \begin{pmatrix} \Phi_{2\times2} & & & \\ & \Phi_{2\times2} & & \\ & & \ddots & \\ & & & \Phi_{2\times2} \end{pmatrix}_{n\times n},$$

$$\mathcal{T}_2 = \begin{pmatrix} I_{2\times 2} & (\Phi^{-1}\Psi)_{2\times 2} & & \\ & I_{2\times 2} & \ddots & \\ & & \ddots & (\Phi^{-1}\Psi)_{2\times 2} \\ & & & I_{2\times 2} \end{pmatrix}_{n\times n},$$

and $I_{2\times 2}$ is the 2×2 identity matrix. It is clear that $I_{2\times 2}$ and $(\Phi^{-1}\Psi)_{2\times 2}$ commute. It follows from Lemma 2.2 that

$$\mathcal{T}_2^{-1} = \begin{pmatrix} I_{2\times 2} & -\Phi_{2\times 2}^{-1}\Psi_{2\times 2} & (-\Phi_{2\times 2}^{-1}\Psi_{2\times 2})^2 & \cdots & \cdots & (-\Phi_{2\times 2}^{-1}\Psi_{2\times 2})^{n-1} \\ O_{2\times 2} & I_{2\times 2} & -\Phi_{2\times 2}^{-1}\Psi_{2\times 2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & (-\Phi_{2\times 2}^{-1}\Psi_{2\times 2})^2 \\ \vdots & & & \ddots & I_{2\times 2} & -\Phi_{2\times 2}^{-1}\Psi_{2\times 2} \\ O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & I_{2\times 2} \end{pmatrix}.$$

Now by (2.1),

$$\begin{aligned} \mathcal{T}_{2n}^{-1}(\Phi_{2\times 2}, \Psi_{2\times 2}) &= \mathcal{T}_2^{-1}\mathcal{T}_1^{-1} \\ &= \mathcal{T}_2^{-1} \begin{pmatrix} \Phi_{2\times 2} & & & \\ & \Phi_{2\times 2} & & \\ & & \ddots & \\ & & & \Phi_{2\times 2} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \Phi_{2\times 2}^{-1} & -\Phi_{2\times 2}^{-1}\Psi_{2\times 2}\Phi_{2\times 2}^{-1} & (-\Phi_{2\times 2}^{-1}\Psi_{2\times 2})^2\Phi_{2\times 2}^{-1} & \cdots & \cdots & (-\Phi_{2\times 2}^{-1}\Psi_{2\times 2})^{n-1}\Phi_{2\times 2}^{-1} \\ O_{2\times 2} & \Phi_{2\times 2}^{-1} & -\Phi_{2\times 2}^{-1}\Psi_{2\times 2}\Phi_{2\times 2}^{-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & (-\Phi_{2\times 2}^{-1}\Psi_{2\times 2})^2\Phi_{2\times 2}^{-1} \\ \vdots & & & \ddots & \Phi_{2\times 2}^{-1} & -\Phi_{2\times 2}^{-1}\Psi_{2\times 2}\Phi_{2\times 2}^{-1} \\ O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & \Phi_{2\times 2}^{-1} \end{pmatrix}. \end{aligned}$$

□

3 Determinant, inverse and eigenvalues of the perturbed Toeplitz four band matrix of type I

This section consists of the main results of the paper. Theorem 3.1, Theorem 3.2 and Theorem 3.3 below are devoted to the determinant, inverse and eigenvalues of the matrix $\mathcal{M}$, respectively.

Theorem 3.1. *Let $\mathcal{M}$ be an $n\times n$ (with $n \geq 3$) perturbed Toeplitz four-band matrix of type I given in (1.1). Then*

$$\det \mathcal{M} = \begin{cases} 2^{\frac{n-3}{2}}(-f)^{n-2}(ad-bc) & \text{if } n \text{ is odd}, \\ 2^{\frac{n-2}{2}} f^{n-2}(ad-bc) & \text{if } n \text{ is even}. \end{cases} \tag{3.1}$$

Proof. Let

$$\mathfrak{P} = \text{circ}\,(0, 0, \cdots, 0, 1) \tag{3.2}$$

be an $n \times n$ circulant permutation matrix. It is obvious that

$$\mathfrak{P}^{-1} = \mathfrak{P}^{T}. \tag{3.3}$$

We partition $\mathfrak{P}\mathcal{M}\mathfrak{P}^{-1}$ into blocks as follows

$$\begin{aligned}
\mathfrak{P}\mathcal{M}\mathfrak{P}^{-1} &= \left(\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right) \\
&= \left(\begin{array}{cc|ccccccccc}
d & c & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & \cdots & \cdots & t_3 & t_2 & t_1 \\
b & a & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & \cdots & \cdots & s_3 & s_2 & s_1 \\
\hline
0 & 0 & -f & -f & -f & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & f & -f & -f & -f & \ddots & & & & \vdots \\
\vdots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & f & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & & \ddots & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & & & & \ddots & f & \ddots & \ddots & -f \\
\vdots & \vdots & \vdots & & & & & \ddots & 0 & \ddots & -f \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & f & -f
\end{array}\right)_{n\times n}.
\end{aligned} \tag{3.4}$$

Then

$$\det \mathcal{M} = \det \mathfrak{P}\mathcal{M}\mathfrak{P}^{-1} = \det D_1 \det D_4. \tag{3.5}$$

It is clear that

$$\det D_1 = ad - bc, \quad \det \mathfrak{P} = \det \mathfrak{P}^{-1} = (-1)^{n-1}. \tag{3.6}$$

We proceed to evaluate $\det D_4$.
If n is even, then we partition D_4 into blocks

$$D_4 = \left(\begin{array}{cc|cc|cc|cc|cc}
-f & -f & -f & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
f & -f & -f & -f & 0 & 0 & \cdots & \cdots & 0 & 0 \\
\hline
0 & 0 & -f & -f & \ddots & & \ddots & & \vdots & \vdots \\
0 & 0 & f & -f & & \ddots & & \ddots & \vdots & \vdots \\
\hline
\vdots & \vdots & \ddots & & \ddots & & \ddots & & 0 & 0 \\
\vdots & \vdots & & \ddots & & \ddots & & \ddots & 0 & 0 \\
\hline
\vdots & \vdots & & & \ddots & & -f & -f & -f & 0 \\
\vdots & \vdots & & & & \ddots & f & -f & -f & -f \\
\hline
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & -f & -f \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & f & -f
\end{array}\right)_{\frac{n-2}{2}\times\frac{n-2}{2}},$$

and obtain

$$\det D_4 = (2f^2)^{\frac{n-2}{2}} = 2^{\frac{n-2}{2}} f^{n-2}. \tag{3.7}$$

If n is odd, then we partition D_4 into blocks

$$D_4 = \left(\begin{array}{cc:cc:cc:cc:cc:c} -f & -f & -f & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ f & -f & -f & -f & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \hdashline 0 & 0 & -f & -f & \ddots & & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & f & -f & & \ddots & & \ddots & \vdots & \vdots & \vdots \\ \hdashline \vdots & \vdots & \ddots & & \ddots & & \ddots & & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & & \ddots & & \ddots & 0 & 0 & 0 \\ \hdashline \vdots & \vdots & & & \ddots & & -f & -f & -f & 0 & 0 \\ \vdots & \vdots & & & & \ddots & f & -f & -f & -f & 0 \\ \hdashline 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & -f & -f & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & f & -f & 0 \\ \hdashline 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & -f \end{array}\right)_{\frac{n-3}{2} \times \frac{n-3}{2}},$$

and obtain

$$\det D_4 = -f(2f^2)^{\frac{n-3}{2}} = 2^{\frac{n-3}{2}}(-f)^{n-2}. \tag{3.8}$$

Combining (3.5), (3.6), (3.7) and (3.8) yields $\det \mathcal{M}$ as in (3.1). □

Remark The result of Theorem 3.1 is somewhat surprising as the determinant of $\mathcal{M}$ is independent of $s_i,\ t_i$ $(1 \le i \le n-2)$. Moreover, it is clear from the theorem that $\mathcal{M}$ given in (1.1) is nonsingular.

Theorem 3.2. *Let $\mathcal{M}$ be a $2n \times 2n$ (with $n \ge 2$) perturbed Toeplitz four-band matrix of type I given in (1.1). Then*

$$\mathcal{M}^{-1} = \left(\begin{array}{c:cccccc:c} \frac{d}{ad-bc} & k_1 & k_2 & k_3 & \cdots & \cdots & k_{2n-2} & \frac{-b}{ad-bc} \\ \hdashline 0 & (\Lambda_1)_{2\times 2} & (\Lambda_2)_{2\times 2} & (\Lambda_3)_{2\times 2} & \cdots & \cdots & (\Lambda_{n-1})_{2\times 2} & 0 \\ \vdots & O_{2\times 2} & (\Lambda_1)_{2\times 2} & (\Lambda_2)_{2\times 2} & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & (\Lambda_3)_{2\times 2} & \vdots \\ \vdots & \vdots & & & \ddots & (\Lambda_1)_{2\times 2} & (\Lambda_2)_{2\times 2} & \vdots \\ 0 & O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & (\Lambda_1)_{2\times 2} & 0 \\ \hdashline \frac{-c}{ad-bc} & m_1 & m_2 & m_3 & \cdots & \cdots & m_{2n-2} & \frac{a}{ad-bc} \end{array}\right)_{2n\times 2n}, \tag{3.9}$$

where

$$(\Lambda_j)_{2\times 2} = \frac{1}{2^j f}\begin{pmatrix} U_{j+1} & U_j \\ U_{j+2} & U_{j+1} \end{pmatrix}, \quad 1 \le j \le n-1, \tag{3.10}$$

$U_j = -U_{j-1} - 2U_{j-2}$, $U_0 = 0$, $U_1 = 1$*;*

$$m_j = \begin{cases} \dfrac{\sum_{k=1}^{\frac{j+1}{2}}\left[\frac{U_{\frac{j+5}{2}-k}}{2^{\frac{j+3}{2}-k}}(cs_{2n-2k} - at_{2n-2k}) + \frac{U_{\frac{j+7}{2}-k}}{2^{\frac{j+3}{2}-k}}(cs_{2n-1-2k} - at_{2n-1-2k})\right]}{f(ad-bc)} & \text{if } j \text{ is odd}, \\[2ex] \dfrac{\sum_{k=1}^{\frac{j}{2}}\left[\frac{U_{\frac{j+2}{2}-k}}{2^{\frac{j+2}{2}-k}}(cs_{2n-2k} - at_{2n-2k}) + \frac{U_{\frac{j+4}{2}-k}}{2^{\frac{j+2}{2}-k}}(cs_{2n-1-2k} - at_{2n-1-2k})\right]}{f(ad-bc)} & \text{if } j \text{ is even}, \end{cases} \tag{3.11}$$

$$k_j = \begin{cases} \dfrac{\sum_{k=1}^{\frac{j+1}{2}} [\frac{U_{\frac{j+5}{2}-k}}{2^{\frac{j+3}{2}-k}}(bt_{2n-2k} - ds_{2n-2k}) + \frac{U_{\frac{j+7}{2}-k}}{2^{\frac{j+3}{2}-k}}(bt_{2n-1-2k} - ds_{2n-1-2k})]}{f(ad-bc)} & \textit{if } j \textit{ is odd,} \\[2ex] \dfrac{\sum_{k=1}^{\frac{j}{2}} [\frac{U_{\frac{j+2}{2}-k}}{2^{\frac{j+2}{2}-k}}(bt_{2n-2k} - ds_{2n-2k}) + \frac{U_{\frac{j+4}{2}-k}}{2^{\frac{j+2}{2}-k}}(bt_{2n-1-2k} - ds_{2n-1-2k})]}{f(ad-bc)} & \textit{if } j \textit{ is even,} \end{cases} \tag{3.12}$$

and $1 \le j \le 2n-2$.

Proof. Let $\mathbb{P} = \operatorname{circ}(0, 0, \cdots, 0, 1)$ be a $2n \times 2n$ circulant permutation matrix. We partition $\mathbb{P}\mathcal{M}\mathbb{P}^{-1}$ into blocks as follows

$$\begin{aligned} \mathbb{P}\mathcal{M}\mathbb{P}^{-1} &= \left(\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right) \\ &= \left(\begin{array}{cc|ccccccccc} d & c & t_{2n-2} & t_{2n-3} & t_{2n-4} & \cdots & \cdots & \cdots & t_3 & t_2 & t_1 \\ b & a & s_{2n-2} & s_{2n-3} & s_{2n-4} & \cdots & \cdots & \cdots & s_3 & s_2 & s_1 \\ \hline 0 & 0 & -f & -f & -f & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & f & -f & -f & -f & \ddots & & & & \vdots \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & f & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & f & \ddots & \ddots & -f & 0 \\ \vdots & \vdots & \vdots & & & & \ddots & 0 & \ddots & -f & -f \\ \vdots & \vdots & \vdots & & & & & \ddots & \ddots & -f & -f \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & f & -f \end{array}\right)_{2n\times 2n} \\ &= \left(\begin{array}{cc|ccccc} d & c & t_{2n-2} & t_{2n-3} & \cdots & t_2 & t_1 \\ b & a & s_{2n-2} & s_{2n-3} & \cdots & s_2 & s_1 \\ \hline \multicolumn{2}{c|}{O_{2\times2}} & (F_1)_{2\times2} & (F_2)_{2\times2} & O_{2\times2} & \cdots & O_{2\times2} \\ \multicolumn{2}{c|}{\vdots} & O_{2\times2} & (F_1)_{2\times2} & (F_2)_{2\times2} & \ddots & \vdots \\ \multicolumn{2}{c|}{\vdots} & \vdots & \ddots & \ddots & \ddots & O_{2\times2} \\ \multicolumn{2}{c|}{\vdots} & \vdots & & \ddots & \ddots & (F_2)_{2\times2} \\ \multicolumn{2}{c|}{O_{2\times2}} & O_{2\times2} & \cdots & \cdots & O_{2\times2} & (F_1)_{2\times2} \end{array}\right)_{2n\times 2n}, \end{aligned} \tag{3.13}$$

where

$$(F_1)_{2\times2} = f\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad (F_2)_{2\times2} = f\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.$$

It is obvious that D_4 is invertible, then by Lemma 2.1 we get

$$(\mathbb{P}\mathcal{M}\mathbb{P}^{-1})^{-1} = \left(\begin{array}{c|c} D_1^{-1} & -D_1^{-1}D_2D_4^{-1} \\ \hline O & D_4^{-1} \end{array}\right). \tag{3.14}$$

It is clear that

$$D_1^{-1} = \begin{pmatrix} \frac{a}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{d}{ad-bc} \end{pmatrix}. \tag{3.15}$$

Next, we go on to find the D_4^{-1} and $-D_1^{-1}D_2D_4^{-1}$.

Since D_4 is a block bidiagonal Toeplitz matrix, by Lemma 2.3 we obtain

$$D_4^{-1} = \begin{pmatrix} F_1^{-1} & -F_1^{-1}F_2F_1^{-1} & (-F_1^{-1}F_2)^2F_1^{-1} & \cdots & \cdots & (-F_1^{-1}F_2)^{n-2}F_1^{-1} \\ O_{2\times 2} & F_1^{-1} & -F_1^{-1}F_2F_1^{-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & (-F_1^{-1}F_2)^2F_1^{-1} \\ \vdots & & & \ddots & F_1^{-1} & -F_1^{-1}F_2F_1^{-1} \\ O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & F_1^{-1} \end{pmatrix}_{(n-1)\times(n-1)}$$

$$= \begin{pmatrix} (\Lambda_1)_{2\times 2} & (\Lambda_2)_{2\times 2} & (\Lambda_3)_{2\times 2} & \cdots & \cdots & (\Lambda_{n-1})_{2\times 2} \\ O_{2\times 2} & (\Lambda_1)_{2\times 2} & (\Lambda_2)_{2\times 2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & (\Lambda_3)_{2\times 2} \\ \vdots & & & \ddots & (\Lambda_1)_{2\times 2} & (\Lambda_2)_{2\times 2} \\ O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & (\Lambda_1)_{2\times 2} \end{pmatrix}_{(n-1)\times(n-1)}, \tag{3.16}$$

where

$$(\Lambda_j)_{2\times 2} = (-F_1^{-1}F_2)^{j-1}F_1^{-1} = \frac{1}{2f}\begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix}^{j-1}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

It could be verified that

$$\begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}^j = \begin{pmatrix} -2U_{j-1} & U_j \\ -2U_j & U_{j+1} \end{pmatrix},$$

where $U_j = -U_{j-1} - 2U_{j-2}$, $U_0 = 0$, $U_1 = 1$.

Hence,

$$(\Lambda_j)_{2\times 2} = \frac{1}{2^j f}\begin{pmatrix} -2U_{j-2} & U_{j-1} \\ -2U_{j-1} & U_j \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2^j f}\begin{pmatrix} U_{j+1} & U_j \\ U_{j+2} & U_{j+1} \end{pmatrix}.$$

Next, we calculate

$$-D_1^{-1}D_2D_4^{-1} = \begin{pmatrix} m_1 & m_2 & m_3 & \cdots & \cdots & \cdots & m_{2n-3} & m_{2n-2} \\ k_1 & k_2 & k_3 & \cdots & \cdots & \cdots & k_{2n-3} & k_{2n-2} \end{pmatrix}, \tag{3.17}$$

where

$$m_j = \begin{cases} \dfrac{\sum_{k=1}^{\frac{j+1}{2}}[\frac{U_{\frac{j+5}{2}-k}}{2^{\frac{j+3}{2}-k}}(cs_{2n-2k} - at_{2n-2k}) + \frac{U_{\frac{j+7}{2}-k}}{2^{\frac{j+3}{2}-k}}(cs_{2n-1-2k} - at_{2n-1-2k})]}{f(ad-bc)} & \text{if } j \text{ is odd,} \\[2ex] \dfrac{\sum_{k=1}^{\frac{j}{2}}[\frac{U_{\frac{j+2}{2}-k}}{2^{\frac{j+2}{2}-k}}(cs_{2n-2k} - at_{2n-2k}) + \frac{U_{\frac{j+4}{2}-k}}{2^{\frac{j+2}{2}-k}}(cs_{2n-1-2k} - at_{2n-1-2k})]}{f(ad-bc)} & \text{if } j \text{ is even,} \end{cases}$$

$$k_j = \begin{cases} \dfrac{\sum_{k=1}^{\frac{j+1}{2}}[\frac{U_{\frac{j+5}{2}-k}}{2^{\frac{j+3}{2}-k}}(bt_{2n-2k} - ds_{2n-2k}) + \frac{U_{\frac{j+7}{2}-k}}{2^{\frac{j+3}{2}-k}}(bt_{2n-1-2k} - ds_{2n-1-2k})]}{f(ad-bc)} & \text{if } j \text{ is odd,} \\[2ex] \dfrac{\sum_{k=1}^{\frac{j}{2}}[\frac{U_{\frac{j+2}{2}-k}}{2^{\frac{j+2}{2}-k}}(bt_{2n-2k} - ds_{2n-2k}) + \frac{U_{\frac{j+4}{2}-k}}{2^{\frac{j+2}{2}-k}}(bt_{2n-1-2k} - ds_{2n-1-2k})]}{f(ad-bc)} & \text{if } j \text{ is even,} \end{cases}$$

and $1 \le j \le 2n-2$.

Combining (3.15), (3.16) and (3.17), we obtain

$$(\mathbb{P}\mathcal{M}\mathbb{P}^{-1})^{-1} = \left(\begin{array}{cc|cccccc}
\frac{a}{ad-bc} & \frac{-c}{ad-bc} & m_1 & m_2 & m_3 & \cdots & m_{2n-3} & m_{2n-2} \\
\frac{-b}{ad-bc} & \frac{d}{ad-bc} & k_1 & k_2 & k_3 & \cdots & k_{2n-3} & k_{2n-2} \\ \hline
0 & 0 & (\Lambda_1)_{2\times2} & (\Lambda_2)_{2\times2} & (\Lambda_3)_{2\times2} & \cdots & \cdots & (\Lambda_{n-1})_{2\times2} \\
0 & 0 & O_{2\times2} & (\Lambda_1)_{2\times2} & (\Lambda_2)_{2\times2} & \ddots & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & (\Lambda_3)_{2\times2} \\
\vdots & \vdots & \vdots & & & \ddots & (\Lambda_1)_{2\times2} & (\Lambda_2)_{2\times2} \\
0 & 0 & O_{2\times2} & \cdots & \cdots & \cdots & O_{2\times2} & (\Lambda_1)_{2\times2}
\end{array}\right)_{n\times n}. \tag{3.18}$$

The desired result follows from the simple observation $\mathcal{M} = \mathbb{P}^{-1}(\mathbb{P}\mathcal{M}\mathbb{P}^{-1})^{-1}\mathbb{P}$ and (3.18). □

Remark Theorem 3.2 tells us that the inverse of $\mathcal{M}$ is independent of s_j, t_j $(1 \le j \le 2n-2)$. Unfortunately, we have been unable to give a formula for the inverse of $\mathcal{M}$ when the matrix size is odd.

Corollary Let

$$\mathbb{M} = \begin{pmatrix}
a & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b \\
0 & -f & -f & -f & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & f & -f & \ddots & \ddots & \ddots & & & & \vdots \\
0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & \ddots & f & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & f & -f & -f & -f & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & -f & -f & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & f & -f & 0 \\
c & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & d
\end{pmatrix}.$$

Then

$$\mathbb{M}^{-1} = \left(\begin{array}{c|cccccc|c}
\frac{d}{ad-bc} & 0 & 0 & 0 & \cdots & \cdots & 0 & \frac{-b}{ad-bc} \\ \hline
0 & (\Lambda_1)_{2\times2} & (\Lambda_2)_{2\times2} & (\Lambda_3)_{2\times2} & \cdots & \cdots & (\Lambda_{n-1})_{2\times2} & 0 \\
\vdots & O_{2\times2} & (\Lambda_1)_{2\times2} & (\Lambda_2)_{2\times2} & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & (\Lambda_3)_{2\times2} & \vdots \\
\vdots & \vdots & & & \ddots & (\Lambda_1)_{2\times2} & (\Lambda_2)_{2\times2} & \vdots \\
0 & O_{2\times2} & \cdots & \cdots & \cdots & O_{2\times2} & (\Lambda_1)_{2\times2} & 0 \\ \hline
\frac{-c}{ad-bc} & 0 & 0 & 0 & \cdots & \cdots & 0 & \frac{a}{ad-bc}
\end{array}\right)_{2n\times2n}, \tag{3.19}$$

where

$$(\Lambda_j)_{2\times2} = \frac{1}{2^j f}\begin{pmatrix} U_{j+1} & U_j \\ U_{j+2} & U_{j+1} \end{pmatrix}, \quad 1 \le j \le n-1, \tag{3.20}$$

$U_j = -U_{j-1} - 2U_{j-2}$, and $U_0 = 0$, $U_1 = 1$.

Theorem 3.3. *Let $\mathcal{M}$ be an $n \times n$ perturbed Toeplitz four-band matrix of type I given in (1.1).*

(1) Assume that $(a-d)^2+4bc \geq 0$. *If n is odd, then the eigenvalues* λ_k $(1 \leq k \leq n)$ *of* $\mathcal{M}$ *are*

$$\lambda_k = \begin{cases} f(-1+i) & 1 \leq k \leq \frac{n-3}{2}, \\ f(-1-i) & \frac{n-3}{2}+1 \leq k \leq n-3, \\ \frac{(a+d)+\sqrt{4bc+(a-d)^2}}{2} & k = n-2, \\ \frac{(a+d)-\sqrt{4bc+(a-d)^2}}{2} & k = n-1, \\ -f & k = n; \end{cases} \tag{3.21}$$

if n is even, then the eigenvalues λ_k $(1 \leq k \leq n)$ *of* $\mathcal{M}$ *are*

$$\lambda_k = \begin{cases} f(-1+i) & 1 \leq k \leq \frac{n-2}{2}, \\ f(-1-i) & \frac{n-2}{2}+1 \leq k \leq n-2, \\ \frac{(a+d)+\sqrt{4bc+(a-d)^2}}{2} & k = n-1, \\ \frac{(a+d)-\sqrt{4bc+(a-d)^2}}{2} & k = n, \end{cases} \tag{3.22}$$

where $i=\sqrt{-1}$.

(2) Assume that $(a-d)^2+4bc < 0$. *If n is odd, then the eigenvalues* λ_k $(1 \leq k \leq n)$ *of* $\mathcal{M}$ *are*

$$\lambda_k = \begin{cases} f(-1+i) & 1 \leq k \leq \frac{n-3}{2}, \\ f(-1-i) & \frac{n-3}{2}+1 \leq k \leq n-3, \\ \frac{(a+d)+i\sqrt{-(a-d)^2-4bc}}{2} & k = n-2, \\ \frac{(a+d)-i\sqrt{-(a-d)^2-4bc}}{2} & k = n-1, \\ -f & k = n; \end{cases} \tag{3.23}$$

if n is even, then the eigenvalues λ_k $(1 \leq k \leq n)$ *of* $\mathcal{M}$ *are*

$$\lambda_k = \begin{cases} f(-1+i) & 1 \leq k \leq \frac{n-2}{2}, \\ f(-1-i) & \frac{n-2}{2}+1 \leq k \leq n-2, \\ \frac{(a+d)+i\sqrt{-(a-d)^2-4bc}}{2} & k = n-1, \\ \frac{(a+d)-i\sqrt{-(a-d)^2-4bc}}{2} & k = n, \end{cases} \tag{3.24}$$

where $i=\sqrt{-1}$.

Proof. The key observation is the similarity transformation (3.4). Since $\mathfrak{P}\mathcal{M}\mathfrak{P}^{-1}$ is a block upper triangular matrix, the eigenvalues of $\mathfrak{P}\mathcal{M}\mathfrak{P}^{-1}$ are the union of the eigenvalues of the main diagonal blocks $A_1, A_2, \cdots, A_{[\frac{n+1}{2}]}$.

If n is odd, then the main diagonal block matrices are $A_1 = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$, $A_2 = A_3 = \cdots = A_{\frac{n-1}{2}} = \begin{pmatrix} -f & -f \\ f & -f \end{pmatrix}$, $A_{[\frac{n+1}{2}]} = \begin{pmatrix} -f \end{pmatrix}$.

If n is even, then the main diagonal block matrices are $A_1 = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$, $A_2 = A_3 = \cdots = A_{\frac{n}{2}} = \begin{pmatrix} -f & -f \\ f & -f \end{pmatrix}$.

We leave the simple calculation to the interested reader. □

4 Determinant and inverse of the perturbed Hanakel four-band matrix of type I

In this section, we present the determinant and inverse of the matrix $\mathcal{N}$ given in (1.2).

Theorem 4.1. *Let $\mathcal{N}$ be an $n \times n$ (with $n \geq 3$) perturbed Hankel four-band matrix of type I given in (1.2). Then*

$$\det \mathcal{N} = \begin{cases} (-1)^{\frac{n^2+n-4}{2}} \cdot 2^{\frac{n-3}{2}} \cdot f^{n-2}(ad-bc) & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n(n-1)}{2}} \cdot 2^{\frac{n-2}{2}} \cdot f^{n-2}(ad-bc) & \text{if } n \text{ is even.} \end{cases} \tag{4.1}$$

Proof. It follows from (1.2) that $\det \mathcal{N} = \det \hat{\mathbf{I}}_n \det \mathcal{M}$. Then we obtain (4.1) by using Theorem 3.1 and $\det \hat{\mathbf{I}}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Theorem 4.2. *Let N be a $2n \times 2n$ (with $n \geq 2$) perturbed Hankel four-band matrix of type I given in (1.2). Then*

$$N^{-1} = \left(\begin{array}{c:ccccccc:c} \frac{-b}{ad-bc} & k_{2n-2} & k_{2n-3} & k_{2n-4} & \cdots & \cdots & k_1 & \frac{d}{ad-bc} \\ \hdashline 0 & (Y_{n-1})_{2\times 2} & \cdots & \cdots & (Y_3)_{2\times 2} & (Y_2)_{2\times 2} & (Y_1)_{2\times 2} & 0 \\ \vdots & \vdots & & \iddots & \iddots & \iddots & O_{2\times 2} & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots \\ \vdots & (Y_3)_{2\times 2} & \iddots & \iddots & \iddots & & \vdots & \vdots \\ \vdots & (Y_2)_{2\times 2} & \iddots & \iddots & & & \vdots & \vdots \\ 0 & (Y_1)_{2\times 2} & O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & 0 \\ \hdashline \frac{a}{ad-bc} & m_{2n-2} & m_{2n-3} & m_{2n-4} & \cdots & \cdots & m_1 & \frac{-c}{ad-bc} \end{array}\right)_{2n\times 2n},$$

where

$$(Y_j)_{2\times 2} = (\Lambda_j)_{2\times 2}\hat{\mathbf{I}}_2, \; 1 \leq j \leq n-1,$$

$(\Lambda_j)_{2\times 2}$ is the same as (3.10), m_j $(1 \leq j \leq 2n-2)$ is the same as (3.11) and k_j $(1 \leq j \leq 2n-2)$ is the same as (3.12).

Proof. We obtain this conclusion by using $N^{-1} = M^{-1}\hat{\mathbf{I}}_{2n}^{-1} = M^{-1}\hat{\mathbf{I}}_{2n}$ and Theorem 3.2, where

$$\hat{\mathbf{I}}_{2n} = \begin{pmatrix} & & & 1 \\ & & \hat{\mathbf{I}}_2 & \\ & \iddots & & \\ \hat{\mathbf{I}}_2 & & & \\ 1 & & & \end{pmatrix}_{2n\times 2n}. \tag{4.2}$$

□

5 Determinant and inverse of the perturbed Hankel four-band matrix of type II

In this section, we present the determinant and inverse of the matrix $\mathcal{P}$ given in (1.3).

Theorem 5.1. *Let $\mathcal{P}$ be an $n \times n$ (with $n \geq 3$) perturbed Hankel four-band matrix of type II given in (1.3). Then*

$$\det \mathcal{P} = \begin{cases} (-1)^{\frac{n^2+n-4}{2}} \times 2^{\frac{n-3}{2}} f^{n-2}(ad-bc) & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n(n-1)}{2}} \times 2^{\frac{n-2}{2}} f^{n-2}(ad-bc) & \text{if } n \text{ is even.} \end{cases} \tag{5.1}$$

Proof. By (1.3) we have $\det \mathcal{P} = \det \mathcal{M} \det \hat{\mathbf{I}}_n$. Then the desired result follows from Theorem 3.1 and $\det \hat{\mathbf{I}}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Theorem 5.2. *Let P be a $2n \times 2n$ (with $n \geq 2$) perturbed Hankel four-band matrix of type II given in (1.3). Then*

$$P^{-1} = \left(\begin{array}{c|cccccc|c} \frac{-c}{ad-bc} & m_1 & m_2 & m_3 & \cdots & \cdots & m_{2n-2} & \frac{a}{ad-bc} \\ \hline 0 & O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & (\Gamma_1)_{2\times 2} & 0 \\ \vdots & \vdots & & & \iddots & \iddots & (\Gamma_2)_{2\times 2} & \vdots \\ \vdots & \vdots & & \iddots & \iddots & \iddots & (\Gamma_3)_{2\times 2} & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots \\ \vdots & O_{2\times 2} & \iddots & \iddots & \iddots & & \vdots & \vdots \\ 0 & (\Gamma_1)_{2\times 2} & (\Gamma_2)_{2\times 2} & (\Gamma_3)_{2\times 2} & \cdots & \cdots & (\Gamma_{n-1})_{2\times 2} & 0 \\ \hline \frac{d}{ad-bc} & k_1 & k_2 & k_3 & \cdots & \cdots & k_{2n-2} & \frac{-b}{ad-bc} \end{array}\right)_{2n\times 2n},$$

where

$$(\Gamma_j)_{2\times 2} = \hat{\mathbf{I}}_2 (\Lambda_j)_{2\times 2}, \ 1 \leq j \leq n-1,$$

$(\Lambda_j)_{2\times 2}$ is the same as (3.10), m_j $(1 \leq j \leq 2n-2)$ is the same as (3.11) and k_j $(1 \leq j \leq 2n-2)$ is the same as (3.12).

Proof. We obtain this conclusion by using $P^{-1} = \hat{\mathbf{I}}_{2n}^{-1} M^{-1} = \hat{\mathbf{I}}_{2n} M^{-1}$ and Theorem 3.2, where $\hat{\mathbf{I}}_{2n}$ is the same as (4.2). □

6 Determinant and inverse of the perturbed Toeplitz four-band matrix of type II

In this section, we present the determinant and inverse of the matrix $\mathcal{Q}$ given in (1.4).

Theorem 6.1. *Let $\mathcal{Q}$ be an $n \times n$ (with $n \geq 3$) perturbed Toeplitz four-band matrix of type II given in (1.4). Then*

$$\det \mathcal{Q} = \begin{cases} 2^{\frac{n-3}{2}} (-f^{n-2})(ad-bc) & \text{if } n \text{ is odd}, \\ 2^{\frac{n-2}{2}} f^{n-2} (ad-bc) & \text{if } n \text{ is even}. \end{cases} \tag{6.1}$$

Proof. By (1.4) we have $\det \mathcal{Q} = \det \hat{\mathbf{I}}_n \det \mathcal{M} \det \hat{\mathbf{I}}_n$. Then the desired result follows from Theorem 3.1 and $\det \hat{\mathbf{I}}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Theorem 6.2. *Let Q be a $2n \times 2n$ (with $n \geq 2$) perturbed Toeplitz four-band matrix of type II given in (1.4). Then*

$$Q^{-1} = \left(\begin{array}{c|cccccc|c} \frac{a}{ad-bc} & m_{2n-2} & m_{2n-3} & m_{2n-4} & \cdots & \cdots & m_1 & \frac{-c}{ad-bc} \\ \hline 0 & (\Re_1)_{2\times 2} & O_{2\times 2} & \cdots & \cdots & \cdots & O_{2\times 2} & 0 \\ \vdots & (\Re_2)_{2\times 2} & (\Re_1)_{2\times 2} & \ddots & & & \vdots & \vdots \\ \vdots & (\Re_3)_{2\times 2} & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & (\Re_1)_{2\times 2} & O_{2\times 2} & \vdots \\ 0 & (\Re_{n-1})_{2\times 2} & \cdots & \cdots & (\Re_3)_{2\times 2} & (\Re_2)_{2\times 2} & (\Re_1)_{2\times 2} & 0 \\ \hline \frac{-b}{ad-bc} & k_{2n-2} & k_{2n-3} & k_{2n-4} & \cdots & \cdots & k_1 & \frac{d}{ad-bc} \end{array}\right)_{2n\times 2n},$$

where

$$(\Re_j)_{2\times 2} = \hat{\mathbf{I}}_2 (\Lambda_j)_{2\times 2} \hat{\mathbf{I}}_2, \ 1 \le j \le n-1,$$

$(\Lambda_j)_{2\times 2}$ is the same as (3.10), m_j $(1 \le j \le 2n-2)$ is the same as (3.11) and k_j $(1 \le j \le 2n-2)$ is the same as (3.12).

Proof. We obtain this conclusion by using $Q^{-1} = \hat{\mathbf{I}}_{2n}^{-1} M^{-1} \hat{\mathbf{I}}_{2n}^{-1} = \hat{\mathbf{I}}_{2n} M^{-1} \hat{\mathbf{I}}_{2n}$ and Theorem 3.2, where $\hat{\mathbf{I}}_{2n}$ is the same as (4.2). □

7 Algorithm and Numerical example

In this section, algorithm for finding $\det \mathcal{M}$, $\mathcal{M}^{-1}$ and eig$\mathcal{M}$ are given, respectively. An example demonstrates the method which introduced above for the calculation of determinant, inverse and eigenvalues of the matrix $\mathcal{M}$.

Algorithm
Step 1: Compute $\det \mathcal{M}$ by using the formula (3.1);
Step 2: Compute $(\Lambda_j)_{2\times 2}$ $(1 \le j \le n-1)$ via the formula (3.10), respectively;
Step 3: Compute m_j and k_j $(1 \le j \le 2n-2)$ by formulas (3.11) and (3.12);
Step 4: By the formula (3.9), we obtain $\mathcal{M}^{-1}$;
Step 5: By using the formulas (3.21), (3.22), (3.23) and (3.24), we obtain eig$\mathcal{M}$.

Remark Algorithms for finding $\det \mathcal{N}$, N^{-1}, $\det \mathcal{P}$, P^{-1}, $\det \mathcal{Q}$ and Q^{-1} are similar.

Example
We consider a 6×6 matrix $\mathcal{M}$:

$$\mathcal{M} = \begin{pmatrix} 2 & 1 & 2 & 3 & 4 & 3 \\ 0 & -5 & -5 & -5 & 0 & 0 \\ 0 & 5 & -5 & -5 & -5 & 0 \\ 0 & 0 & 0 & -5 & -5 & 0 \\ 0 & 0 & 0 & 5 & -5 & 0 \\ 3 & 4 & 3 & 2 & 1 & 2 \end{pmatrix}_{6\times 6},$$

Step 1: From (3.1), we get $\det \mathcal{M} = -12500$.
Step 2: Computing $(\Lambda_j)_{2\times 2}$ via the formula (3.10), we get

$$(\Lambda_1)_{2\times 2} = \frac{1}{10}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \ (\Lambda_2)_{2\times 2} = \frac{1}{10}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix};$$

Step 3: By formulas (3.11) and (3.12), we obtain m_j and k_j as follows:

$$k_1 = \tfrac{3}{10}, \quad k_2 = -\tfrac{1}{10}, \quad k_3 = -\tfrac{3}{20}, \quad k_4 = \tfrac{1}{20};$$

$$m_1 = -\tfrac{1}{10}, \quad m_2 = \tfrac{1}{10}, \quad m_3 = \tfrac{1}{4}, \quad m_4 = \tfrac{1}{20}.$$

Step 4: By the formula (3.9), we obtain $\mathcal{M}^{-1}$

$$\mathcal{M}^{-1} = \begin{pmatrix} -\frac{2}{5} & \frac{3}{10} & -\frac{1}{10} & -\frac{3}{20} & \frac{1}{20} & \frac{3}{5} \\ 0 & -\frac{1}{10} & \frac{1}{10} & -\frac{1}{20} & -\frac{1}{20} & 0 \\ 0 & -\frac{1}{10} & -\frac{1}{10} & \frac{3}{20} & -\frac{1}{20} & 0 \\ 0 & 0 & 0 & -\frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & 0 \\ \frac{3}{5} & -\frac{1}{10} & \frac{1}{10} & \frac{1}{4} & \frac{1}{20} & -\frac{2}{5} \end{pmatrix}.$$

Step 5: As for eig$\mathcal{M}$, from (3.22), we obtain $\lambda_1 = \lambda_2 = -5 + 5i$, $\lambda_3 = \lambda_4 = -5 - 5i$, $\lambda_5 = 5$, $\lambda_6 = -1$.

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