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Computational methods for difference families in finite abelian groups

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Abstract: Our main objective is to show that the computational methods, developed previously to search for difference families in cyclic groups, can be fully extended to the more general case of arbitrary finite abelian groups. In particular the power spectral density test and the method of compression can be used to help the search.

Keywords: Difference families, discrete Fourier transform, periodic autocorrelation, power spectral density, Goethals–Seidel array

MSC: 05B10, 05B20

1 Introduction

Many important tools for the construction of various types of Hadamard matrices, D-optimal designs, periodic Golay pairs, etc., via cyclic difference families, such as the power spectral density (PSD) test and the method of compression, have been used extensively by many researchers, see e.g. [7]. Our main objective is to show that these tools can be fully extended to the more general case of arbitrary finite abelian groups. In particular the PSD test and the method of compression can be used to help the search.

In the first four sections we recall some basic definitions and facts about difference families in finite abelian groups, the characters of finite abelian groups, and the finite Fourier transformation. Our main results are Theorems 1-3 in section 6. They describe the close connection between the periodic autocorrelation and power spectral density functions in the context of the so-called complementary functions. In section 7 we give two methods for the implementation of the PSD-test. Section 8 describes the compression method and section 9 explains how to use the difference families to construct the desired special matrices. Section 10 contains a variety of toy-examples whose main purpose is to illustrate how our results can be applied in practise. Finally in section 11 we point out that in some applications it may be useful to alter slightly the definition of difference families by considering them as ordered tuples and allowing blocks to be empty or improper.

2 Difference families

Let G be a finite abelian group of order v . We write it multiplicatively and denote its identity element by e .

Definition 1. We say that an ordered t -tuple (X_1, X_2, \dots, X_t) , $t \geq 1$, of proper nonempty subsets of G is a difference family if the sets $S_a = \{(x, ax, i) : x, ax \in X_i, i = 1, 2, \dots, t\}$ with $a \in G \setminus \{e\}$ have the same

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cardinality. In that case we denote this cardinality by λ and we refer to

$$(v; k_1, k_2, \dots, k_t; \lambda), \quad (1)$$

where $k_i = |X_i|$, as the parameter set of this difference family.

A simple counting argument shows that these parameters must satisfy the equation

$$\sum_{i=1}^t k_i(k_i - 1) = \lambda(v - 1). \quad (2)$$

We shall also need an additional parameter, n , defined by the equation

$$n = \sum_{i=1}^t k_i - \lambda. \quad (3)$$

Let us assume that (X_1, X_2, \dots, X_t) is a difference family in G with the parameters displayed above. Then we also say that the sets X_i are its *base blocks*. There are two simple types of transformations which we can perform on the difference families in G with fixed index set $\{1, 2, \dots, t\}$ and fixed value of the parameter n .

For the first type let π be a permutation of the index set $\{1, 2, \dots, t\}$. Then the t -tuple $(X_{\pi 1}, X_{\pi 2}, \dots, X_{\pi t})$ is also a difference family. Its parameter set is obtained from (1) by substituting πk_i for k_i for each index i . The parameter λ does not change.

For the second type we select an index, say j , and replace X_j with its complement $G \setminus X_j$. We obtain again a difference family. Its parameter set is obtained from (1) by substituting $v - k_j$ for k_j and $\lambda + v - 2k_j$ for λ .

By performing a finitely many transformations of the two types described above, we can replace the original difference family with one whose parameter set satisfies the following additional conditions

$$v/2 \geq k_1 \geq k_2 \geq \dots \geq k_t \geq 1. \quad (4)$$

Note that this implies that $v \geq 2$.

3 Difference families and the group algebra

Let \mathcal{R} be the group algebra of G over the complex numbers, \mathbf{C} . The elements of \mathcal{R} are formal linear combinations $\sum_{x \in G} c_x x$, with complex coefficients $c_x \in \mathbf{C}$. Thus G is a vector space basis of \mathcal{R} , and \mathcal{R} has dimension v . The linear map $\varepsilon : \mathcal{R} \rightarrow \mathbf{C}$ such that $\varepsilon(x) = 1$ for all $x \in G$ is an algebra homomorphism known as the *augmentation*. Note that

$$\varepsilon\left(\sum_{x \in G} c_x x\right) = \sum_{x \in G} c_x.$$

The algebra \mathcal{R} has an involution, “*”, which acts on the scalars as the complex conjugation and acts as the inversion map on group elements. Thus

$$\left(\sum_{x \in G} c_x x\right)^* = \sum_{x \in G} \bar{c}_x x^{-1}, \quad c_x \in \mathbf{C}.$$

For any subset $X \subseteq G$, by abuse of notation, we also denote by X the sum of all elements of X in \mathcal{R} . It will be clear from the context which of these two meanings is used. For instance, we have

$$X = \sum_{x \in X} x.$$

The X on the lefthand side is an element of \mathcal{R} , while on the righthand side it is a subset of G .

In section 10 we shall use the symmetric and skew subsets of G . We define them as follows. A subset $X \subseteq G$ is *symmetric* if $X^* = X$. A subset $Y \subseteq G$ is *skew* if G is a disjoint union of Y , Y^* and $\{e\}$.

For any element $X \in \mathcal{R}$ we define its *norm*, $N(X)$, by $N(X) = XX^*$. The proof of the following lemma is straightforward and we omit it.

Lemma 1. Let (X_1, X_2, \dots, X_t) be a t -tuple of proper nonempty subsets of G with cardinalities $k_i = |X_i|$, and let λ be a nonnegative integer. Then (X_1, X_2, \dots, X_t) is a difference family in G with the parameter set $(v; k_1, k_2, \dots, k_t; \lambda)$ if and only if

$$\sum_{i=1}^t N(X_i) = n \cdot e + \lambda G, \quad (5)$$

where n is defined by (3).

To any function $f : G \rightarrow \mathbf{C}$ we assign an element, $a_f \in \mathcal{R}$, by setting

$$a_f = \sum_{x \in G} f(x)x.$$

The conjugate \bar{f} of f is defined by $\bar{f}(x) = \overline{f(x)}$, $x \in G$.

The periodic autocorrelation function PAF_f of f is defined by

$$\text{PAF}_f(x) = \sum_{y \in G} f(xy)\bar{f}(y), \quad x \in G. \quad (6)$$

We claim that

$$N(a_f) = \sum_{x \in G} \text{PAF}_f(x)x. \quad (7)$$

Indeed, this follows from the identities

$$\begin{aligned} N(a_f) &= N\left(\sum_{x \in G} f(x)x\right) \\ &= \sum_{x, y \in G} f(x)\bar{f}(y)xy^{-1} \\ &= \sum_{z \in G} \left(\sum_{y \in G} f(yz)\bar{f}(y)\right)z \\ &= \sum_{z \in G} \text{PAF}_f(z)z. \end{aligned}$$

4 Characters of finite abelian groups

In this section we recall some well known facts. For more details see for instance [10, 12].

Let $L^2(G)$ denote the finite-dimensional complex Hilbert space of all complex-valued functions on G with the inner product

$$\langle f, g \rangle = \sum_{x \in G} f(x)\bar{g}(x). \quad (8)$$

For $a \in G$ let $\delta_a : G \rightarrow \mathbf{C}$ be the function defined by $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. These delta functions form an orthonormal (o.n.) basis of $L^2(G)$.

A character χ of G is a group homomorphism $G \rightarrow \mathbf{T}$, where $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ is the circle group, i.e., the group of complex numbers of modulus 1 (with multiplication as the group operation). If χ and ψ are two characters of G , then their product $\chi\psi$ defined, as usual, by $\chi\psi(x) = \chi(x)\psi(x)$, $x \in G$, is also a character of G . It follows easily that the characters of G form an abelian group, called the *dual group* of G , which we denote by \hat{G} . Its identity element is the *trivial character*, θ , which maps all elements $x \in G$ to $1 \in \mathbf{T}$. The group \hat{G} is finite, and it has the same order as G . Moreover, these two groups are isomorphic, $\hat{\hat{G}} \cong G$.

If $\alpha : H \rightarrow G$ is a homomorphism of finite abelian groups and χ is a character of G then the composite $\chi \circ \alpha : H \rightarrow \mathbf{T}$ is a character of H . In particular, the map $\alpha : G \rightarrow G$ defined by $\alpha(x) = x^{-1}$, $x \in G$, is an

automorphism of G and we have $\chi \circ \alpha(x) = \chi(x^{-1}) = \chi(x)^{-1} = \bar{\chi}(x)$. Thus if χ is a character of G then its complex conjugate $\bar{\chi}$ is also a character of G .

We claim that if χ is a nontrivial character of G then

$$\sum_{x \in G} \chi(x) = 0, \quad \chi \neq \theta. \quad (9)$$

Indeed, there exists $a \in G$ such that $\chi(a) \neq 1$. Then $\chi(a) \sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(ax) = \sum_{x \in G} \chi(x)$ and our claim follows.

Next we claim that if χ and ψ are different characters of G then they are orthogonal, i.e.,

$$\langle \chi, \psi \rangle = 0, \quad \chi \neq \psi. \quad (10)$$

Indeed, we have $\langle \chi, \psi \rangle = \sum_x \chi(x)\bar{\psi}(x) = \sum_x \chi\bar{\psi}(x) = 0$ since the character $\chi\bar{\psi}$ is nontrivial.

For $\chi \in \hat{G}$ we have $\langle \chi, \chi \rangle = \sum_{x \in G} \chi(x)\bar{\chi}(x) = \sum_{x \in G} 1 = |G| = \nu$. Thus the functions $\chi/\sqrt{\nu}$, with $\chi \in \hat{G}$, form another o.n. basis of $L^2(G)$.

5 Discrete Fourier transform, DFT

The *discrete Fourier transform* of a function $f \in L^2(G)$ is the function DFT_f , also written as $\text{DFT}(f)$, in $L^2(\hat{G})$ defined by the formula

$$\text{DFT}_f(\chi) = \sum_{x \in G} f(x)\bar{\chi}(x) = \langle f, \chi \rangle, \quad \chi \in \hat{G}. \quad (11)$$

Thus we have a linear map $\text{DFT} : L^2(G) \rightarrow L^2(\hat{G})$ to which we refer as the *discrete Fourier transform* on G . For the proofs of the following four basic properties of this transform we again refer to [12].

(a) DFT is a vector space isomorphism.

(b) $\text{DFT}_{f * g} = \text{DFT}_f \cdot \text{DFT}_g$ where $f * g$ is the *convolution* of f and g defined by the formula

$$f * g(x) = \sum_{y \in G} f(y)g(xy^{-1}).$$

(c) For $f \in L^2(G)$,

$$f = \frac{1}{\nu} \sum_{\chi \in \hat{G}} \text{DFT}_f(\chi)\chi.$$

(d) Define the inner product in $L^2(\hat{G})$ by $\langle H, K \rangle = \sum_{\chi \in \hat{G}} H(\chi)\bar{K}(\chi)$. Then

$$\|\text{DFT}_f\|^2 = \nu \|f\|^2, \quad f \in L^2(G).$$

The property (c) is known as the *inversion formula* as it shows how to recover the function f from its Fourier transform DFT_f . It follows from (c) that the *inverse Fourier transform*, DFT^{-1} , is given by the formula

$$\text{DFT}^{-1}(\phi) = \frac{1}{\nu} \sum_{\chi \in \hat{G}} \phi(\chi)\chi, \quad \phi \in L^2(\hat{G}). \quad (12)$$

The property (d) shows that the linear map $(1/\sqrt{\nu})\text{DFT} : L^2(G) \rightarrow L^2(\hat{G})$ is an isometry.

As an example let us compute the DFT of the cyclic group $G = \langle g \rangle$ of order ν with a generator g . For each $j \in \{0, 1, \dots, \nu - 1\}$ there is a unique character χ_j of G such that $\chi_j(g) = \omega^j$, where $\omega = e^{2\pi i/\nu}$. Hence we have

$\hat{G} = \{\chi_j : j = 0, 1, \dots, v-1\}$. Note that the trivial character is $\theta = \chi_0$. Let $f \in L^2(G)$ be arbitrary. Then

$$\begin{aligned} \text{DFT}_f(\chi_j) &= \langle f, \chi_j \rangle \\ &= \sum_{k=0}^{v-1} f(g^k) \bar{\chi}_j(g^k) \\ &= \sum_{k=0}^{v-1} f(g^k) \omega^{jk} \\ &= \sum_{k=0}^{v-1} \omega^{-jk} f(g^k). \end{aligned}$$

The *power spectral density* of a function $f \in L^2(G)$ is the function $\text{PSD}_f \in L^2(\hat{G})$ defined by

$$\text{PSD}_f(\chi) = |\text{DFT}_f(\chi)|^2. \quad (13)$$

In various places, in the case when G is cyclic, the following simple lemma is referred to as the “Wiener-Khinchin theorem”, see [7, Theorem 1] and its references, and also [4].

Lemma 2. $\text{PSD}_f = \text{DFT}(\text{PAF}_f)$, $f \in L^2(G)$.

Proof. For $\chi \in \hat{G}$ we have

$$\begin{aligned} \text{PSD}_f(\chi) &= \left| \sum_{x \in G} f(x) \bar{\chi}(x) \right|^2 \\ &= \sum_{x, y \in G} f(x) \bar{f}(y) \bar{\chi}(x) \bar{\chi}(y^{-1}) \\ &= \sum_{x, y \in G} f(x) \bar{f}(y) \bar{\chi}(xy^{-1}) \\ &= \sum_{z \in G} \left(\sum_{y \in G} f(yz) \bar{f}(y) \right) \bar{\chi}(z) \\ &= \sum_{z \in G} \text{PAF}_f(z) \bar{\chi}(z) \\ &= \text{DFT}(\text{PAF}_f)(\chi). \end{aligned}$$

□

6 Complementary functions

The notion of complementary sequences plays an important role in combinatorics, see e.g. [8, 11]. By analogy, we define the complementary functions on G .

Definition 2. We say that the functions $f_1, f_2, \dots, f_t \in L^2(G)$ are complementary if

$$\sum_{i=1}^t \text{PAF}_{f_i} = (\alpha_0 - \alpha) \delta_e + \alpha \theta. \quad (14)$$

or, equivalently,

$$\sum_{i=1}^t \text{PAF}_{f_i}(s) = \begin{cases} \alpha_0, & \text{if } s = e; \\ \alpha, & \text{otherwise,} \end{cases} \quad (15)$$

for some constants α_0 and α (the PAF-constants).

Note that

$$\alpha_0 = \sum_{i=1}^t \sum_{x \in G} |f_i(x)|^2 = \sum_{i=1}^t \|f_i\|^2.$$

In the special case when the f_i take values in $\{\pm 1\}$, we have $\alpha_0 = tv$.

By analogy with G , we introduce the notation $\hat{\theta}$ for the trivial character of the group \hat{G} , and denote by δ_θ the function on \hat{G} which takes value 1 at θ and value 0 at all other characters χ of G . It is easy to verify that

$$\text{DFT}_{\delta_e} = \hat{\theta}, \quad \text{DFT}_\theta = v\delta_\theta. \quad (16)$$

Theorem 1. *The functions $f_1, f_2, \dots, f_t \in L^2(G)$ are complementary with PAF-constants α_0 and α if and only if*

$$\sum_{i=1}^t \text{PSD}_{f_i} = (\beta_0 - \beta)\delta_\theta + \beta\hat{\theta}. \quad (17)$$

or, equivalently, for $\chi \in \hat{G}$

$$\sum_{i=1}^t \text{PSD}_{f_i}(\chi) = \begin{cases} \beta_0, & \text{if } \chi = \theta; \\ \beta, & \text{otherwise,} \end{cases} \quad (18)$$

where the constants β_0 and β (the PSD-constants) are given by

$$\beta_0 = \alpha_0 + (v - 1)\alpha, \quad \beta = \alpha_0 - \alpha. \quad (19)$$

Proof. Since at the points θ and $\chi \neq \theta$ the function δ_θ takes values 1 and 0, respectively, while the function $\hat{\theta}$ takes value 1 at all points $\chi \in \hat{G}$, we deduce that the equations (17) and (18) are equivalent.

Assume that the functions f_1, f_2, \dots, f_t are complementary, i.e., that (14) holds. By applying DFT to (14) and by using Lemma 2 and the equations (16), we obtain that

$$\sum_{i=1}^t \text{PSD}_{f_i} = (\alpha_0 - \alpha)\hat{\theta} + \alpha v\delta_\theta.$$

Hence, the equation (17) holds where the constants β_0 and β are defined as in (19).

To prove the converse, we just have to apply the inverse Fourier transform to the equation (17). \square

For any subset $X \subseteq G$ we define a $\{\pm 1\}$ -valued function f_X on G as follows: $f_X(x)$ is equal to -1 if $x \in X$ and is equal to $+1$ otherwise. We also say that the function f_X is *associated with* X .

In the next theorem we show that each difference family with t base blocks gives t complementary functions having values in $\{\pm 1\}$, and we compute their PAF-constants. Their PSD-constants can be computed by using the formulas (19).

Theorem 2. *Let (X_1, X_2, \dots, X_t) be a difference family in G with parameter set (1) and let $f_i = f_{X_i}$, $i = 1, 2, \dots, t$, be their associated functions. Then*

(a)

$$\sum_i N(a_{f_i}) = 4n \cdot e + (tv - 4n)G;$$

(b) the $f_i : G \rightarrow \{\pm 1\}$ are non-constant complementary functions with PAF-constants

$$\alpha_0 = tv, \quad \alpha = tv - 4n;$$

(c)

$$\sum_{i=1}^t (v - 2k_i)^2 = 4n + v(tv - 4n).$$

Proof. If we view the X_i and G as elements of \mathcal{R} , then we have $xG = G$ for $x \in G$, $X_iG = GX_i = k_iG$ for each i , and $GG = vG$. Since $a_{f_i} = G - 2X_i$, we have

$$N(a_{f_i}) = (G - 2X_i)(G - 2X_i^*) = (v - 4k_i)G + 4N(X_i). \quad (20)$$

The assertion (a) follows by adding up these equations and by applying Lemma 1.

The assertion (b) follows from (a) by using the formula (7) and by comparing the coefficients of $x \in G$ on both sides.

The assertion (c) follows from (a) by applying the augmentation map ε . \square

The converse of the assertion (b) is also valid.

Theorem 3. *Let $f_i : G \rightarrow \{\pm 1\}$, $i = 1, 2, \dots, t$, be non-constant complementary functions with PAF-constants α_0 and α . Then the $X_i = \{x \in G : f_i(x) = -1\}$, $i = 1, 2, \dots, t$, form a difference family of G with parameters $(v; k_1, k_2, \dots, k_t; \lambda)$ where $k_i = |X_i|$ and $\lambda = \sum k_i - (tv - \alpha)/4$.*

Proof. Since the functions f_i are non-constant, the X_i are proper non-empty subsets of G . By the hypothesis, the formula (14) holds true. Hence, for $x \in G$ we have

$$\sum \text{PAF}_{f_i}(x) = (\alpha_0 - \alpha)\delta_e(x) + \alpha.$$

By multiplying this equation by x , summing up over all $x \in G$, and by using (7), we obtain that

$$\sum N(a_{f_i}) = (\alpha_0 - \alpha)e + \alpha G.$$

Since $\alpha_0 = tv$, by using (20) we obtain that

$$\sum N(X_i) = \frac{tv - \alpha}{4}e + \left(\sum k_i - \frac{tv - \alpha}{4} \right) G.$$

It follows that $tv - \alpha$ is divisible by 4 and the assertion of the theorem follows from Lemma 1. \square

7 PSD-test

Suppose that we want to search for a difference family (X_1, X_2, \dots, X_t) in G having the parameter set (1). An exhaustive search can be carried out only when the order, v , of G is relatively small. For larger v one uses some randomized or heuristic procedure to generate candidates for the base blocks X_i . One can often improve such a procedure by using a test, known as the *PSD-test*, to discard some of the candidates for the set X_i . This test is based on Theorems 1 and 2. First, the functions f_i associated with X_i are complementary with PAF-constants $\alpha_0 = tv$ and $\alpha = tv - 4n$. Second, for any nontrivial character χ of G we must have

$$\sum_i \text{PSD}_{f_i}(\chi) = \beta = 4n, \quad \chi \neq \theta.$$

(Recall that $n = \sum k_i - \lambda$.)

Since $\text{PSD}_{f_i}(\chi) = |\text{DFT}_{f_i}(\chi)|^2 \geq 0$, we can discard the candidate X_i if for some $\chi \neq \theta$ we have $\text{PSD}_{f_i}(\chi) > 4n$. In that case we say that X_i (or f_i) fails the PSD-test. This test is most effective when $t = 2$.

There is another method for performing the PSD-test. For that we need to assign to $X \subseteq G$ the function $\Phi_X : G \rightarrow \mathbf{Z}$ defined by

$$N(X) = XX^* = \sum_{x \in G} \Phi_X(x)x.$$

As $N(X)^* = N(X)$, we have $\Phi_X(x^{-1}) = \Phi_X(x)$ for all $x \in G$.

Next we set $\Phi_i = \Phi_{X_i}$ for $i = 1, 2, \dots, t$. By (7) and (20) we have

$$\text{PAF}_{f_i} = (v - 4k_i)\theta + 4\Phi_i.$$

Since $\text{DFT}_\theta = \nu\delta_\theta$, by applying the DFT-transform to the above equation and by using Lemma 2, we obtain that

$$\text{PSD}_{f_i} = (\nu - 4k_i)\nu\delta_\theta + 4\text{DFT}_{\Phi_i}.$$

By evaluating both sides at a nontrivial character χ , we obtain that $\text{PSD}_{f_i}(\chi) = 4\text{DFT}_{\Phi_i}(\chi)$. Since $\alpha_0 = \nu$, by adding up these equations and by using Theorem 1 we obtain that

$$\sum_i \text{DFT}_{\Phi_i}(\chi) = n, \quad \chi \neq \theta.$$

Hence, X_i passes the PSD-test if and only if for all nontrivial characters χ of G we have

$$\text{DFT}_{\Phi_i}(\chi) \leq n, \quad \chi \neq \theta.$$

Since $\Phi_i(e) = k_i$, $\chi(e) = 1$ and $\text{DFT}_{\Phi_i}(\chi) = \langle \Phi_i, \chi \rangle$ this inequality can be written as follows

$$\sum_{x \in G \setminus \{e\}} \Phi_i(x) \Re \chi(x) \leq n - k_i, \quad \chi \neq \theta.$$

($\Re \chi$ denotes the real part of χ .)

8 Compression of complementary functions

Let M be a subgroup of order m of G and $H = G/M$, the corresponding quotient group of order $d = \nu/m$. We denote by σ the canonical map $G \rightarrow H$. Further, we denote by θ_H and $\hat{\theta}_H$ the trivial characters of H and its dual group \hat{H} , respectively. Finally $\hat{\sigma} : \hat{H} \rightarrow \hat{G}$ will be the dual map of σ defined by $\hat{\sigma}(\phi) = \phi\sigma$, $\phi \in \hat{H}$.

For any $f \in L^2(G)$ we define $f^M \in L^2(H)$ by

$$f^M(xM) = \sum_{z \in M} f(xz), \quad x \in G.$$

We say that f^M is the M -compression of f and that m is the *compression factor*. Note that $\tilde{f}^M = \overline{f^M}$. We choose a set Y of coset representatives of M in G . Then each $x \in G$ can be written uniquely as $x = yz$ with $y \in Y$ and $z \in M$.

Lemma 3. For $f \in L^2(G)$ we have $\text{PAF}_{f^M} = (\text{PAF}_f)^M$, i.e.,

$$\text{PAF}_{f^M}(xM) = \sum_{z \in M} \text{PAF}_f(xz), \quad x \in G. \quad (21)$$

Proof. We have

$$\begin{aligned} \text{PAF}_{f^M}(xM) &= \sum_{y \in Y} f^M(xyM) \tilde{f}^M(yM) \\ &= \sum_{y \in Y} \sum_{p, w \in M} f(xyp) \tilde{f}(yw). \end{aligned}$$

After setting $p = wz$, we obtain that

$$\begin{aligned} \text{PAF}_{f^M}(xM) &= \sum_{z \in M} \sum_{y \in Y} \sum_{w \in M} f(xzyw) \tilde{f}(yw) \\ &= \sum_{z \in M} \text{PAF}_f(xz) \\ &= (\text{PAF}_f)^M(xM). \end{aligned}$$

□

Theorem 4. Let $f_1, f_2, \dots, f_t \in L^2(G)$ be complementary functions with PAF constants α_0 and α . Then the functions $f_1^M, f_2^M, \dots, f_t^M \in L^2(H)$ are complementary with PAF constants

$$\alpha_0^M = \alpha_0 + (m-1)\alpha, \quad \alpha^M = m\alpha. \quad (22)$$

Proof. By Lemma 3 we have

$$\text{PAF}_{f_i}^M(xM) = \sum_{z \in M} \text{PAF}_{f_i}(xz).$$

By adding these equations and by using the equation (17), we obtain that

$$\begin{aligned} \sum_{i=1}^t \text{PAF}_{f_i}^M(xM) &= \sum_{z \in M} ((\alpha_0 - \alpha)\delta_e(xz) + \alpha\theta(xz)) \\ &= (\alpha_0 - \alpha)\delta_M(xM) + m\alpha. \end{aligned}$$

(The delta function $\delta_M \in L^2(H)$ takes value 1 at the point $M \in H$ and 0 at all other points of H .) Hence, the theorem is proved. \square

Corollary 1. Let f_1, f_2, \dots, f_t be as in Theorem 4. Then the PSD-constants β_0^M and β^M of $f_1^M, f_2^M, \dots, f_t^M$ are the same as the PSD-constants β_0 and β of f_1, f_2, \dots, f_t .

Proof. This follows from the formulas (22) by applying Theorem 1 to the group H . \square

An alternative proof of this corollary can be given by using the equation (18), the equalities

$$\delta_\theta \circ \hat{\sigma} = \delta_{\theta_H}, \quad \hat{\theta} \circ \hat{\sigma} = \hat{\theta}_H \quad (23)$$

and the following lemma.

Lemma 4. For $f \in L^2(G)$ we have $\text{DFT}_f \circ \hat{\sigma} = \text{DFT}_{f^M}$.

Proof. Let Y be a set of coset representatives of M in G . Then for $\phi \in \hat{H}$ we have

$$\begin{aligned} \text{DFT}_f \circ \hat{\sigma}(\phi) &= \text{DFT}_f(\phi\sigma) \\ &= \langle f, \phi\sigma \rangle \\ &= \sum_{x \in G} f(x)\bar{\phi}(xM) \\ &= \sum_{y \in Y} \sum_{z \in M} f(yz)\bar{\phi}(yM) \\ &= \sum_{y \in Y} f^M(yM)\bar{\phi}(yM) \\ &= \langle f^M, \phi \rangle \\ &= \text{DFT}_{f^M}(\phi). \end{aligned}$$

\square

This lemma generalizes [3, Theorem 3.1] which applies only to the case when G is cyclic. Thus, our lemma can be useful when one wants to construct a difference family in G by using the compression method [4].

9 Regular representation

Since the algebra \mathcal{R} comes equipped with a natural basis, G , we can identify the algebra of linear transformations on \mathcal{R} with $M_\nu(\mathbf{C})$, the algebra of complex matrices of order ν . For this one has to choose an ordering

of G , however we will suppress the ordering and will label the rows and columns of these matrices by the elements of G . We have defined an involution on \mathcal{R} , and there is also one on this matrix algebra, namely the conjugate transpose map. The regular representation of \mathcal{R} is the homomorphism of algebras with involution $\text{Mat} : \mathcal{R} \rightarrow M_\nu(\mathbf{C})$ which assigns to $X \in \mathcal{R}$ the matrix of the linear transformation $\mathcal{R} \rightarrow \mathcal{R}$ sending $Y \rightarrow XY$, $Y \in \mathcal{R}$. In particular, we have $\text{Mat}(X^*) = \text{Mat}(X)^*$, $X \in \mathcal{R}$.

The matrices $A = [a_{x,y}] \in \text{Mat}(\mathcal{R})$ are G -invariant, i.e., they satisfy the condition

$$a_{xz,yz} = a_{x,y}, \quad x, y, z \in A. \quad (24)$$

For any $x \in G$ we have $Gx = G = \sum_{y \in G} y$. This implies that $\text{Mat}(G) = J_\nu$, the all-one matrix of order ν .

Let us use the hypotheses and notation of Theorem 2. By setting $A_i = \text{Mat}(a_{f_i})$ for $i = 1, 2, \dots, t$ and by applying Mat to the equation (a) of Theorem 2 we obtain the following matrix identity

$$\sum_{i=1}^t A_i A_i^T = 4nI_\nu + (t\nu - 4n)J_\nu. \quad (25)$$

Note that $A_i^* = A_i^T$ because the A_i are $\{\pm 1\}$ -matrices.

We shall need later the permutation matrix $R = [r_{x,y}]$ of order ν whose entries are defined by the formula $r_{x,y} = \delta_e(xy)$, $x, y \in G$. This matrix is involutory, $R^2 = I_\nu$, but not G -invariant in general. However, for any G -invariant matrix $A = [a_{x,y}]$ we have

$$RAR = A^T. \quad (26)$$

Indeed, the (x, y) -entry of RAR is $\sum_{u,v \in G} \delta_e(xu) a_{u,v} \delta_e(vy) = a_{x^{-1}, y^{-1}} = a_{y,x}$.

As far as we know, the above matrix R has been first introduced in this context in [13, Lemma 8].

10 Some special classes of difference families

For the construction of various type of Hadamard matrices and related combinatorial designs several special classes of difference families are widely used mostly over finite cyclic groups in which case they can be viewed as $\{\pm 1\}$ -sequences. Many of them make sense over noncyclic finite abelian groups. We list several such families and provide explicit examples.

10.1 DO-matrices and DO-difference families ($\alpha = 2$)

Consider the $\{\pm 1\}$ -matrices M of order m . Those among them which have maximum determinant are known as D -optimal matrices or DO -matrices. If $m = 1, 2$ or m is a multiple of 4 then D -optimal matrices are just the Hadamard matrices. They have determinant $m^{m/2}$. When $m = 2\nu$ with $\nu > 1$ odd, then there are no Hadamard matrices of order m . In that case it is well known that

$$\det M \leq 2^\nu (2\nu - 1)(\nu - 1)^{\nu-1}$$

and that this inequality is strict if $2\nu - 1$ is not a sum of two squares. We are interested here only in the case when $2\nu - 1$ is a sum of two squares. Henceforth we assume in this section that ν satisfies this condition.

For the known results on DO -matrices of order $m = 2\nu$ we refer to [9, V.3] and [3]. One can construct DO -matrices of order 2ν from difference families (X_1, X_2) in the group G having the parameter sets $(\nu; r, s; \lambda)$, where $\nu/2 \geq r \geq s \geq 1$ and $\nu = 2n + 1$, $n = r + s - \lambda$. We refer to these difference families as DO -difference families. Recall from Theorem 2 the associated functions $f_i = f_{X_i}$, $i = 1, 2$, and from section 9 the matrices $A_i = \text{Mat}(a_{f_i})$. Since in this case $t = 2$ and $\nu = 2n + 1$, we have $\alpha = 2$ and, by using the equation (25), we obtain that $A_1 A_1^T + A_2 A_2^T = 2(\nu + 1)I_\nu - 2J_\nu$. It follows that

$$\begin{bmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{bmatrix} \quad (27)$$

is a DO-matrix.

If the base block X_1 is symmetric, then the matrix A_1 is symmetric. Hence, if we multiply by -1 the second block-row in the above matrix, then that matrix becomes a symmetric DO-matrix

$$\begin{bmatrix} A_1 & A_2 \\ A_2^T & -A_1^T \end{bmatrix}. \tag{28}$$

Let us give a simple example. In $G = \mathbf{Z}_3 \times \mathbf{Z}_3$ we have the difference family $X_1 = \{(0, 0), (1, 1), (2, 1)\}$, $X_2 = \{(0, 1), (0, 2)\}$ with the parameter set $(9; 3, 2; 1)$, $n = 4$, $v = 9 = 2n + 1$. Moreover, the block X_2 is symmetric. This gives a symmetric DO-matrix of order 18. We order the elements of $\mathbf{Z}_3 \times \mathbf{Z}_3$ lexicographically. Since we used in previous sections the multiplicative notation for the group G , let us write x for $(0, 1)$ and y for $(1, 0)$. Then $x^3 = y^3 = e$ and $xy = yx$. The values of the function f_1 at the basis elements $e, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2$ are $-1, 1, 1, 1, -1, 1, 1, -1, 1$, respectively. The values of f_2 are $1, -1, -1, 1, 1, 1, 1, 1, 1$. The corresponding elements $a_{f_i} \in \mathcal{R}$ are

$$\begin{aligned} a_{f_1} &= -e + x + x^2 + y - xy + x^2y + y^2 - xy^2 + x^2y^2, \\ a_{f_2} &= e - x - x^2 + y + xy + x^2y + y^2 + xy^2 + x^2y^2. \end{aligned}$$

Since $a_{f_1}e = a_{f_1}$, the first column of the matrix $A_1 = \text{Mat}(a_{f_i})$ is the transpose of the row $[-1, 1, 1, 1, -1, 1, 1, -1, 1]$. Similarly, the first column of A_2 is the transpose of the row $[1, -1, -1, 1, 1, 1, 1, 1, 1]$. Further, by using some linear algebra, one finds that

$$A_1 = \begin{bmatrix} P_1 & P_2 & P_3 \\ P_3 & P_1 & P_2 \\ P_2 & P_3 & P_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ Q_3 & Q_1 & Q_2 \\ Q_2 & Q_3 & Q_1 \end{bmatrix},$$

where $P_1 = J_3 - 2I_3$, $P_2 = P_3 = J_3 - 2C_3^T$, $Q_1 = 2I_3 - J_3$, $Q_2 = Q_3 = J_3$, and C_3 is the circulant matrix with first row $[0, 1, 0]$.

We refer to matrices like A_i as multicirculants. More precisely, let F be any field and let n_1, n_2, \dots, n_r be positive integers. Then we define recursively *multicirculant* matrices over F of type (n_1, n_2, \dots, n_r) as follows:

- (a) if $r = 1$ they are just circulant matrices over F of order n_1 ;
- (b) if $r > 1$ they are block-circulant matrices of order n_1 whose blocks are multicirculant matrices over F of type (n_2, \dots, n_r) .

Since our block X_2 (and the matrix block A_2) is symmetric, we first switch A_1 and A_2 and then plug them into the array (28). In this way we obtain a symmetric DO-matrix of order 18. One can easily verify that $A_1A_1^T + A_2A_2^T = 16I_9 + 2J_9$.

10.2 Periodic Golay pairs ($\alpha = 0$)

If (X_1, X_2) is a difference family in G with parameters $(v; k_1, k_2; \lambda)$ and $v = 2n$ then, by Theorem 2, the functions $f_i = f_{X_i}$, $i = 1, 2$, are complementary with PAF-constants $\alpha_0 = 2v$ and $\alpha = 0$. When G is cyclic, the functions f_1 and f_2 may be viewed as $\{\pm 1\}$ -sequences $f_i(0), f_i(1), \dots, f_i(v - 1)$ and are known as *periodic Golay pairs*. For more information about such sequences and in particular the existence question for specified length v see [5].

There is a necessary arithmetic condition for the existence of periodic Golay pairs of length v , a special case of a theorem of Arasu and Xiang [1]. This condition is not satisfied when $v = 18$. Consequently there are no periodic Golay pairs of length 18. Equivalently, there are no cyclic difference families with parameter set $(18; 9, 6; 6)$. However, the theorem of Arasu and Xiang is applicable only to cyclic groups G and may fail for other groups. Indeed, we have found that in the non-cyclic group $G = \mathbf{Z}_3 \times \mathbf{Z}_6$ there exist difference families with parameter set $(18; 9, 6; 6)$. Let us give an example

$$\begin{aligned} X_1 &= \{(0, 0), (0, 1), (0, 4), (1, 0), (1, 2), (1, 5), (2, 2), (2, 3), (2, 4)\}, \\ X_2 &= \{(0, 0), (0, 1), (0, 3), (0, 5), (1, 0), (2, 0)\}. \end{aligned}$$

Let A_1 and A_2 be the matrices defined as in section 9. With suitable indexing, these matrices are multicirculants and since $t = 2$ and $v = 2n$ the equation (25) shows that $A_1A_1^T + A_2A_2^T = 36I_{18}$. Hence, the matrix

$$\begin{bmatrix} A_2 & A_1 \\ A_1^T & -A_2^T \end{bmatrix}$$

is a symmetric Hadamard matrix of order 36 made up from two multicirculants.

10.3 Legendre pairs ($\alpha = -2$)

If $q \equiv 3 \pmod{4}$ is a prime power then the nonzero squares in a finite field F_q of order q form a difference set in the additive group of F_q . The parameters of this difference set are $(q; (q-1)/2; (q-3)/4)$. If we use two copies of this difference set, we obtain a difference family with parameters $(q; (q-1)/2, (q-1)/2; (q-3)/2)$.

By generalizing, we say that a difference family (X_1, X_2) in G (a finite abelian group of order v) having the parameter set $(v; (v-1)/2, (v-1)/2; (v-3)/2)$ is a *Legendre pair*. In the case when G is cyclic, they were introduced first in the paper [7] where they were called “generalized Legendre pairs”. It was shown in the same paper [7, Theorem 2] that such pairs give Hadamard matrices of order $2v+2$. Moreover if one of the blocks, say X_1 , is symmetric or skew then the resulting Hadamard matrix can be made symmetric or skew-Hadamard, respectively, see the arrays H_s and H_k below. All these facts remain valid over arbitrary finite abelian groups.

Let (X_1, X_2) be a Legendre pair with the above parameter set. The parameter n for this parameter set is $n = (v-1) - (v-3)/2 = (v+1)/2$. By Theorem 2 the associated functions f_i of X_i , $i = 1, 2$, are complementary with PAF-constants $\alpha_0 = 2v$ and $\alpha = 2v - 4n = -2$.

As nontrivial examples we give two Legendre pairs in the non-cyclic group $\mathbf{Z}_5 \times \mathbf{Z}_5$, which we identify with the additive group of the finite field $\mathbf{Z}_5[x]/(x^2 + 2)$ of order 25. The block X_1 is symmetric in the first pair and skew in the second pair.

$$\begin{aligned} X_1 &= \{\pm x, \pm 2x, \pm(1+2x), \pm(1+4x), \pm(2+3x), \pm(2+4x)\} \\ X_2 &= \{1, 3, 1+2x, 2+2x, 4+x, 4+4x, \pm(1+3x), \pm(3+x), \pm(3+2x)\}; \end{aligned}$$

$$\begin{aligned} X_1 &= \{1, 2, x, 2x, 1+2x, 1+3x, 1+4x, 2+3x, 3+3x, 2+4x, 3+4x, 4+4x\} \\ X_2 &= \{1, 3, 4, 2x, 1+2x, 1+4x, 2+x, 2+3x, 3+2x, 4+2x, 4+3x, 4+4x\}. \end{aligned}$$

The multicirculants A_i , $i = 1, 2$, associated with the base blocks X_i of the first resp. second family should be plugged into the array H_s resp. H_k below to obtain a symmetric resp. skew Hadamard matrix of order 52.

$$H_s = \begin{bmatrix} - & - & + \cdots + & + \cdots + \\ - & + & + \cdots + & - \cdots - \\ \hline + & + & & \\ \vdots & \vdots & A_1 & A_2 \\ + & + & & \\ \hline + & - & & \\ \vdots & \vdots & A_2^T & -A_1^T \\ + & - & & \end{bmatrix}, \quad H_k = \begin{bmatrix} + & - & + \cdots + & + \cdots + \\ + & + & + \cdots + & - \cdots - \\ \hline - & - & & \\ \vdots & \vdots & A_1 & A_2 \\ - & - & & \\ \hline - & + & & \\ \vdots & \vdots & -A_2^T & A_1^T \\ - & + & & \end{bmatrix}$$

11 Goethals-Seidel quadruples ($\alpha = 0$)

In our definition of difference families in a finite abelian group G of order v , we have the rather unnatural restriction that no base block X_i can be \emptyset or G . The usual justification for that restriction is that such blocks are trivial. The trivial blocks can be discarded and the number, t , of base blocks lowered. However in some applications one does not have the freedom of changing the parameter t . For that reason, it is necessary to allow the possibility of trivial blocks in such applications. In this section we shall examine one such application. To avoid confusion, we warn the reader that in the remainder of this section we deviate from Definition 1 by permitting the base blocks to be trivial. Since we deal with the “trivial” cases, we also permit G to be trivial, i.e., we may have $v = 1$.

In our recent paper [6] we have introduced a special class of difference families to which we refer as the Goethals-Seidel families. Namely, we say that a difference family (X_0, X_1, X_2, X_3) is a *Goethals-Seidel family* (or quadruple) if its parameters $(v; k_0, k_1, k_2, k_3; \lambda)$ and $n = \sum k_i - \lambda$ satisfy the additional condition $n = v$. This additional condition is equivalent to $\alpha = 0$, see Theorem 2. It is also equivalent to

$$\sum_{i=0}^3 k_i = \lambda + v. \tag{29}$$

For convenience, we arrange here the k_i so that

$$0 \leq k_0 \leq k_1 \leq k_2 \leq k_3 \leq v/2. \tag{30}$$

By plugging the associated matrices A_i into the well-known Goethals–Seidel array (GS-array)

$$H = \begin{bmatrix} A_0 & A_1R & A_2R & A_3R \\ -A_1R & A_0 & -A_3^T R & A_2^T R \\ -A_2R & A_3^T R & A_0 & -A_1^T R \\ -A_3R & -A_2^T R & A_1^T R & A_0 \end{bmatrix}, \tag{31}$$

we obtain a Hadamard matrix H . This can be easily verified by using the equation (26).

If some $k_i = 0$ then $X_i = \emptyset$ and the corresponding matrix block $A_i = J_v$, the all-one matrix of order v . We are here interested only in the cases when at least one of the matrix blocks A_i is equal to J_v . As we assume that (30) holds, this means that $k_0 = 0$. Then the equation (2) takes the form

$$\sum_{i=1}^3 k_i(k_i - 1) = \lambda(v - 1). \tag{32}$$

If $v = 1$ then this equation says nothing about λ and we shall use equation (29) to compute λ . The equation (32) can be written as

$$\sum_{i=1}^3 \left(\frac{v}{2} - k_i\right)^2 = \frac{v}{4}(4 - v). \tag{33}$$

It implies that $v \leq 4$. Thus there are four cases to consider according to the value of $v = 1, 2, 3, 4$. In the first three cases the group G is necessarily cyclic and we shall assume that $G = \mathbf{Z}_v$.

Case $v = 1$. Then (30) implies that $k_1 = k_2 = k_3 = 0$. Thus all $X_i = \emptyset$, and all four blocks $A_i = J_1 = I_1$, the identity matrix of order 1. We can plug these blocks into the GS-array to obtain a Hadamard matrix of order 4. (Note that in this case the equation (29) implies that $\lambda = -1$.)

Case $v = 2$. Then the inequalities (30) and the equation (33) imply that $k_1 = 0$ and $k_2 = k_3 = 1$. Now $X_0 = X_1 = \emptyset$ and we can set $X_2 = X_3 = \{0\}$. Thus $A_0 = A_1 = J_2$ and $A_2 = A_3 = J_2 - 2I_2$. We can plug these blocks into the GS-array to get a Hadamard matrix of order 8. (In this case $\lambda = 0$.)

Case $v = 3$. The inequalities (30) and the equation (33) imply that $k_1 = k_2 = k_3 = 1$. Now $X_0 = \emptyset$ and we can set $X_1 = X_2 = X_3 = \{0\}$. The four matrix blocks are $A_0 = J_3$ and $A_1 = A_2 = A_3 = J_3 - 2I_3$. This gives a Hadamard matrix of order 12. (In this case again $\lambda = 0$.)

Case $v = 4$. The inequalities (30) and the equation (33) imply that $k_1 = k_2 = k_3 = 2$. It follows that $\lambda = 2$.

If $G = \mathbf{Z}_4$, a cyclic group, there is a GS-difference family with $X_0 = \emptyset$ and $X_i = \{0, i\}$ for $i = 1, 2, 3$.

There is also a possibility that G is a Klein four-group, which we identify with the additive group $\{0, 1, x, 1+x\}$, $x^2 = 1+x$, of the finite field F_4 of order 4. The required GS-family exists, e.g.

$$X_0 = \emptyset, X_1 = \{0, 1\}, X_2 = \{0, x\}, X_3 = \{0, 1+x\}.$$

For this difference family we give all details for the construction of the corresponding Hadamard matrix. If we label the rows and columns of the matrix blocks A_i with group elements $0, 1, x, 1+x$ then $A_0 = J_4$ and

$$A_1 = \begin{bmatrix} - & - & + & + \\ - & - & + & + \\ + & + & - & - \\ + & + & - & - \end{bmatrix}, \quad A_2 = \begin{bmatrix} - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{bmatrix}, \quad A_3 = \begin{bmatrix} - & + & + & - \\ + & - & - & + \\ + & - & - & + \\ - & + & + & - \end{bmatrix},$$

where we write \pm instead of ± 1 . Since $u+u=0$ for all $u \in G$, we have $\delta_{0,y+z} = 1$ if and only if $y = z$. Thus $R = I_4$ in this case. For the same reason, each X_i (and each A_i) is symmetric. By plugging the A_i into the GS-array, we obtain the symmetric Hadamard matrix

$$H = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix}.$$

It is easy to check that $A_0^2 = 4A_0$ and $A_i^2 = -4A_i$ for $i \neq 0$. Further we have $A_i A_j = 0$ whenever $i \neq j$ and $A_0 - A_1 - A_2 - A_3 = 4I_4$. It is now easy to verify that H is indeed a Hadamard matrix.

This H is an example of a Hadamard matrix of Bush-type (see [2]) because its order is a square $4v = m^2$, when partitioned into blocks of size m , the diagonal blocks are all equal to J_m and each off-diagonal block has all row and column sums 0.

In the case $G = \mathbf{Z}_4$ we also obtain a Bush-type Hadamard matrix of order 16. (The details are left to the reader.)

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