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The location of classified edges due to the change in the geometric multiplicity of an eigenvalue in a tree

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Abstract: Given a combinatorially symmetric matrix A whose graph is a tree T and its eigenvalues, edges in T can be classified in four categories, based upon the change in geometric multiplicity of a particular eigenvalue, when the edge is removed. We investigate a necessary and sufficient condition for each classification of edges. We have similar results as the case for real symmetric matrices whose graph is a tree. We show that a g-2-Parter edge, a g-Parter edge and a g-downer edge are located separately from each other in a tree, and there is a g-neutral edge between them. Furthermore, we show that the distance between a g-downer edge and a g-2-Parter edge or a g-Parter edge is at least 2 in a tree. Lastly we give a combinatorially symmetric matrix whose graph contains all types of edges.

Keywords: Edges, Eigenvalues, Graph, Multiplicity, Tree

MSC: 15A18, 05C50, 13H15, 05C05

1 Introduction

If T is a simple, undirected tree on n vertices, we denote the set of all n-by-n real symmetric matrices by $\mathcal{S}(T)$, and the set of all combinatorially symmetric matrices (i.e. $A = (a_{ij})$ and $a_{ji} \neq 0$ iff $a_{ij} \neq 0$) over a field $\mathbb{F}$ by $\mathcal{F}(T)$, the graph of whose off-diagonal entries is T. $\mathrm{m}_A(\lambda)$ and $\mathrm{gm}_A(\lambda)$ denote the algebraic multiplicity and geometric multiplicity of an eigenvalue λ of A respectively, and the set of eigenvalues of A by $\sigma(A)$. When we remove a vertex u from T, the remaining graph is denoted by $T(u)$, and corresponding submatrix of A by $A(u)$, which is the principal submatrix of A, resulting from deletion of the row and column corresponding to u. When an edge $\{i, j\}$ is removed from T, we denote the remaining graph by $T(e_{ij})$, and a corresponding matrix by $A(e_{ij})$. When T_0 is a induced subgraph of T, $A[T_0]$ denotes the principal submatrix of A corresponding to T_0. For an identified matrix $A \in \mathcal{F}(T)$, we often speak interchangeably about the graph and the matrix, for convenience.

Geometric multiplicity of an eigenvalue of $A \in \mathcal{F}(T)$ can change at most by 1 when a vertex is removed from T [5, Lemma 1]. If $\mathrm{gm}_{A(v)}(\lambda) - \mathrm{gm}_A(\lambda) = 1$ (resp. 0, −1), then v is called a geometrically Parter (resp. geometrically neutral, geometrically downer) vertex of T for λ in A, or g-Parter (resp. g-neutral, g-downer) for short. If A is a Hermitian matrix, then we simply call Parter (resp. neutral, downer). We call the classification of a vertex as g-Parter, g-neutral or g-downer, the *g-status* of that vertex for a given eigenvalue of a matrix in $\mathcal{F}(T)$.

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Given $A \in \mathcal{F}(T)$, and λ an eigenvalue of A, we may denote the g-status of a vertex in a natural numerical way, g-Parter should be 1, g-neutral 0 and g-downer −1. Specifically, define

$$S_A(i) = \mathrm{gm}_{A(i)}(\lambda) - \mathrm{gm}_A(\lambda),$$

so that the g-status number of a vertex is $1, 0$ and -1, depending upon whether the vertex is g-Parter, g-neutral or g-downer.

Let T_0 be a branch at vertex v in T that contains the vertex u_0 adjacent to v. If $\mathrm{gm}_{A[T_0(u_0)]}(\lambda) = \mathrm{gm}_{A[T_0]}(\lambda) - 1$, then T_0 is called a *g-downer branch* at v for the eigenvalue λ and u_0 is called a *g-downer neighbor* at v for λ [6]. To identify a g-Parter vertex for λ in T, a g-downer branch plays an important role in a tree.

Lemma 1. [6] *Let T be a tree, v a vertex of T, $A \in \mathcal{F}(T)$ and $\lambda \in \sigma(A)$. Then v is a g-Parter vertex for λ if and only if there is a g-downer branch for λ at v.*

Next we consider the change in geometric multiplicity of an eigenvalue of $A \in \mathcal{F}(T)$ when an edge is removed from the tree T. Since removal of an edge from a graph may change the rank of a matrix by as much as 2 (in either direction), it may change the geometric multiplicity of an eigenvalue by as much as 2 (in either direction). So, we say that an edge $\{i, j\}$ is *g-2-Parter* (resp. *g-Parter, g-neutral, g-downer, g-2-downer*) for λ , if for $A \in \mathcal{F}(T)$ and $A(e_{ij}) \in \mathcal{F}(T(e_{ij}))$,

$$\mathrm{gm}_{A(e_{ij})}(\lambda) - \mathrm{gm}_A(\lambda) = 2 \text{ (resp. } 1, 0, -1, -2).$$

In an analogous way, we may numerically classify edges:

$$S_A(e_{ij}) = \mathrm{gm}_{A(e_{ij})}(\lambda) - \mathrm{gm}_A(\lambda),$$

so that the g-status number of an edge is $2, 1, 0, -1$ or -2, depending upon whether the edge is g-2-Parter, g-Parter, g-neutral, g-downer or g-2-downer. We note that there is a relation between the g-status number of an edge and the g-status number of the incident vertices [8]:

$$S_A(e_{ij}) = S_A(i) - S_{A(e_{ij})}(i) = S_A(j) - S_{A(e_{ij})}(j). \tag{1}$$

When $A \in \mathcal{F}(T)$ is Hermitian, the range of change in the algebraic multiplicity of an eigenvalue is investigated in [4], when one edge is removed from a tree T. Even when $A \in \mathcal{F}(T)$ is not Hermitian, we can observe that the range of change in the geometric multiplicity of an eigenvalue is the same as in Hermitian case, when one edge is removed from T. So we note that a g-2-downer edge does not occur in $A \in \mathcal{F}(T)$ when the graph is a tree. Because if $\mathrm{gm}_{A(e_{ij})}(\lambda) = \mathrm{gm}_A(\lambda) - 2$, then i must be g-downer in A and g-Parter in $A(e_{ij})$ by (1), since the g-status number of g-2-downer edge is -2. Then, there must be a g-downer branch at i in $A(e_{ij})$ by Lemma 1. But, since i is g-downer in A, there is no g-downer branch at i in A, which is a contradiction. So, $\mathrm{gm}_{A(e_{ij})}(\lambda) = \mathrm{gm}_A(\lambda) - 2$ does not occur in $A \in \mathcal{F}(T)$.

Lemma 2. *Let T be a tree, $A \in \mathcal{F}(T)$, $\lambda \in \mathbb{R}$ and $\{i, j\}$ an edge of T.*

$$\mathrm{gm}_A(\lambda) - 1 \le \mathrm{gm}_{A(e_{ij})}(\lambda) \le \mathrm{gm}_A(\lambda) + 2.$$

Therefore an edge in a tree relative to $A \in \mathcal{F}(T)$ and an identified eigenvalue of A can be classified in four classes based upon the change in the geometric multiplicity of a particular eigenvalue, when the edge is removed from T. For a real symmetric matrix $A \in \mathcal{S}(T)$ whose graph is a tree, the classification of edges in a tree was investigated regarding to the change in the algebraic multiplicity of a particular eigenvalue in [4] and [7] etc. Here, we consider combinatorially symmetric matrices whose graph is a tree, and we investigate the change in the geometric multiplicity of a particular eigenvalue when an edge is removed.

We give a necessary and sufficient condition for each classification of edges in a tree T relative to $A \in \mathcal{F}(T)$. We have the similar results even in $\mathcal{F}(T)$ as the ones for the edges relative to $A \in \mathcal{S}(T)$. Furthermore, we clarified that a g-2-Parter edge, a g-Parter edge and a g-downer edge are not incident each other and that there is a g-neutral edge between them. We observe that the distance between a g-downer edge and a g-2-Parter edge or g-Parter edge is at least 2, that is, there are at least two g-neutral edges between them. We notice that these are also the similar results as the ones in Hermitian case.

Let T be a tree, and $A \in \mathcal{F}(T)$. Let v be a g-Parter vertex for an eigenvalue λ of $A \in \mathcal{F}(T)$. If there is only one g-downer branch at v for λ, we call v a *singly g-Parter vertex* for λ, and if there is more than one g-downer branch at v, we call v a *multiply g-Parter vertex* for λ.

When A is a real symmetric matrix whose graph is a tree and λ is an eigenvalue of A, a necessary and sufficient condition for each classified edge in T relative to $A \in \mathcal{S}(T)$ and λ is given in [4] or [7].

Theorem 3. [7] *Let T be a tree, $A \in \mathcal{S}(T)$, $\lambda \in \mathbb{R}$ and $\{i, j\}$ an edge of T.*

(i) *The edge $\{i, j\}$ is 2-Parter for λ if and only if i and j are both singly Parter for λ in A, and each is the downer neighbor for the other.*

(ii) *The edge $\{i, j\}$ is Parter for λ if and only if i is singly Parter for λ such that j is the downer neighbor for i, and j is neutral for λ, or vice versa for i and j.*

(iii) *The edge $\{i, j\}$ is neutral for $\lambda \in \sigma(A)$ if and only if i is Parter for λ such that there is a downer branch at i that does not contain j, or both i and j are neutral for λ in A. Here i and j are interchangeable.*

(iv) *The edge $\{i, j\}$ is downer for λ if and only if i and j are both downer vertices for λ in A.*

2 Classification of edges by change in geometric multiplicity

When A is a combinatorially symmetric matrix whose graph is a tree T and λ is an eigenvalue of A, the geometric multiplicity of λ of $A \in \mathcal{F}(T)$ can change when an edge in T is removed. We consider a necessary and sufficient condition for each classified edge in T relative to $A \in \mathcal{F}(T)$ and an identified eigenvalue of A. Here we note that we consider generally non-symmetric matrices $A \in \mathcal{F}(T)$.

Theorem 4. *Let T be a tree, $A \in \mathcal{F}(T)$, $\lambda \in \mathbb{F}$ and $\{i, j\}$ an edge of T. The edge $\{i, j\}$ is g-2-Parter for λ relative to A if and only if i and j are both g-Parter for λ in A, and each is the g-downer neighbor for the other.*

Proof. For sufficiency, we suppose that i and j are g-Parter vertices relative to $A \in \mathcal{F}(T)$ for λ and each is the g-downer neighbor for the other. Since i and j are both g-Parter in A and they are both g-downer in $A(e_{ij})$, the g-status number of the edge e_{ij} becomes 2 by (1). For necessity, we suppose that $\{i, j\}$ is g-2-Parter relative to A for λ. Then, the status of i and j must be g-Parter in A and g-downer in $A(e_{ij})$ by (1). Since the graph is a tree, j (resp. i) is g-downer in $A(i)$ (resp. $A(j)$). □

Next, we consider conditions for a g-Parter edge and g-downer edge respectively for an eigenvalue relative to $A \in \mathcal{F}(T)$.

Theorem 5. *Let T be a tree, $A \in \mathcal{F}(T)$, $\lambda \in \mathbb{F}$ and $\{i, j\}$ an edge of T. The edge $\{i, j\}$ is g-Parter for λ relative to A if and only if i is g-Parter and j is g-neutral for λ in A, and j is the g-downer neighbor for i. Here i and j are interchangeable.*

Proof. For sufficiency, we suppose that i is g-Parter and j is g-neutral for λ in A, and j is the g-downer neighbor at i. Then, j is g-downer in $A(e_{ij})$ since T is a tree. Thus the g-status number of the edge $\{i, j\}$ is 1 by (1). So the edge $\{i, j\}$ is g-Parter in A.

For the necessity portion, we suppose that the edge $\{i, j\}$ is g-Parter for λ in A. Then, we can consider two possibilities of the g-status of the incident vertex, as g-neutral or g-Parter from (1). If the g-status of i is g-Parter, then the g-status of i in $A(e_{ij})$ is g-neutral. Thus, there is no g-downer branch for λ at i in $A(e_{ij})$. Since i was g-Parter in A, the g-downer branch at i in A has to include j. So j is the g-downer neighbor for i. Since T is a tree, j is also g-downer in $A(e_{ij})$. From (1), the g-status of j in A has to be g-neutral, since the edge $\{i, j\}$ is g-Parter.

Next when the edge $\{i, j\}$ is g-Parter, if the g-status of i is g-neutral, then the g-status of i in $A(e_{ij})$ is g-downer by (1). Since T is a tree, i is also g-downer in $A(j)$. Thus, i is a g-downer neighbor for j in A. Therefore, j has to be g-Parter by Lemma 1. That concludes the assertion. □

Theorem 6. *Let T be a tree, $A \in \mathcal{F}(T)$, $\lambda \in \sigma(A)$ and $\{i, j\}$ an edge of T. The edge $\{i, j\}$ is g-downer for λ relative to A if and only if both i and j are g-downer vertices for λ in A.*

Proof. For sufficiency, we suppose that i and j are g-downer for λ in A. To reach a contradiction, suppose that the edge $\{i, j\}$ is g-neutral or g-Parter for λ in A. If the edge $\{i, j\}$ is g-neutral, then the g-status of j in $A(e_{ij})$ will be g-downer by (1). Then, j is a g-downer neighbor for i in A. Thus, i must be g-Parter in A by Lemma 1, a contradiction. Next, if the edge $\{i, j\}$ is g-Parter, then it is easy to find that the g-status of i cannot be g-downer in A from (1). So, the g-status of the edge is g-downer.

For necessity portion, we suppose that the edge $\{i, j\}$ is g-downer for λ in A. From (1), the g-status of the incident vertex cannot be g-Parter. If i is g-neutral in A, then the g-status of i in $A(e_{ij})$ has to be g-Parter by (1). Then there must be a g-downer branch at i in $A(e_{ij})$. But it is also a downer branch at i in A, so i must be g-Parter in A, a contradiction. Therefore, the incident vertices cannot be g-neutral in A. Thus, the g-statuses of the incident vertices have to be g-downer in A.

□

Next, we give a condition for a g-neutral edge for an eigenvalue relative to $A \in \mathcal{F}(T)$.

Theorem 7. *Let T be a tree, $A \in \mathcal{F}(T)$, $\lambda \in \sigma(A)$ and $\{i, j\}$ an edge of T. The edge $\{i, j\}$ is g-neutral for λ relative to A if and only if there is a g-downer branch at i that does not include j, or both i and j are g-neutral for λ in A. Here i and j are interchangeable.*

Proof. For sufficiency, we suppose that there is a g-downer branch at i that does not include j. Then i is g-Parter for λ in A from Lemma 1. We note that the branch is also a downer branch at i in $A(e_{ij})$. So, i is g-Parter even in $A(e_{ij})$. Therefore, the edge $\{i, j\}$ is g-neutral by (1). Next we suppose that both i and j are g-neutral for λ in A. In this case, the edge $\{i, j\}$ cannot be g-2-Parter, g-Parter or g-downer from Theorem 4, 5 and 6. So the edge $\{i, j\}$ is g-neutral for λ in A.

For necessity portion, we suppose that the edge $\{i, j\}$ is g-neutral for λ in A. In case that i is g-Parter in A, i is g-Parter in $A(e_{ij})$ by (1), since the edge $\{i, j\}$ is g-neutral. Then there is a g-downer branch at i that does not include j. In case that i is g-downer in A, i is g-downer in $A(e_{ij})$. Then i is a g-downer neighbor for j, so j has to be g-Parter in A. We note that j is also g-Parter in $A(e_{ij})$, since the edge $\{i, j\}$ is g-neutral. Then there is a g-downer branch at j that does not include i. Lastly, in case that i is g-neutral in A, there is not possibility such that j is g-downer in A, because if so, j will be g-downer in $A(e_{ij})$, then i must be g-Parter, a contradiction. So, when i is g-neutral, then j is g-neutral or g-Parter. When j is g-Parter in A, j is g-Parter in $A(e_{ij})$, then there is a g-downer branch at j that does not include j. That conludes the assertion.

□

3 Location of classified edges

We consider the position of the classified edges in a tree. We observe some constraints about the positions of four kinds of edges in a tree.

Lemma 8. *Let T be a tree, $A \in \mathcal{F}(T)$ and $\lambda \in \mathbb{R}$. All edges adjacent to a g-2-Parter edge are g-neutral for λ in A.*

Proof. Let an edge $\{i, j\}$ be a g-2-Parter edge in A, and $\{i, k\}$ be another edge incident to i in A. Then i is g-Parter in A and j is a downer neighbor at i, from Theorem 4. Thus there is a g-downer branch at i that does not include k. Then, the edge $\{i,k\}$ is g-neutral by Theorem 7. Here i and j are interchangeable, therefore all edges incident to a g-2-Parter edge are g-neutral in A. □

We suppose that there is a g-Parter edge $\{i,j\}$ and a g-downer edge $\{k,l\}$ in T relative to $A \in \mathcal{F}(T)$. Then, i is g-Parter and j is g-neutral in T from Theorem 5, here i and j may be interchangeable. Since k and l are g-downers in T, from Theorem 6, a g-Parter edge and a g-downer edge cannot be incident in T.

Lemma 9. *Let T be a tree, $A \in \mathcal{F}(T)$ and $\lambda \in \sigma(A)$. A g-Parter edge and a g-downer edge are not incident each other in T.*

From the two lemmas above, we have the information about the location of g-2-Parter edges, g-Parter edges and g-downer edges in T for an eigenvalue relative to $A \in \mathcal{F}(T)$.

Theorem 10. *Let T be a tree, $A \in \mathcal{F}(T)$ and $\lambda \in \sigma(A)$. A g-2-Parter edge, a g-Parter edge and a g-downer edge for λ in T are located separately from each other, and there is at least one g-neutral edge between them.*

Furthermore, we can investigate the distance between a g-downer edge and a g-2-Parter edge or g-Parter edge. We can observe that there are at least two g-neutral edges between them.

Theorem 11. *Let T be a tree, $A \in \mathcal{F}(T)$ and $\lambda \in \sigma(A)$. The distance between a g-downer edge and a g-2-Parter or a g-Parter edge is at least 2.*

Proof. Let $\{i,j\}$ be a g-downer edge for λ in T relative to $A \in \mathcal{F}(T)$. Then, by Theorem 6, i and j are g-downer vertices in A. A g-downer edge can be adjacent a g-neutral edge besides a g-downer edge. Suppose that $\{i,j\}$ is adjacent to a g-neutral edge $\{j,k\}$. Then, k must be g-Parter from Theorem 7 and k is a g-multiply Parter vertex. Because k is g-Parter in $A(e_{jk})$, since $\{j,k\}$ is g-neutral in A and the g-status of k does not change after removing the edge $\{j,k\}$. So, there is a g-downer branch at k that does not include j. On the other hand, j is also g-downer in $A(e_{jk})$ from the expression (1), since $\{j,k\}$ is g-neutral in A. Thus, there is a g-downer branch at k that includes j. Then k is a multiply g-Parter vertex in A.

Next we observe that all edges incident to k are g-neutral. Let $\{k,l\}$ be an edge incident to k. Then, there is a downer branch at k that does not include l, since k is a multiply g-Parter vertex in A. Thus, $\{k,l\}$ is a g-neutral edge by Theorem 7. So, two g-neutral edges $\{j,k\}$ and $\{k,l\}$ can be incident to a g-downer edge $\{i,j\}$. That means that there are at least two neutral edges between a g-neutral edge and a g-2-Parter or g-Parter edge in T relative to A. □

From the above Theorem, we can deduce the next corollary.

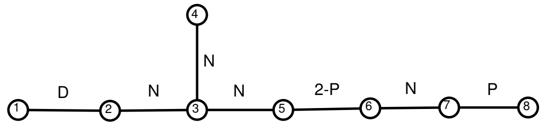
Corollary 12. *Let T be a tree, $A \in \mathcal{F}(T)$ and $\lambda \in \mathbb{R}$. If the diameter of T is less than 4, then a g-downer edge and a g-2-Parter or a g-Parter edge are not contained together in T.*

Next we give an $A \in \mathcal{F}(T)$ whose graph is a tree that contains all kind of edges.

Example 13. *Let*

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix},$$

whose graph is as follows.



A has the eigenvalue 3 with multiplicity 1. When one edge is removed from the tree, we can observe the change in multiplciity of the eigenvalue 3 so that $\mathrm{gm}_{A(e_{12})}(3) = 0$, $\mathrm{gm}_{A(e_{23})}(3) = \mathrm{gm}_{A(e_{34})}(3) = \mathrm{gm}_{A(e_{35})}(3) = \mathrm{gm}_{A(e_{67})}(3) = 1$, $\mathrm{gm}_{A(e_{56})}(3) = 3$ and $\mathrm{gm}_{A(e_{78})}(3) = 2$. The character over each edge denotes the g-status of the edge (2-P: g-2-Parter, P: g-Parter, N: g-neutral and D: g-downer). The matrix A has all kind of edges in the tree, and we can observe that there is at least one g-neutral edge between a g-2-Parter, g-Parter and g-downer edge, further we can see that there are two g-neutral edges beween a g-downer edge and a g-2-Parter edge in a tree.

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References

[1] R. Horn, C.R.Johnson, Matrix Analysis, 2nd Edition. Cambridge University Press, New York, 2013.

[2] C.R. Johnson, A. Leal-Duarte and C.M. Saiago, The Parter-Wiener theorem: refinement and generalization, SIAM J. Matrix Anal. Appl. 25 (2003), 352-361.

[3] C.R. Johnson, A. Leal-Duarte, C.M. Saiago. The change in eigenvalue multiplicity associated with perturbation of a diagonal entry, Linear Multilinear Algebra 60 (2012), 525-532 .

[4] C.R. Johnson, P.R. McMichael, The change in multiplicity of an eigenvalue of a Hermitian matrix associated with the removal of an edge from its graph, Discrete Math. 311 (2011), 166-170.

[5] C.R. Johnson, C. M. Saiago, Geometric Parter-Wiener, etc. theory, Linear Algebra Appl, 537 (2018), 332-347.

[6] C.R. Johnson, C.M. Saiago, Eigenvalues, Multiplicities and Graphs. Cambridge University Press, New York, 2018.

[7] K. Toyonaga, C.R. Johnson, The classification of edges and the change in multiplicity of an eigenvalue of a real symmetric matrix tesulting from the change in an edge value, Spec. Matrices 5 (2017), 51-60.

[8] C.R. Johnson, C.M. Saiago, K. Toyonaga, Classification of vertices and edges with respect to the geometric multiplicity of an eigenvalue in a matrix, with a given graph, over a field, Linear Multilinear Algebra 66 (2018), 2168-2182.