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The integer cp-rank of 2×2 matrices

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Abstract: We show the cp-rank of an integer doubly nonnegative 2×2 matrix does not exceed 11.

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MSC: 15B36, 15B48

Dedicated to Professor Charles R. Johnson.

1 Introduction

A $n \times n$ matrix A is said to be *completely positive*, if there exists a (not necessarily square) nonnegative matrix V such that $A = VV^T$. Completely positive matrices have been widely studied, and they play an important role in various applications. It is a subject to which C.R. Johnson has made an important contribution [7–10]. For further background on completely positive matrices, we refer the reader to the following works and citations therein [2–5].

Clearly, any completely positive matrix is nonnegative and positive semidefinite. We call the family of matrices that are both nonnegative and positive semidefinite *doubly nonnegative*. Doubly nonnegative matrices of order less than 5 are completely positive [14]. However, this is no longer true for matrices of order larger than or equal to 5 [12].

Any $n \times n$ completely positive matrix A has many *cp-factorizations* of the form $A = VV^T$, where V is an $n \times m$ matrix. Note that m is also not unique. We define the *cp-rank* of A to be the minimal possible m. If we demand that V has rational entries, then we say that A has *a rational cp-factorization*. We define the *rational cp-rank* correspondingly. In this note we will study *integer cp-factorizations*, where we demand V to be an integer nonnegative matrix, and *the integer cp-rank*, the minimal number of columns in the integer cp-factorization of a given matrix.

Every rational matrix which lies in the interior of the cone of completely positive matrices has a rational cp-factorization [11], but the question is still open for rational matrices on the boundary of the region. On the other hand, for $n \geq 3$ it it easy to find examples of $n \times n$ integer completely positive matrices that do not have an integer cp-factorization. In [13] the authors answered a question posed in [1], by proving that for $n = 2$ every integer doubly nonnegative matrix has an integer cp-factorization. An alternative proof of this result can be found in [6]. Neither of those proofs offer a bound on the integer cp-rank of such matrices. In this note we prove that the integer cp-rank of 2×2 matrices cannot be larger than 11.

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2 Main Result

The question of determining the completely positive integer rank for a given $n \times n$ completely positive matrix is not trivial, even in the case when $n = 1$. In this case the answer is given by Lagrange's Four-Square Theorem.

Theorem 2.1. *[Lagrange's Four Square Theorem] Every positive integer x can be written as the sum of at most four squares. If x is not of the form*

$$x = 4^r(8k + 7) \tag{1}$$

for some nonnegative integers r and k, then x is the sum of at most three squares.

With Theorem 2.1 rank one matrices are easy to analyse.

Lemma 2.1. *An integer doubly nonnegative matrix A of rank 1 is completely positive, and has the integer cp-rank equal to the integer cp-rank of the greatest common divisor of its diagonal elements.*

Proof. Let $A = (a_{ij})$ be an integer doubly nonnegative matrix of rank 1, and let $d := \gcd(a_{11}, a_{22}, \dots, a_{nn})$. Since $a_{ij} = \sqrt{a_{ii}a_{jj}}$, d also divides all the off-diagonal elements of A. Hence, $A = dB$, where $B = (b_{ij})$ is a rank 1 integer doubly nonnegative matrix that satisfies satisfies $\gcd(b_{11}, b_{22}, \dots, b_{nn}) = 1$.

To complete the proof, we need to show that each diagonal element in B is a perfect square. If this is not true, then there exists i_0 such that $b_{i_0 i_0} = pc_{i_0}^2$, where p is a product of distinct primes. Now $b_{i_0 j}^2 = b_{i_0 i_0} b_{jj} = pc_{i_0}^2 b_{jj}$. Hence $b_{jj} = pc_j^2$ for some positive integer c_j and for all j. This implies that p divides b_{jj} for all j. As this contradicts our assumption, the claim is proved.

Let $b_{ii} = c_{ii}^2$, $b := \begin{pmatrix} c_{11} & c_{22} & \dots & c_{nn} \end{pmatrix}^T \in \mathbb{Z}^n$, and let $d = \sum_{i=1}^{r_0} d_i^2$, where $r_0 \le 4$ is the integer cp-rank of d. Then $A = \sum_{i=1}^{r_0} (d_i b)(d_i b)^T$ is the integer cp-factorization of A. □

Proposition 2.1 below not only gives the first bound on the integer cp-rank of 2×2 matrices, but it also provides an approach that is later refined to improve the bound.

Proposition 2.1. *A 2×2 integer doubly nonnegative matrix has an integer cp-factorization, and an integer cp-rank less than or equal to 12.*

Proof. Let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be an integer doubly nonnegative matrix. First we prove that we can reduce our problem to the case when $a \ge b$ and $c \ge b$. To this end we assume $b > c$, and write $b = \alpha c + b_0$, where $\alpha \ge 1$ is a positive integer, and $b_0 \in \{0, \dots, c-1\}$. Let

$$S(\alpha) := \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$$

We claim that

$$A_0 = S(\alpha)AS(\alpha)^T = \begin{pmatrix} a - 2\alpha b + \alpha^2 c & b - \alpha c \\ b - \alpha c & c \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ b_0 & c_0 \end{pmatrix}$$

is a doubly nonnegative matrix. From $\det S(\alpha) = 1$, we deduce $\det A_0 = \det A \ge 0$. Now, the inequalities $\det A = a_0 c_0 - b_0^2 \ge 0$ and $c_0 = c > 0$, imply that $a_0 = (a - 2\alpha b_0 - \alpha^2 c) \ge 0$.

Since $S(\alpha)^{-1} > 0$, any completely positive factorization of A_0: $A_0 = B_0 B_0^T$, gives us a completely positive factorization of A: $A = (S(\alpha)^{-1}B_0)(S(\alpha)^{-1}B_0)^T$. Clearly, B_0 and $(S(\alpha)^{-1}B_0)$ have the same number of columns, hence the two factorizations give the same bound on the cp-rank. With this we have proved, that to find an integer completely positive factorization for A it is sufficient to solve the problem for A_0, that satisfies $a_0 \ge b_0$ and $c_0 \ge b_0$. If $b > a$, we can repeat the above argument, with the roles of the diagonal elements reversed.

From now on we may assume that our given matrix A satisfies $a \geq b$ and $c \geq b$. Under this assumption we can write:

$$A = \begin{pmatrix} b & b \\ b & b \end{pmatrix} + \begin{pmatrix} a-b & 0 \\ 0 & c-b \end{pmatrix} \tag{2}$$

$$= b\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} a-b & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c-b \end{pmatrix}. \tag{3}$$

By Lemma 2.1 each of the rank 1 matrices in the above sum have the integer cp-rank at most 4, so the integer cp-rank of A is at most 12. □

To reduce the bound for the cp-rank to 11 we look more closely at the family of integers that cannot be written as a sum of less than four squares.

Lemma 2.2. *Let x be a positive integer of the form* (1). *Then $x-2$, $x-6$, $x+2$ and $x+6$ are not of the form* (1).

Proof. Let $x = 4^r(8k+7)$ for some nonnegative integers r and k. Then:

$$\begin{aligned} x &\equiv 7 \pmod 8 \text{ when } r = 0, \\ x &\equiv 4 \pmod 8 \text{ when } r = 1, \\ x &\equiv 0 \pmod 8 \text{ when } r \geq 2. \end{aligned}$$

In each case, it is straightforward to check that $x-6$, $x-2$, $x+2$, $x+6$ are not equivalent to 7, 4 or 0 modulo 8, so they cannot be of the form (1). □

Theorem 2.2. *Let*

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be an integer doubly nonnegative matrix. Then A has an integer cp-factorization, and an integer cp-rank less than or equal to 11.

Proof. From (2) and Theorem 2.1 it is clear that the bound 12 will not be reached unless b, $a-b$ and $c-b$ are all of the form (1). In particular, the result holds for $b \leq 6$, and for $a-b \leq 6$ or $c-b \leq 6$. So we assume b, $a-b$ and $c-b$ are all greater than or equal to 7, and that they all require four squares in Theorem 2.1.

First let us consider the case when $a-b \not\equiv 7 \pmod 8$. In this case $a-b-3 \equiv 1 \pmod 8$ or $a-b-3 \equiv 5 \pmod 8$, so $a-b-3$ is not of the form (1). We write:

$$A = \begin{pmatrix} a-b-3 & 0 \\ 0 & c-b+2 \end{pmatrix} + (b-6)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \end{pmatrix}.$$

Under our assumption, we can write each $a-b-3$, $c-b+2$ and $b-6$ as sums of at most three squares by Lemma 2.2, so the integer completely positive rank of A is at most $3+3+3+1 = 10$. The case, when $c-b \not\equiv 7 \pmod 8$ can be dealt with in a similar way.

Now we assume that $a-b \equiv 7 \pmod 8$ and $c-b \equiv 7 \pmod 8$. We write:

$$A = \begin{pmatrix} a-b+1 & 0 \\ 0 & c-b-2 \end{pmatrix} + (b-2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}.$$

Since $c-b-2$ and $b-2$ are not of the form (1) by Lemma 2.2, the integer completely positive rank of A is at most $4+3+3+1 = 11$. □

Next example shows that the integer cp-rank of a 2×2 matrix can be as high as 9, but we were not able to find examples of 2×2 matrices with cp-rank larger than that.

Example 2.1. *Let*

$$A = \begin{pmatrix} a & 1 \\ 1 & c \end{pmatrix},$$

where a and c are positive integers. Then any integer cp-factorization of A must involve $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$, so the integer cp-rank of A is $1+p+q$, where p, q are the least number of squares of integers needed to represent $a-1$, $c-1$, respectively. In particular, A has the integer cp-rank 9 if a and c are both divisible by 8.

Example 2.2. *Let*

$$B = \begin{pmatrix} a & 2 \\ 2 & c \end{pmatrix}$$

with integers $c \geq a \geq 2$. The decomposition

$$B = \begin{pmatrix} a-2 & 0 \\ 0 & c-2 \end{pmatrix} + 2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

shows that the integer cp-rank of B is at most $3+3+2=8$, unless at least one of $a-2$ and $c-2$ is of the form (1). But if $a-2$ is of the form (1) $a-4$ is not, by Lemma 2.2. In this case the decomposition

$$B = \begin{pmatrix} a-4 & 0 \\ 0 & c-1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix}$$

shows that the integer cp-rank of B is at most $3+4+1=8$. The case, when $c-2$ is of the form (1) (and $a-2$ is not), can be dealt with correspondingly. We conclude that the integer cp-rank of all such B is at most 8.

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