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The almost semimonotone matrices

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Abstract: A (strictly) semimonotone matrix $A \in \mathbb{R}^{n \times n}$ is such that for every nonzero vector $x \in \mathbb{R}^n$ with nonnegative entries, there is an index k such that $x_k > 0$ and $(Ax)_k$ is nonnegative (positive). A matrix which is (strictly) semimonotone has the property that every principal submatrix is also (strictly) semimonotone. Thus, it becomes natural to examine the almost (strictly) semimonotone matrices which are those matrices which are not (strictly) semimonotone but whose proper principal submatrices are (strictly) semimonotone. We characterize the 2×2 and 3×3 almost (strictly) semimonotone matrices and describe many of their properties. Then we explore general almost (strictly) semimonotone matrices, including the problem of detection and construction. Finally, we relate (strict) central matrices to semimonotone matrices.

Keywords: semipositive matrix, P-matrix, central matrix, linear complementarity problem

MSC: 15A48, 90C33

1 Introduction

A matrix $A \in \mathbb{R}^{n \times n}$ is (strictly) semimonotone if for every nonzero nonnegative vector x, there exists a positive entry in x such that the corresponding entry of Ax is nonnegative (positive). In other words, a semimonotone matrix A cannot negate all positive entries of any nonzero nonnegative vector x, while a strictly semimonotone matrix cannot negate or turn to zero all positive entries of any nonzero nonnegative vector x. An overview and explanation of properties and basic facts of semimonotone matrices can be found in [20].

Semimonotone matrices include the nonnegative matrices, the $\mathbf{P}_0$-matrices (matrices all of whose principal minors are nonnegative), and the copositive matrices (see [9]). However, by no means are these classes of matrices the only ones that belong to the class of semimonotone matrices. An example of a $\mathbf{P}_0$ and copositive semimonotone matrix is the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}.$$

Any nonnegative matrix with negative determinant is an example of a semimonotone copositive matrix which is not $\mathbf{P}_0$. An example of a semimonotone matrix which is neither $\mathbf{P}_0$ nor copositive is the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & -1 \\ 6 & -8 & 4 \end{bmatrix}.$$

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Semimonotone matrices have an important connection with the linear complementarity problem, which is the problem of finding a vector $x \in \mathbb{R}^n$ such that

$$\begin{aligned} x &\geq 0 \\ q + Ax &\geq 0 \\ x^T(q + Ax) &= 0, \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. We denote the above problem by LCP(q, A). It is well-known (see [8, 17]) that a matrix A is (strictly) semimonotone if and only if the LCP (q, A) has a unique solution for all $q > 0$ $(q \geq 0)$.

Semimonotone and strictly semimonotone matrices are closely related to semipositive and weakly semipositive matrices. A matrix A is *semipositive*, denoted by $A \in \mathbf{S}$, if there exists an $x > 0$ such that $Ax > 0$. A matrix A is called *weakly semipositive*, denoted by $A \in \mathbf{S_0}$, if there exists a $0 \neq x \geq 0$ such that $Ax \geq 0$. Semipositive matrices have been examined in detail. For many of the basic properties of semipositive matrices, see [2, 11]. More recent results can be found in [5, 7, 14]. The study of semipositive matrices from a qualitative point of view was studied in [12, 13].

It has been shown (see [9]) that a matrix A is (strictly) semimonotone if and only if A and all of its principal submatrices are weakly semipositive (semipositive). The above characterization of semimonotone matrices is valuable in many ways; however, it does not provide an efficient method of detecting semimonotone matrices, since an $n \times n$ matrix has $2^n - 1$ principal submatrices. Thus, finding an efficient detection algorithm for semimonotone matrices remains an open problem. To get closer toward the goal of finding a new characterization of (strictly) semimonotone matrices, we have found it useful to examine the *almost* (strictly) semimonotone matrices, similar to how the almost **P**- and almost **N**-matrices have been studied (see [19]). An *almost (strictly) semimonotone* matrix A is a matrix which is not (strictly) semimonotone but whose proper principal submatrices are (strictly) semimonotone.

In this paper, we prove a variety of results regarding the 2×2 and 3×3 almost (strictly) semimonotone matrices and describe a characterization for these matrices. We discuss a counterexample to show that certain properties that hold for the 2×2 and 3×3 cases do not necessarily hold in general, and then we prove some properties that do hold in the general case. Finally, we look at (strict) central matrices, which are those matrices having a nonzero nonnegative (positive) vector in their nullspace (see [1, 3]), and we explore their connections to semimonotone matrices. In particular, we are interested in the class of matrices which are almost strictly semimonotone and semimonotone since these matrices are strict central.

2 Preliminary results

2.1 Notation

We will use the following notation:

- The i^{th} entry of a vector x is denoted by x_i.
- $\text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$.
- $\sigma(A)$ denotes the spectrum of A .
- $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is the spectral radius of A.
- $\text{adj}(A)$ refers to the adjugate matrix of A.
- Inequalities are entry-wise. For example, if X is an array, then $X \geq 0$ $(X > 0)$ means that its entries are nonnegative (positive).
- If $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$ are nonempty index sets whose entries are in increasing order, then $x[\alpha]$ denotes the vector of entries indexed by α and $A[\alpha, \beta]$ denotes the submatrix of A whose rows and columns are indexed by α and β, respectively. We write $A[\alpha]$ for $A[\alpha, \alpha]$. Any submatrix $A[\alpha]$ is called a *principal submatrix* of A. Also, we say that $A[\alpha]$ is a *proper* principal submatrix of A if α is a nonempty proper subset of $\{1, 2, \ldots, n\}$.

2.2 Definitions of various matrix classes

Here we formally define many of the various matrix classes used throughout this paper.

Definition 2.1. A matrix $A \in \mathbb{R}^{n\times n}$ is *semimonotone*, denoted by $A \in \mathbf{E_0}$, if for all $0 \neq x \geq 0$, there exists a k such that $x_k > 0$ and $(Ax)_k \geq 0$. The matrix A is *strictly semimonotone*, denoted by $A \in \mathbf{E}$, if the latter inequality is strict.

Definition 2.2. A matrix $A \in \mathbb{R}^{n\times n}$ is *almost (strictly) semimonotone* if all proper principal submatrices of A are (strictly) semimonotone and there exists an $x > 0$ such that $Ax < 0$ ($Ax \leq 0$).

Definition 2.3. A matrix $A \in \mathbb{R}^{n\times n}$ is a $\mathbf{P}$-matrix ($\mathbf{P}_0$-matrix), denoted by $A \in \mathbf{P}$ ($A \in \mathbf{P}_0$), if all its principal minors are positive (nonnegative).

Definition 2.4. A matrix $A \in \mathbb{R}^{n\times n}$ is an almost $\mathbf{P}$-matrix (almost $\mathbf{P}_0$-matrix) if all its proper principal submatrices are $\mathbf{P}$-matrices ($\mathbf{P}_0$-matrices) and $\det A \leq 0$ ($\det A < 0$).

Definition 2.5. A matrix $A \in \mathbb{R}^{n\times n}$ is *copositive* if $x^T Ax \geq 0$ for all $x \geq 0$, and A is *strictly copositive* if $x^T Ax > 0$ for all $0 \neq x \geq 0$.

Definition 2.6. A matrix $A \in \mathbb{R}^{m\times n}$ is *semipositive*, denoted by $A \in \mathbf{S}$, if there exists an $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax > 0$. By continuity of a matrix as a linear map, this is equivalent to saying that there exists an $x \in \mathbb{R}^n$ such that $x > 0$ and $Ax > 0$. A matrix $A \in \mathbb{R}^{m\times n}$ is *weakly semipositive*, denoted by $A \in \mathbf{S_0}$, if there exists an $x \in \mathbb{R}^n$ such that $0 \neq x \geq 0$ and $Ax \geq 0$.

Definition 2.7. A matrix $A \in \mathbb{R}^{m\times n}$ is *central* if there exists an $x \in \mathbb{R}^n$ such that $0 \neq x \geq 0$ and $Ax = 0$. The matrix A is said to be *strict central* if there exists an $x > 0$ such that $Ax = 0$.

2.3 Previous results

The following theorem lists some well-known facts about the (strictly) semimonotone matrices (see [9, 17]).

Theorem 2.8.
(1) $A \in \mathbb{R}^{n\times n}$ is (strictly) semimonotone if and only if A and all its proper principal submatrices are weakly semipositive (semipositive).
(2) $A \in \mathbb{R}^{n\times n}$ is (strictly) semimonotone if and only if A^T is (strictly) semimonotone.
(3) $A \in \mathbb{R}^{n\times n}$ is (strictly) semimonotone if and only if the LCP(q, A) has a unique solution for every $q > 0$ ($q \geq 0$).
(4) Every $\mathbf{P}_0$-matrix and every copositive matrix is semimonotone. Every $\mathbf{P}$-matrix and every strictly copositive matrix is strictly semimonotone.

All 2×2 (strictly) semimonotone matrices can be characterized in the following way, as shown in [20, Proposition 3.6].

Theorem 2.9. *Let A be a 2×2 real matrix with nonnegative (positive) diagonal entries. Then A is (strictly) semimonotone if and only if either all entries in A are nonnegative or the determinant of A is nonnegative (positive).*

The following results concerning the almost (strictly) semimonotone matrices were also proved in [20].

Theorem 2.10. *If $A \in \mathbb{R}^{n\times n}$ is almost semimonotone, then A^{-1} exists and $A^{-1} \leq 0$.*

Theorem 2.11. *If $A \in \mathbb{R}^{n\times n}$ is almost strictly semimonotone and singular, then A is semimonotone and adj(A) is either positive or negative.*

Theorem 2.12. *If $A \in \mathbb{R}^{n\times n}$ is almost strictly semimonotone and nonsingular, then A is not semimonotone and $A^{-1} < 0$.*

3 The 2 × 2 and 3 × 3 almost semimonotone matrices

We start by looking at some examples of almost (strictly) semimonotone matrices.

Example 3.1. The matrix

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

is almost semimonotone (and almost strictly semimonotone) since each 1×1 proper principal submatrix of A is positive (and hence strictly semimonotone) but $\det A = -1 < 0$ and so A is not semimonotone by Theorem 2.9. □

Example 3.2. Note that an almost strictly semimonotone matrix can still be semimonotone. For example, the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is almost strictly semimonotone and semimonotone. □

We can easily characterize all 2 × 2 almost (strictly) semimonotone matrices as shown in the next result.

Theorem 3.3. *A matrix $A \in \mathbb{R}^{2\times 2}$ is almost (strictly) semimonotone if and only if A is of the form $A = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$ where $b, c > 0$, $a, d \geq 0$ (resp. $a, d > 0$), and $\det A < 0$ (resp. $\det A \leq 0$).*

Proof. This follows from Theorem 2.9. □

Let us now focus on the 3 × 3 almost (strictly) semimonotone matrices.

Theorem 3.4. *Suppose $A \in \mathbb{R}^{3\times 3}$ with all proper principal submatrices semimonotone. Then A is almost semimonotone if and only if adj$(A) \geq 0$ and $\det A < 0$.*

Proof. ($\Rightarrow$) Suppose $A \in \mathbb{R}^{3\times 3}$ is almost semimonotone. Then A^{-1} exists and $A^{-1} \leq 0$. Thus, since $\operatorname{adj}(A) = (\det A)\, A^{-1}$, adj$(A)$ is either nonnegative or nonpositive. Also, note that clearly one of the 2 × 2 principal submatrices of A must contain at least one negative entry and hence have nonnegative determinant by Theorem 2.9. Thus, $\operatorname{adj}(A) \geq 0$ and $\det A < 0$.

($\Leftarrow$) Suppose $A \in \mathbb{R}^{3\times 3}$ with all proper principal submatrices semimonotone and suppose $\operatorname{adj}(A) \geq 0$ and $\det A < 0$. It follows that $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) \leq 0$. Also note A^{-1} cannot have a zero row since it is invertible. Choose any $x > 0$. Then $A^{-1}x = y < 0$. Thus, $A(-y) = -x < 0$ where $-y > 0$, and so A is not semimonotone. Thus A is almost semimonotone. □

Corollary 3.5. *Suppose $A \in \mathbb{R}^{3\times 3}$ has all proper principal submatrices semimonotone. Then A is semimonotone if and only if adj$(A) \geq 0$ or $\det A \geq 0$.*

Remark 3.6. The above corollary implies that a matrix $A \in \mathbb{R}^{3\times3}$ with all proper principal submatrices semimonotone can have negative determinant and still be semimonotone as long as at least one 2×2 submatrix of A has negative determinant. For example, the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}$$

has negative determinant and all proper principal submatrices semimonotone. Notice that the principal submatrix $A([2,3])$ has negative determinant. It follows that A is semimonotone.

Theorem 3.7. *Suppose $A \in \mathbb{R}^{3\times3}$ with all proper principal submatrices strictly semimonotone. Then A is not semimonotone if and only if $adj(A) > 0$ and* $\det A < 0$.

Proof. The proof is very similar to the proof of Theorem 3.4. □

Remark 3.8. From Theorem 3.4, note that a matrix $A \in \mathbb{R}^{3\times3}$ which is almost semimonotone is almost $\mathbf{P}_0$. Similarly, from Theorem 3.7, note that a matrix $A \in \mathbb{R}^{3\times3}$ which is almost strictly semimonotone but not semimonotone is almost $\mathbf{P}$.

Theorem 3.4 shows that every 3×3 almost semimonotone matrix has negative determinant. This allows us to prove that every 3×3 almost semimonotone matrix has exactly one negative eigenvalue which corresponds to a positive eigenvector. However, before proving this fact, we need the following lemma which holds for a general almost semimonotone matrix.

Lemma 3.9. *If $A \in \mathbb{R}^{n\times n}$ is almost semimonotone, then A^{-1} is irreducible.*

Proof. Suppose A is almost semimonotone and suppose that A^{-1} is reducible. Then there is a permutation matrix P such that $PA^{-1}P^T = \begin{bmatrix} E & F \\ 0 & G \end{bmatrix}$. Thus, $PAP^T = \begin{bmatrix} (E)^{-1} & * \\ 0 & (G)^{-1} \end{bmatrix}$ which must be almost semimonotone since A is almost semimonotone. However, E^{-1} and G^{-1} are permutation similarities of semimonotone principal submatrices of A and are thus semimonotone (see [20, Theorem 4.3]). Thus, PAP^T is semimonotone (see [20, Theorem 4.7]), a contradiction. □

Theorem 3.10. *If $A \in \mathbb{R}^{3\times3}$ is almost semimonotone, then A has exactly one negative eigenvalue corresponding to a positive eigenvector.*

Proof. Suppose $A \in \mathbb{R}^{3\times3}$ is almost semimonotone. By Theorem 2.10 and Theorem 3.9, $A^{-1} \leq 0$ and A^{-1} is irreducible. By the Perron Frobenius Theorem for irreducible nonnegative matrices (see [2]), $-A^{-1}$ has a positive eigenvalue associated with a positive eigenvector. Thus, A has a negative eigenvalue associated with a positive eigenvector. By Theorem 3.4, $\det A < 0$ and so A has an odd number of negative eigenvalues (since all complex eigenvalues must come in conjugate pairs). Since $\operatorname{tr}(A) \geq 0$, A has exactly one negative eigenvalue. □

Adopting the terminology in [16], we make the following definitions.

Definition 3.11. A vector x is *unisigned* provided $x \geq 0$ or $x \leq 0$. Otherwise it is said to be *non-unisigned*.

Definition 3.12. A matrix $A \in \mathbb{R}^{n\times n}$ *reverses the sign* of a vector $x \in \mathbb{R}^n$ if $x_i(Ax)_i \leq 0$ for all $i \in \{1, 2, \ldots, n\}$.

Recall that if A is an almost semimonotone matrix, then there is an $x > 0$ such that $Ax < 0$ and hence A reverses the sign of a positive vector (or a negative vector). One may ask whether or not an almost semimonotone matrix can reverse the sign of a non-unisigned vector. For the 3×3 case, we have the following results.

Theorem 3.13. *If $A \in \mathbb{R}^{3\times3}$ is almost strictly semimonotone but not semimonotone, then A cannot reverse the sign of any non-unisigned vector.*

Proof. Suppose $A \in \mathbb{R}^{3\times3}$ is almost strictly semimonotone but not semimonotone. Then A is almost **P** and thus $A^{-1} < 0$ is an **N**-matrix (see [19, Theorem 3.3]). It follows that A^{-1} does not reverse the sign of any non-unisigned vector (see [18, Theorem 2]). ☐

Corollary 3.14. *If $A \in \mathbb{R}^{3\times3}$ is almost strictly semimonotone but not semimonotone, then SAS is strictly semimonotone for each signature matrix $S \neq \pm I$.*

In the case of 3×3 almost semimonotone matrices, we have a similar result. Let us first make the following definition.

Definition 3.15. A matrix $A \in \mathbb{R}^{n\times n}$ *completely reverses the sign* of a vector $x \in \mathbb{R}^n$ if $x_i(Ax)_i < 0$ for all $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq 0$.

Note that a $\mathbf{P}_0$-matrix cannot completely reverse the sign of any nonzero vector (see [9, Theorem 3.4.2]).

Theorem 3.16. *If $A \in \mathbb{R}^{3\times3}$ is almost semimonotone, then A cannot completely reverse the sign of any non-unisigned vector with a zero entry, and it cannot reverse (or completely reverse) the sign of any non-unisigned vector with no zero entries.*

Proof. Suppose $A \in \mathbb{R}^{3\times3}$ is almost semimonotone. Then A is almost $\mathbf{P}_0$ and thus A cannot completely reverse the sign of any non-unisigned vector with a zero entry. Now suppose that A reverses the sign of some non-unisigned vector v with no zero entries. Without loss of generality, suppose $v = \begin{bmatrix} v_1 \\ v_2 \\ -v_3 \end{bmatrix}$. Then $Av = -Dv$ where $D = \text{diag}(d_1, d_2, d_3)$ has nonnegative diagonal entries. Now note, since A is almost semimonotone, there is an $x > 0$ such that $Ax = -y < 0$. Let c be such that $x + cv \geq 0$ with the first two entries positive and the third entry zero. Then $A(x + cv) = -y + cAv$ has the first two entries negative, a contradiction since $A([1, 2])$ is semimonotone. ☐

Corollary 3.17. *If $A \in \mathbb{R}^{3\times3}$ is almost semimonotone, then SAS is semimonotone for each signature matrix $S \neq \pm I$.*

4 General almost semimonotone matrices

As shown in the previous section, if $A \in \mathbb{R}^{n\times n}$ is an almost strictly semimonotone matrix which is not semimonotone and $n = 2$ or $n = 3$, then A has the following properties:
(1) A is almost **P**.
(2) A cannot reverse the sign of any non-unisigned vector.
(3) SAS is strictly semimonotone for each signature matrix $S \neq \pm I$.
(4) $\det A < 0$ and $\text{adj}(A) > 0$.
(5) A has exactly one negative eigenvalue corresponding to a positive eigenvector.
Also if $A \in \mathbb{R}^{n\times n}$ is an almost semimonotone matrix with $n = 2$ or $n = 3$, then A has the following properties:
(1) A is almost $\mathbf{P}_0$.
(2) A cannot completely reverse the sign of any non-unisigned vector.
(3) SAS is semimonotone for each signature matrix $S \neq \pm I$.
(4) $\det A < 0$ and $\text{adj}(A) \geq 0$.
(5) A has exactly one negative eigenvalue corresponding to a positive eigenvector.

This leads one to wonder if every almost (strictly) semimonotone matrix has these properties. It turns out, however, that properties (1) - (3) do *not* hold for 4×4 and larger almost (strictly) semimonotone matrices. A counterexample is given by

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 1 & 1 & -4 & 6 \\ -1 & 1 & 2 & -4 \\ -1 & -3 & 1 & 1 \end{bmatrix}.$$

All proper principal submatrices of A are $\mathbf{P}_0$ except $A([1,2])$ which has negative determinant. Since $A([1,2])$ has positive entries, it belongs to $\mathbf{E}$. Thus, all proper principal submatrices of A are in $\mathbf{E_0}$. Now note A is almost semimonotone since A^{-1} exists and $A^{-1} \leq 0$. However, A is clearly not almost $\mathbf{P}_0$ since $\det A([1,2]) < 0$. Also, note that A reverses the sign of the non-unisigned vector $x = \begin{bmatrix} 27 & -15 & 1 & -1 \end{bmatrix}^T$ since $Ax = \begin{bmatrix} -3 & 2 & -36 & 18 \end{bmatrix}^T$. Notice in the previous example, the proper principal submatrices of A were not all in $\mathbf{E}$, leading one to perhaps wonder if properties (1), (2), and (3) may be true if we require the proper principal submatrices to be in $\mathbf{E}$. However, these properties would still not always hold. Consider $A + \epsilon I$ where $\epsilon = 0.1$. This matrix has all proper principal submatrices in $\mathbf{E}$, yet one can show properties (1), (2), and (3) do not hold.

A counterexample has not yet been found to disprove properties (4) and (5) for larger matrices, nor has a proof of this result been found. We conjecture that the following result holds.

Conjecture 4.1. *Suppose $A \in \mathbb{R}^{n\times n}$ is almost semimonotone. Then* $\det A < 0$ *and adj*$(A) \geq 0$. (*In the case when A is almost strictly semimonotone but not semimonotone, adj*$(A) > 0$.) *Further, A has exactly one negative eigenvalue (of multiplicity one) which corresponds to a positive eigenvector.*

Although we cannot (dis)prove the above conjecture at this time, we can prove the following facts.

Theorem 4.2. *If $A \in \mathbb{R}^{n\times n}$ is almost semimonotone, then A has a negative eigenvalue (of multiplicity one), which is the smallest eigenvalue of A in absolute value, associated with a positive eigenvector. That is, $Ax = -\lambda x$ where $x > 0$ and $\lambda = \min_i\{|\lambda_i| : \lambda_i \in \sigma(A)\}$. Also, there are no other nonnegative eigenvectors of A.*

Proof. We will prove the result in the case when A is almost semimonotone; the case when A is almost strictly semimontone and not semimonotone is similar. Since A is almost semimonotone, $A^{-1} \leq 0$. Thus, $-A^{-1} \geq 0$. Also note $-A^{-1}$ is irreducible from Lemma 3.9. Thus, by the Perron Frobenius Theorem, the spectral radius of $-A^{-1}$ is an eigenvalue corresponding to a positive eigenvector, and there are no other nonnegative eigenvectors of $-A^{-1}$. So, $-A^{-1}x = \frac{1}{\lambda}x$ where $x > 0$ and $\frac{1}{\lambda}$ is the largest eigenvalue of $-A^{-1}$ in absolute value. Thus, $Ax = -\lambda x$, where λ is the smallest eigenvalue of A in absolute value, and A has no other nonnegative eigenvectors. □

Although it has been shown that an almost semimonotone matrix can reverse the sign of a non-unisigned vector, it turns out to be the case that the following is true.

Theorem 4.3. *Suppose A is almost semimonotone or almost strictly semimonotone and not semimonotone. Then A cannot reverse the sign of any non-unisigned vector x with exactly one positive entry, or exactly one negative entry, and no zero entries.*

Proof. Without loss of generality, we will prove that A cannot reverse the sign of the non-unisigned vector

$$v = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_{n-1} \\ v_n \end{bmatrix}, \qquad v_1, v_2, \dots, v_n > 0.$$

Suppose A reverses the sign of v. That is, suppose

$$Av = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ -y_n \end{bmatrix}, \qquad y_1, y_2, \dots, y_n \geq 0.$$

Now, since A is almost semimonotone, there is an $x > 0$ such that $Ax = -z < 0$. Note there exists a $c > 0$ such that $x - cv$ has all its first $n-1$ entries positive and its nth entry zero. Thus, since $A[1, 2, \dots, n-1]$ is semimonotone, one of the first $n-1$ entries of $A(x - cv)$ must be nonnegative. However, note that

$$A(x - cv) = \begin{bmatrix} -z_1 - cy_1 \\ -z_2 - cy_2 \\ \vdots \\ -z_{n-1} - cy_{n-1} \\ * \end{bmatrix}$$

all of whose entries are negative except possibly the nth entry, a contradiction. □

One of the biggest challenges with semimonotone and almost semimonotone matrices is the problem of construction and detection. It has been shown (see [22]) that the problem of deciding whether an arbitrary square matrix with integer or rational entries is semimonotone or strictly semimonotone is co-NP-complete, similar to how the analogous decision problem for **P**-matrices is co-NP-complete (see [6]). Recently, the problem of constructing generic **P**-matrices has been solved (see [21]). Finding a characterization for (strictly) semimonotone matrices, as well as for the almost (strictly) semimonotone matrices, that would allow an efficient method for construction/detection remains an open problem.

Here we describe one possible method of construction for semimonotone and almost (strictly) semimonotone matrices which involves starting with a random matrix A and adding positive multiples of the identity to the principal submatrices of A until all principal submatrices of A are semimonotone. However, this method relies on looking at $2^n - 1$ different principal submatrices and is thus quite computationally intensive.

1. Choose an arbitrary real $n \times n$ matrix $A = [a_{ij}]$.
2. Replace a_{ii} with $(a_{ii} + \max\{-a_{ii}, 0\})$ for each $i = 1, 2, \dots, n$ so that we have added just enough to each diagonal entry to make the diagonal entries nonnegative.
3. Then look at each submatrix $A[\alpha]$ where $|\alpha| = 2$. If a principal submatrix $A[\alpha]$ has a positive eigenvector corresponding to a negative eigenvalue $-\lambda$, we replace $A[\alpha]$ with $(A[\alpha] + \lambda I)$ and then go on to check the next 2×2 principal submatrix. After this step is completed, all 1×1 and 2×2 principal submatrices of A are semimonotone.
4. Proceed in this fashion for the 3×3 principal submatrices of A and so on until we have replaced each $(n-1) \times (n-1)$ principal submatrix of A with new principal submatrices which are semimonotone. If A has a positive eigenvector corresponding to a negative eigenvalue $-\lambda$, A is almost semimonotone and $A + \lambda I$ is semimonotone. If not, A is already semimonotone.

5 Central matrices

A matrix $A \in \mathbb{R}^{m\times n}$ is *(strict) central* if there exists a nonzero nonnegative (positive) vector $x \in \mathbb{R}^n$ such that $Ax = 0$. The *sign matrix* of a matrix A is a matrix with entries in $\{0, -1, 1\}$ obtained from A by replacing each entry by its sign. The *qualitative class* of a matrix A consists of all matrices with the same sign matrix as A. We say that a matrix A is *(strict) sign-central* if each matrix in the qualitative class of A is (strict) central. The notion of sign-central matrices were discussed in [1] and [4] while strict central and strict sign-central matrices were investigated in [3]. Tight sign-central matrices, an extension of sign-central matrices, were discussed in [15]. In this section, we are mostly concerned with (strict) central matrices, rather than (strict) sign-central matrices. In particular, we are interested in exploring connections between semimonotone matrices and (strict) central matrices.

5.1 Basic facts and properties about (strict) central matrices

Note that every square central or strict central matrix must be singular. Also, if a matrix A is strict central, then each row must either be zero or have at least one positive and at least one negative entry. Some authors (e.g. [1]) refer to such vectors as *balanced*. Thus, if A is a strict central matrix, then each row of A must be balanced. If A consists of a single row, then A is strict central if and only if its row is balanced.

Let us next recall Stiemke's Theorem (see e.g. [9]) which can be stated in the following way:

Theorem 5.1. *Given $A \in \mathbb{R}^{m\times n}$, the system $Ax = 0$, $x > 0$ has a solution if and only if the system $A^Ty \geq 0$, $A^Ty \neq 0$ has no solution.*

The following lemma is a direct consequence of Stiemke's Theorem:

Lemma 5.2. *If $A \in \mathbb{R}^{m\times n}$ is strict central, then $A^Tx \geq 0$ implies $A^Tx = 0$.*

We will also need the notion of minimal semipositivity, defined next.

Definition 5.3. A matrix A is *minimally semipositive* if A is semipositive and no column-deleted submatrix of A is semipositive.

In [11, Theorem 3.4], it is proven that a matrix $A \in \mathbb{R}^{n\times n}$ is minimally semipositive if and only if A^{-1} exists and $A^{-1} \geq 0$. We use the same terminology as in [10] to define a notion similar to minimal semipositivity for $\mathbf{S_0}$-matrices.

Definition 5.4. A matrix $A \in \mathbb{R}^{m\times n}$ is *irreducibly* $\mathbf{S_0}$ if A is $\mathbf{S_0}$ and no column-deleted submatrix of A is $\mathbf{S_0}$.

As shown in [10, Theorem 3.12], a matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular and irreducibly $\mathbf{S_0}$ if and only if it is inverse positive.

Similar to how we define minimally semipositive matrices and irreducibly $\mathbf{S_0}$-matrices, we now define minimally strict central matrices in the following way.

Definition 5.5. A *minimally strict central matrix* is a strict central matrix $A \in \mathbb{R}^{m\times n}$ such that each matrix obtained from A by the deletion of a column is not strict central.

Note that a minimally strict central matrix A has the property that there exists no nonzero nonnegative vector x with at least one zero entry such that $Ax = 0$. In other words, A is strict central but otherwise not central. Thus, no proper principal submatrix of a minimally strict central matrix can be strict central. Also, note there can be no such thing as a “minimally central" matrix which is not minimally strict central.

Theorem 5.6. *Let $A \in \mathbb{R}^{m\times n}$. If A or $-A$ is irreducibly $\mathbf{S_0}$, then any column-deleted submatrix of A (or $-A$) is not strict central. In addition, if A is square and nonsingular, then any row-deleted submatrix of A (or $-A$) is strict central.*

Proof. Let A or $-A \in \mathbb{R}^{m\times n}$ be irreducibly $\mathbf{S_0}$. Suppose a column-deleted submatrix of A (or $-A$) is strict central. But then this column-deleted submatrix is $\mathbf{S_0}$, a contradiction since A or $-A$ is irreducibly $\mathbf{S_0}$.

Now suppose A is also square and nonsingular. Then $A^{-1} > 0$ or $A^{-1} < 0$. WLOG, suppose $A^{-1} > 0$. Thus, if b_i is the ith column of A^{-1}, then $b_i > 0$ and $Ab_i = e_i$ where e_i is the ith column of the $n \times n$ identity matrix. Thus, the submatrix obtained from A by deleting the ith row of A is strict central. Since this holds for any i, any row-deleted submatrix of A (or $-A$) is strict central. □

From this theorem, we can write down a corollary, although it is an extremely obvious result.

Corollary 5.7. *If $A \in \mathbb{R}^{m\times n}$ is irreducibly $\mathbf{S_0}$ and strict central, then A is minimally strict central.*

We also have the following facts. The proofs are obvious and omitted.

Theorem 5.8. *If $A \in \mathbb{R}^{n\times n}$ is a central matrix, then $A + D$ is $\mathbf{S_0}$ for any diagonal matrix D with nonnegative diagonal entries.*

Theorem 5.9. *If $A \in \mathbb{R}^{n\times n}$ is strict central, then $A + D$ is semipositive for any diagonal matrix D with positive diagonal entries.*

5.2 Central matrices in connection with semimonotone matrices

First, notice that any strictly semimonotone matrix cannot be central. Indeed, if $A \in \mathbb{R}^{n\times n}$ is a central matrix, then there is a $0 \neq x \geq 0$ such that $Ax = 0$, which would imply that a principal submatrix of A is not strictly semimonotone. However, it is certainly the case that a semimonotone matrix can be central. For example, the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is semimonotone and strict central.

It becomes interesting to examine the matrices which belong to $\mathbf{E_0} \setminus \mathbf{E}$ since all semimonotone matrices which are central lie in this class. Not all matrices in $\mathbf{E_0} \setminus \mathbf{E}$ are central, however. For example,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is in the class $\mathbf{E_0} \setminus \mathbf{E}$ but is clearly not central.

Theorem 5.10. *Let $A \in \mathbb{R}^{n\times n}$. Then $A \in \mathbf{E_0} \setminus \mathbf{E}$ if and only if $A \in \mathbf{E_0}$ and a principal submatrix of A is strict central.*

Proof. ($\Rightarrow$) Suppose $A \in \mathbf{E_0} \setminus \mathbf{E}$. Then A is semimonotone and there exists a $0 \neq y \geq 0$ such that for each $y_k > 0$, $(Ay)_k = 0$. Now let $\alpha = \text{supp}(y)$, where $\text{supp}(y) = \{i \mid y_i \neq 0\}$ denotes the support of the vector y. Then $y[\alpha] > 0$ and $A[\alpha]y[\alpha] = 0$, and so we see that $A[\alpha]$ is a strict central matrix.

($\Leftarrow$) Now suppose that $A \in \mathbf{E_0}$ and $A[\alpha]$ is strict central where $\alpha \subseteq \{1, 2, \ldots, n\}$. Then there is an $x[\alpha] > 0$ such that $A[\alpha]x[\alpha] = 0$. Thus, $A[\alpha] \notin \mathbf{E}$ and hence $A \notin \mathbf{E}$. Thus, $A \in \mathbf{E_0} \setminus \mathbf{E}$. □

Corollary 5.11. *Let $A \in \mathbb{R}^{n\times n}$. Then $A \in \mathbf{E}$ if and only if $A \in \mathbf{E_0}$ and no principal submatrix of A is strict central.*

Note that A is central if and only if A or a proper principal submatrix of A is strict central. Hence, we can also express the above corollary as follows.

Corollary 5.12. *Let $A \in \mathbb{R}^{n\times n}$. Then $A \in \mathbf{E}$ if and only if $A \in \mathbf{E_0}$ and A is not central.*

The following result also follows from Theorem 5.10.

Corollary 5.13. *If $A \in \mathbb{R}^{n\times n}$ is almost $\mathbf{E}$, then $A \in \mathbf{E_0} \setminus \mathbf{E}$ if and only if A is strict central.*

The next result shows that every semimonotone matrix which is almost $\mathbf{P}$ and has negative determinant must be strictly semimonotone.

Theorem 5.14. *If $A \in \mathbb{R}^{n\times n}$ is semimonotone and almost $\mathbf{P}$ but not $\mathbf{P}_0$, then $A \in \mathbf{E}$.*

Proof. Suppose $A \in \mathbb{R}^{n\times n}$ is semimonotone and almost $\mathbf{P}$ but not $\mathbf{P}_0$, and suppose $A \notin \mathbf{E}$. By Corollary 5.13, A is strict central. Hence $\det A = 0$ and A is $\mathbf{P}_0$, a contradiction. □

Now we recall the following theorem which was proven in [20, Theorem 4.8].

Theorem 5.15. *Suppose $A \in \mathbb{R}^{n\times n}$ with all proper principal submatrices (strictly) semimonotone. Then A is (strictly) semimonotone if and only if for all diagonal matrices $D \in \mathbb{R}^{n\times n}$ with positive (nonnegative) diagonal entries, $A + D$ does not have a positive nullvector.*

From Theorem 5.15, one can easily prove the following.

Theorem 5.16. *Suppose $A \in \mathbb{R}^{n\times n}$ has all proper principal submatrices (strictly) semimonotone. Then A is (strictly) semimonotone if and only if for all diagonal matrices D with positive (nonnegative) diagonal entries, $A + D$ is not strict central.*

We also have the following result.

Theorem 5.17. *Suppose $A \in \mathbb{R}^{n\times n}$ has all proper principal submatrices (strictly) semimonotone. Then A is (strictly) semimonotone if and only if there exists a diagonal matrix D with nonnegative (positive) diagonal entries such that $A - D$ is (strict) central.*

Proof. (⇒) Suppose A is (strictly) semimonotone. Then A is $\mathbf{S_0}$ (resp. semipositive). Thus, there exists a $0 \neq x \geq 0$ (resp. $x > 0$) such that $Ax = Dx \geq 0$ (resp. $Ax = Dx > 0$) where D is a diagonal matrix with nonnegative (resp. positive) diagonal entries. Then $(A - D)x = 0$. Thus, $A - D$ is (strict) central.

(⇐) Suppose A has all proper principal submatrices (strictly) semimonotone and suppose that there exists a diagonal matrix D with nonnegative (positive) diagonal entries such that $A - D$ is (strict) central. Then there is a $0 \neq x \geq 0$ (resp. $x > 0$) such that $(A - D)x = 0$. Thus, $Ax = Dx \geq 0$ (resp. $Ax = Dx > 0$), and hence A is $\mathbf{S_0}$ (resp. semipositive). Thus, A is (strictly) semimonotone. □

Next, we show that every matrix $A \in \mathbf{E_0} \setminus \mathbf{E}$ with all proper principal submatrices belonging to $\mathbf{E}$ is minimally strict central.

Theorem 5.18. *Suppose $A \in \mathbb{R}^{n\times n}$ is almost $\mathbf{E}$ and $A \in \mathbf{E_0} \setminus \mathbf{E}$. Then A and A^T are minimally strict central.*

Proof. Suppose $A \in \mathbb{R}^{n\times n}$ is almost $\mathbf{E}$ and $A \in \mathbf{E_0} \setminus \mathbf{E}$. Then so is A^T. Thus, both $-A$ and $-A^T$ are irreducibly $\mathbf{S_0}$ (see [20, Lemma 5.5]) and strict central. Thus, by Corollary 5.7, $-A$ and $-A^T$ are minimally strict central. The result follows. □

From Lemma 5.2, we can prove the following.

Theorem 5.19. *Suppose $A \in \mathbf{E_0}$ and that A is strict central. Then A^T is central.*

Proof. Suppose $A \in \mathbf{E_0}$ and that A is strict central. Since $A^T \in \mathbf{E_0}$, there is a $0 \neq x \geq 0$ such that $A^T x \geq 0$. By Lemma 5.2, $A^T x = 0$. Thus, A^T is central. □

Remark 5.20. It is not necessarily the case that if $A \in \mathbf{E_0}$ and A is strict central, then A^T is strict central. For example, consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Note that $A \in \mathbf{E_0}$ and is strict central. However, A^T is not strict central.

Now let's recall Ville's Theorem (see e.g. [9]).

Theorem 5.21. *Given $A \in \mathbb{R}^{m \times n}$, the system $Ax > 0$, $x > 0$ has a solution if and only if the system $A^T y \leq 0$, $0 \neq y \geq 0$, has no solution.*

We can now prove the following fact.

Theorem 5.22. *Suppose $A \in \mathbf{E_0}$ and A is strict central. Then A is weakly semipositive but not semipositive.*

Proof. Suppose $A \in \mathbf{E_0}$ and A is strict central. Suppose A is semipositive. Then, by Ville's Thoerem, $A^T y \leq 0$, $0 \neq y \geq 0$, has no solution. Hence, A^T cannot be central, a contradiction from Theorem 5.19. □

Note we do need the condition that A is semimonotone in the above theorem. For example, the matrix $A = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$, which is not semimonotone, is both strict central and semipositive.

We also have the following result in the case of almost $\mathbf{E}$ and $\mathbf{E_0}$ matrices.

Theorem 5.23. *Suppose $A \in \mathbb{R}^{n \times n}$ is almost $\mathbf{E}$ and strict central. Then if $x \neq 0$ and $Ax \geq 0$, then $Ax = 0$ and $x > 0$ or $x < 0$.*

Proof. Suppose $A \in \mathbb{R}^{n \times n}$ is almost $\mathbf{E}$ and strict central. Then A^T is almost $\mathbf{E}$ and A^T is strict central. Now, suppose $x \neq 0$ and $Ax \geq 0$. By Lemma 5.2, $Ax = 0$. Since A is almost $\mathbf{E}$ and singular, it follows (see [20, Theorem 5.8]) that $\operatorname{adj}(A)$ is either positive or negative and hence $\operatorname{rank}(A) = n - 1$. Thus, $x > 0$ or $x < 0$. □

We end this section by observing some connections between almost semimonotone matrices and central matrices. First, note that any almost semimonotone matrix cannot be central since an almost semimonotone matrix is invertible. However, proper principal submatrices of almost semimonotone matrices can certainly be central, as long as the matrix is not almost strictly semimonotone. The following theorem follows from Theorem 5.10.

Theorem 5.24. *Suppose $A \in \mathbb{R}^{n \times n}$ is almost semimonotone. Then A is almost strictly semimonotone if and only if no principal submatrix of A is central.*

The next two results show that any row-deleted submatrix of an almost $\mathbf{E_0}$ (almost $\mathbf{E}$ and not $\mathbf{E_0}$) matrix is (strict) central.

Theorem 5.25. *If $A \in \mathbb{R}^{n \times n}$ is almost semimonotone, then the submatrix obtained from A by deleting any row from A is central.*

Proof. Suppose $A \in \mathbb{R}^{n \times n}$ is almost semimonotone. Without loss of generality, we will show that the submatrix obtained by deleting the first row of A is central. Notice that, since A^{-1} exists and $A^{-1} \leq 0$, for any

$y = \begin{bmatrix} y_1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T$ where $y_1 < 0$, $x = A^{-1}y$ is nonnegative and nonzero. Thus, $Ax = y$ where $0 \neq x \geq 0$ and y has zero entries except for the first entry. It follows that the submatrix obtained by deleting the first row of A is central. □

Theorem 5.26. *If $A \in \mathbb{R}^{n\times n}$ is almost strictly semimonotone but not semimonotone, then the submatrix obtained from A by deleting any row from A is strict central.*

Proof. The proof is similar to the proof of Theorem 5.25. □

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