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On identities involving generalized harmonic, hyperharmonic and special numbers with Riordan arrays

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Abstract: In this paper, by means of the summation property to the Riordan array, we derive some identities involving generalized harmonic, hyperharmonic and special numbers. For example, for $n \geq 0$,

$$\sum_{k=0}^{n} \frac{B_k}{k!} H(n,k,\alpha) = \alpha H(n+1,1,\alpha) - H(n,1,\alpha),$$

and for $n > r \geq 0$,

$$\sum_{k=r}^{n-1} (-1)^k \frac{s(k,r)\, r!}{\alpha^k k!} H_{n-k}(\alpha) = (-1)^r H(n,r,\alpha),$$

where Bernoulli numbers B_n and Stirling numbers of the first kind $s(n,r)$.

Keywords: the generalized harmonic number, Riordan arrays, Stirling number

MSC: 11B99, 11C20, 15A23

1 Introduction

The harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ frequently arise in combinatorial problems and in various branches of number theory. These numbers have been generalized by several authors. One of them is the generalized harmonic numbers $H_n(\alpha)$ [5] defined by, for every ordered pair $(\alpha, n) \in \mathbb{R}^+ \times \mathbb{N}$,

$$H_0(\alpha) = 0, \quad H_n(\alpha) = \sum_{k=1}^{n} \frac{1}{k\alpha^k},$$

and the generating function of these numbers is

$$\sum_{n=1}^{\infty} H_n(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{1-x}. \tag{1.1}$$

In [9], the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ are defined by

$$H_n^r(\alpha) = \sum_{k=1}^{n} H_k^{r-1}(\alpha), \quad n, r \geq 1,$$

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where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$ and $H_n^r(\alpha) = 0$ for $r < 0$ or $n \le 0$. The generating function of these numbers is

$$\sum_{n=1}^{\infty} H_n^r(\alpha)x^n = -\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r}. \tag{1.2}$$

In [3], the generalized harmonic numbers of rank r, $H(n, r, \alpha)$ are defined as for $n \ge 1$ and $r \ge 0$,

$$H(n, r, \alpha) = \sum_{1 \le n_0+n_1+\ldots+n_r \le n} \frac{1}{n_0 n_1 \ldots n_r \alpha^{n_0+n_1+\ldots+n_r}}$$

or equivalently, as

$$H(n, r, \alpha) = \frac{(-1)^{r+1}}{n!}\left(\frac{d^n}{dx^n}\frac{\left[\ln\left(1-\frac{x}{\alpha}\right)\right]^{r+1}}{1-x}\right)\Bigg|_{x=0}.$$

The generating function of the generalized harmonic numbers of rank r, $H(n, r, \alpha)$ is given by

$$\sum_{n=r+1}^{\infty} H(n, r, \alpha)\, x^n = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}. \tag{1.3}$$

For $\alpha = 1$, $H(n, r, 1) = H(n, r)$ were introduced in their works [6, 10].

The Bernoulli numbers B_n are defined as

$$B_0 = 0, \quad (n+1)\,B_n = -\sum_{k=0}^{n-1}\binom{n+1}{k}B_k,$$

and the generating function of the Bernoulli numbers is

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}. \tag{1.4}$$

The Daehee numbers of order r, showed by D_n^r, are defined by the generating functions to be

$$\left(\frac{\ln(1+x)}{x}\right)^r = \sum_{n=0}^{\infty} D_n^r \frac{x^n}{n!}. \tag{1.5}$$

For $r = 1$, $D_n^1 = D_n$ is Daehee numbers. It is clear that

$$D_0 = 1, D_1 = -\frac{1}{2}, \ldots, D_n = (-1)^n \frac{n!}{n+1}.$$

$x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{n}} = x(x-1)\ldots(x-n+1)$ for $n \ge 1$ and $x^{\underline{0}} = 1$.

It is well known that Stirling numbers play an important role in combinatorial analysis. The Stirling numbers of the first kind $s(n, k)$ and of the second kind $S(n, k)$ are given by

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S(n,k)x^{\underline{k}},$$

where for $n \ge 0$, $s(n, 0) = S(n, 0) = \delta_{n0}$, δ_{nk} is the Kronecker delta. Recurrence relations of these numbers are given by

$$\begin{aligned} s(n+1,k) &= s(n,k-1) + ns(n,k), \\ S(n+1,k) &= S(n,k-1) + kS(n,k), \end{aligned}$$

respectively, where $n \ge 0$ and $k > 0$.

The generating functions of these numbers are given by

$$\sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!} = \frac{(\ln(1+x))^k}{k!} \tag{1.6}$$

and

$$\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}, \tag{1.7}$$

respectively.

The signed Stirling numbers of the first kind $|s(n,k)|$ are defined such that the number of permutations of n elements which contain exactly k permutation cycles is the nonnegative number

$$|s(n,k)| = (-1)^{n-k}s(n,k).$$

This means that $s(n,k)=0$ for $k>n$ and $s(n,n)=1$ and the generating function of these numbers [1, 7, 15] is given by

$$\sum_{n=k}^{\infty} |s(n,k)|\frac{x^n}{n!} = \frac{(-\ln(1-x))^k}{k!}.$$

The numbers associated with $s(n,k)$ are given as follows: For $n<k$,

$$\rho(n,k) = \frac{|s(k,k-n)|}{\binom{k-1}{n}},$$

and for $n \geq k$,

$$\rho(n,k) = n!k\sigma_n(k),$$

where $\sigma_n(x)$ is Stirling polynomial [7]. The generating function of these numbers is

$$\sum_{n=0}^{\infty} \frac{\rho(n,k)}{n!}x^n = \left(\frac{x}{1-e^{-x}}\right)^k.$$

Recently, by using the concepts of Riordan arrays, there are some identities related to special numbers and binomial coefficients [1, 13, 14].

Let $f(x)$ be a formal power series in the determinate. $f(x)$ has the form

$$f(x) = \sum_{n=0}^{\infty} f_n x^n.$$

Riordan array is an infinite, lower triangular array $R=(g(x),f(x))=[r_{n,k}]_{n,k\geq 0}$ defined by a pair of generating functions $g(x)$ and $f(x)$ such that

$$r_{n,k} = [x^n]\, g(x)(xf(x))^k, \tag{1.8}$$

where $[x^n]$ denotes the coefficient of x^n in $f(x)$. If $g(0)\neq 0, f(0)\neq 0$, Riordan array is called proper, otherwise it is called improper. The set of proper Riordan arrays is denoted by $\mathcal{R}$ and the set of improper Riordan arrays is denoted by $\mathcal{R}'$. It is known [11] that $\langle \mathcal{R}, *\rangle$ forms a group under matrix multiplication $*$ with the identity $I=(1,1)$:

$$(g(x),f(x))*(h(x),l(x)) = (g(x)h(xf(x)), f(x)l(xf(x))). \tag{1.9}$$

Basically, the concept of a Riordan array is used in a constructive way to find the generating function of many combinatorial sums. The summation property for a Riordan array $R=(g(x),f(x))=[r_{n,k}]_{n,k\geq 0}$ is

$$\sum_{k=0}^{n} r_{n,k}h_k = [x^n]\, g(x)h(xf(x)), \tag{1.10}$$

where $h(x)=\sum_{n=0}^{\infty} h_n x^n$.

Riordan arrays have special structure. If $R=(g(x),f(x))=[r_{n,k}]_{n,k\geq 0}$ is a proper Riordan array, then every element $r_{n+1,k+1}$ of R can be expressed as a linear combination of the elements in preceding row starting from the preceding column, and every element in column 0 can be expressed as a linear combination of all elements of the preceding row [8]:

$$
\begin{aligned}
r_{n+1,k+1} &= \sum_{j=0}^{\infty} a_j r_{n,k+j}, \quad n,k=0,1,\ldots,\\
r_{n+1,0} &= \sum_{j=0}^{\infty} z_j r_{n,j}, \quad n=0,1,\ldots.
\end{aligned}
$$

The coefficients $a_0,\ a_1,\ a_2,\ldots$ and $z_0,\ z_1,\ z_2,\ldots$ are called by the A–sequence and Z–sequence of Riordan array, respectively. If $A(x)$ and $Z(x)$ are the generating functions of corresponding sequences, then it can be proven that $f(x)$ and $g(x)$ are solutions of the functional equations, respectively:

$$
\begin{aligned}
f(x) &= A(xf(x)),\\
g(x) &= \frac{g(0)}{1-xZ(xf(x))}.
\end{aligned}
$$

The relations can be inverted to formulas for the A–sequence and Z–sequence, respectively.

From special cases, it is seen that the Riordan array method is powerful for dealing with harmonic numbers identities.

In [13], the author obtained some identities involving binomial coefficients and harmonic numbers by Riordan arrays. For example, for $n \geq 0$,

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}^2 H_k &= \binom{2n}{n}\left(2H_n - H_{2n}\right),\\
\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k} kH_k &= m\binom{n+m-1}{n+1}\left(H_n - H_{n+m-1} + H_{m-2}\right).
\end{aligned}
$$

In [14], the author established several general summation formulas, from which series of harmonic number identities are obtained. For example,

$$
\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k} H_{k+r}^2 &= \frac{1}{n}\left(2H_{n-1} - H_{n+r} - H_r\right)\binom{n+r}{r}^{-1},\\
\sum_{k=0}^{n}\binom{n+r-k-1}{n-k}\binom{k+s}{k} H_{k+s} &= \binom{n+r+s}{r+s}\left(H_{n+r+s} - H_{r+s} + H_s\right).
\end{aligned}
$$

In [1], the authors obtained many relations between Stirling numbers of both kinds and other generalized harmonic numbers. For example, for any positive integers n and r,

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!} H(n+1,k) &= H_n,\\
\sum_{k=r}^{n-1}\frac{r!}{k!} s(k,r) H_{n-k}^{k+1} &= H(n,r).
\end{aligned}
$$

In [2], the authors established some new identities involving Stirling numbers of both kinds. These identities were obtained via Riordan arrays with nonzero real number x. Some well-known identities were obtained as special cases of the new identities for nonzero real number x. For example, for nonnegative integer r,

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k} k^r x^{n-k} &= \sum_{k=0}^{r} k!\binom{n}{k} S(r,k)(1+x)^{n-k},\\
S(r,n) &= \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} k^r.
\end{aligned}
$$

2 Some identities with Riordan arrays

In this section, we establish some identities related to generalized harmonic numbers of rank r, $H(n, r, \alpha)$, generalized hyperharmonic numbers $H_n^r(\alpha)$ and Stirling numbers of both kinds via Riordan arrays.

Firstly, by (1.3), we can consider

$$r_{n,k} = H(n, k, \alpha) \text{ and } R = \left(\frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x}, \frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{x} \right) \in \mathcal{R}'. \tag{2.1}$$

Thus, to find the following identities in Theorem 1 and Theorem 2, we will apply the summation property (1.10) to the Riordan array.

Theorem 1. *For any positive integer n, we have*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} H(n, k, \alpha) = H_n(\alpha) - \frac{1}{\alpha} H_{n-1}(\alpha)$$

and

$$\sum_{k=0}^{n} \frac{B_k}{k!} H(n, k, \alpha) = \alpha H(n+1, 1, \alpha) - H(n, 1, \alpha).$$

Proof. Taking $h(x) = e^{-x}$ in (1.10), we have

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-1)^k}{k!} H(n, k, \alpha) &= \left[x^n\right] \frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x} e^{-\left(-\ln\left(1 - \frac{x}{\alpha}\right)\right)} \\ &= \left[x^n\right] \frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x} \left(1 - \frac{x}{\alpha}\right) \\ &= \left[x^n\right] \frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x} - \frac{1}{\alpha} \left[x^{n-1}\right] \frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x}, \end{aligned}$$

and from (1.1), equals

$$\left[x^n\right] \sum_{n=0}^{\infty} H_n(\alpha) x^n - \frac{1}{\alpha} \left[x^{n-1}\right] \sum_{n=0}^{\infty} H_n(\alpha) x^n = H_n(\alpha) - \frac{1}{\alpha} H_{n-1}(\alpha).$$

Thus, the proof of the first equality is complete. With the help of (1.4), the other equality is easily given. □

Theorem 2. *Let n, m be nonnegative integers. For $n \geq m$,*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} S(k, m) H(n, k, \alpha) = \frac{H_{n-m}(\alpha)}{(-\alpha)^m m!},$$

and

$$\sum_{k=0}^{n} H(n, k, \alpha) \frac{\rho(k, m)}{k!} = \alpha^m H(n+m, m, \alpha).$$

Proof. Considering $h(x) = \left(e^{-x} - 1\right)^m$ in (1.10), we get

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-1)^k}{k!} m! S(k, m) H(n, k, \alpha) &= \left[x^n\right] \frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x} \left(-\frac{x}{\alpha}\right)^m \\ &= \left(-\frac{1}{\alpha}\right)^m \left[x^{n-m}\right] \frac{-\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x}. \end{aligned}$$

By (1.1), we have

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} m! S(m, k) H(n, k, \alpha) = \left(-\frac{1}{\alpha}\right)^m \left[x^{n-m}\right] \sum_{n=0}^{\infty} H_n(\alpha) x^n$$

$$= \frac{H_{n-m}(\alpha)}{(-\alpha)^m}.$$

Thus, the desired result is obtained. For the proof of the other identity, putting $h(x) = \left(\frac{x}{1-e^{-x}}\right)^m$ in (1.10), we get

$$\begin{aligned}\sum_{k=0}^{n} H(n,k,\alpha)\frac{\rho(k,m)}{k!} &= [x^n]\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{\frac{x}{\alpha}}\right)^m \\ &= \alpha^m\left[x^{n+m}\right]\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{m+1}}{1-x}.\end{aligned}$$

By (1.3), we have

$$\begin{aligned}\sum_{k=0}^{n} H(n,k,\alpha)\frac{\rho(k,m)}{k!} &= \alpha^m\left[x^{n+m}\right]\sum_{n=0}^{\infty} H(n,m,\alpha)x^n \\ &= \alpha^m H(n+m,m,\alpha),\end{aligned}$$

as claimed result. □

By (1.6), we have $r_{n,k} = \frac{(-1)^{n-k}s(n,k)k!}{\alpha^n n!}$ and $R = \left(1, \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{x}\right) \in \mathcal{R}$. Thus, to find the following identities in Theorem 3, we will apply the summation property (1.10) to the Riordan array.

Theorem 3. *For positive integers n, m, we have*

$$\sum_{k=0}^{n}(-1)^k s(n,k)B_k = D_n + nD_{n-1},$$

and

$$\sum_{k=0}^{n}(-1)^k s(n,k)\rho(k,m) = D_n^m.$$

Proof. With the help of the $h(x) = \frac{x}{e^x-1}$ in (1.10) and the generating function of Daehee numbers, we can get

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(-1)^{n-k}s(n,k)k!}{\alpha^n n!}\frac{B_k}{k!} \\ &= [x^n]\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{\frac{x}{\alpha-x}} = \frac{1}{\alpha}[x^n]\frac{\ln\left(1-\frac{x}{\alpha}\right)}{-\frac{x}{\alpha}}(\alpha-x) \\ &= \frac{1}{\alpha}\left\{[x^n]\,\alpha\sum_{n=0}^{\infty}\frac{(-1)^n D_n}{\alpha^n n!}x^n - \left[x^{n-1}\right]\sum_{n=0}^{\infty}\frac{(-1)^n D_n}{\alpha^n n!}x^n\right\} \\ &= \frac{(-1)^n D_n}{\alpha^n n!} - \frac{(-1)^{n-1}nD_{n-1}}{\alpha^n n!},\end{aligned}$$

as claimed. Similarly, if we take $h(x) = \left(\frac{x}{1-e^{-x}}\right)^m$ in (1.10), we have the other proof. □

From (1.2), we have $r_{n,k} = H_{n-k}^{k}(\alpha)$ and $R = \left(-\ln\left(1-\frac{x}{\alpha}\right), \frac{1}{1-x}\right) \in \mathcal{R}'$. Hence, the following theorem is clearly given.

Theorem 4. *For nonnegative integers n, m, we have*

$$\sum_{k=0}^{n}\binom{m}{k}H_{n-k}^{k}(\alpha) = H_n^m(\alpha).$$

Proof. Putting $h(x) = (1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$ in (1.10), we can get

$$\begin{aligned}\sum_{k=0}^{n} \binom{m}{k} H_{n-k}^{k}(\alpha) &= [x^n]\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)\left(1+\frac{x}{1-x}\right)^m \\ &= [x^n]\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^m} \\ &= H_n^m(\alpha),\end{aligned}$$

as claimed. □

Theorem 5. *For any positive integer n, we have*

$$\sum_{k=0}^{n} \frac{H(n,k,\alpha)}{(k+1)!} = \frac{\alpha^n - 1}{\alpha^n(\alpha-1)}, \quad \alpha \neq 1,$$

$$\sum_{k=0}^{n} (-1)^k \frac{H(n,k,\alpha)}{(k+1)!} = \frac{1}{\alpha},$$

and

$$\sum_{k=0}^{n} (-1)^k \frac{H(n+1,k+1,\alpha)}{(k+1)!} = \frac{H_n(\alpha)}{\alpha}.$$

Proof. From $h(x) = \frac{e^x-1}{x} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n$ and using (2.1), we have

$$\begin{aligned}\sum_{k=0}^{n} \frac{H(n,k,\alpha)}{(k+1)!} &= \left[x^n\right]\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\frac{x}{(x-\alpha)\ln\left(1-\frac{x}{\alpha}\right)} \\ &= \left[x^{n-1}\right]\frac{1}{(x-1)(x-\alpha)} = \frac{\alpha^n-1}{\alpha^n(\alpha-1)},\end{aligned}$$

which proves the first one. Similarly, if we take

$$h(x) = \frac{1-e^{-x}}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} x^n$$

and

$$h(x) = 1 - e^{-x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} x^n,$$

in (1.10), respectively, from (2.1), we have the proofs of other results. □

Theorem 6. *Let n, r be positive integers such that $n > r$. We have*

$$\begin{aligned}(-1)^r H(n,r,\alpha) &= \sum_{k=r}^{n-1} (-1)^k \frac{s(k,r)\,r!}{\alpha^k k!} H_{n-k}(\alpha), \\ H(n,r,\alpha) &= \sum_{k=r}^{n-1} \frac{H(k,r-1,\alpha)}{(n-k)\,\alpha^{n-k}}, \\ H_{n-r}^{r+1}(\alpha) &= \sum_{k=r}^{n-1} \binom{k}{r} \frac{1}{(n-k)\,\alpha^{n-k}}.\end{aligned}$$

Proof. From (1.9), we have

$$\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}, \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{x}\right) = \left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}, 1\right) * \left(1, \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{x}\right),$$

and considering Riordan arrays related to $H(n,r,\alpha)$, $H_{n-r}(\alpha)$ and $(-1)^{n-r}s(n,r)\frac{r!}{\alpha^n n!}$, the first identity is obtained from matrix multiplication. Similarly, by Riordan arrays $\left(\frac{1}{1-x},\frac{1}{1-x}\right)=\left[\binom{n}{r}\right]$ and $\left(-\ln\left(1-\frac{x}{\alpha}\right),1\right)=\left[\frac{1}{(n-r)\alpha^{n-r}}\right]$, the other identities are given. □

Theorem 7. *For nonnegative integer n, we have*

$$H_{n+1}(\alpha)-H_{n+1}=\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{1}{k}\left(\frac{1}{\alpha}-1\right)^k.$$

Proof. It is known that

$$(n+1)\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{k+1}=\frac{(1+x)^{n+1}-1}{x}.$$

From this, integrating both sides of the above equation from -1 to $(-1+1/\alpha)$, we write

$$\begin{aligned}\int_{-1}^{-1+1/\alpha}\frac{(1+x)^{n+1}-1}{x}dx &= \int_{-1}^{-1+1/\alpha}\sum_{k=0}^{n}(1+x)^k\,dx=\sum_{k=0}^{n}\left.\frac{(1+x)^{k+1}}{k+1}\right|_{-1}^{-1+1/\alpha}\\ &= \sum_{k=0}^{n}\frac{1}{(k+1)\alpha^{k+1}}=H_{n+1}(\alpha)\end{aligned}$$

and using $H_{n+1}=\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\frac{1}{k+1}$ [4, 12],

$$\begin{aligned}&\int_{-1}^{-1+1/\alpha}(n+1)\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{k+1}dx\\ &= (n+1)\sum_{k=0}^{n}\binom{n}{k}\left.\frac{x^{k+1}}{(k+1)^2}\right|_{-1}^{-1+1/\alpha}\\ &= (n+1)\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)^2}\left(\left(-1+\frac{1}{\alpha}\right)^{k+1}-(-1)^{k+1}\right)\\ &= (n+1)\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)^2}\left(-1+\frac{1}{\alpha}\right)^{k+1}+H_{n+1}.\end{aligned}$$

Thus, from aboving, the proof is complete. □

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References

[1] G.S. Cheon, M.E.A. El-Mikkawy, Generalized harmonic numbers with Riordan arrays, J. Number Theory, 128 (2008), 413-425.
[2] G.S. Cheon, M.E.A. El-Mikkawy and H-G. Seol, New identities for Stirling numbers via Riordan arrays, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 13(4) (2006), 311-318.
[3] Ö. Duran, N. Ömür and S. Koparal, On sums with generalized, harmonic, hyperharmonic and special numbers, Miskolc Math. Notes, 21(2) (2020), 791-803
[4] O. Furdui, Series involving products of two harmonic numbers, Math. Mag., 84(5) (2011), 371-377.
[5] M. Genčev, Binomial sums involving harmonic numbers, Math. Slovaca, 61(2) (2011), 215-226.
[6] A. Gertsch, Generalized harmonic numbers, C. R. Acad. Sci. Paris, Sér. I Math., 324 (1) (1997), 7-10.

[7] R.L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics, Addison-Wesley Publishing Company, Second Edition, 1994.

[8] D. Merlini, D. G. Rogers, R. Sprugnoli and M. C. Verri, On some alternative characterizations of Riordan arrays, Can. J. Math., 49(2) (1997), 301-320.

[9] N. Ömür and G. Bilgin, Some applications of the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, Adv. Appl. Math. Sci., 17(9) (2018), 617-627.

[10] J.M. Santmyer, A Stirling like sequence of rational numbers, Discrete Math., 171 (1997), 229-235.

[11] L.W. Shapiro, S. Getu, W.J. Woan, L.C. Woodson, The Riordan group, Discrete Appl. Math., 34 (1991), 229-239.

[12] Y. Şimşek, Combinatorial sums and binomial identities associated with the Beta-type polynomials, Hacettepe J. Math. and Stat., 47(5) (2018), 1144-1155.

[13] S.W. Wuyungaowa, Sums of involving the harmonic numbers and the binomial coefficients, American J. Computational Math., 5 (2015), 96-105.

[14] W. Wang, Riordan arrays and harmonic number identities, Comput. Math. with Appl., 60 (2010), 1494-1509.

[15] C-X. Zhao and F-Z. Zhao, Asymptotic expansions of certain sums involving powers of binomial coefficients, J. Integer Seq., 13(2010), 1-13.