Research Article

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Doubly constrained totally positive line insertion

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Abstract: It is shown that in any TP matrix, a line (row or column) with two specified entries in any positions (and the others appropriately chosen) may be inserted in any position, as long as the two entries are consistent with total positivity. This generalizes an unconstrained result previously proven, and the two may not generally be increased to three or more. Applications are given, and this fact should be useful in other completion problems, as the unconstrained result has been.

Keywords: TP-completable patterns; Line insertion; Matrix completion problem; Partial matrix; Totally positive matrix

MSC: 15A83; 15B48

1 Introduction

An m -by- n matrix is *totally positive* (*totally nonnegative*), or TP (TN), if all its minors are positive (nonnegative). It has long been known that any TP matrix may be bordered (either on the side, top, or bottom - see page 185 of [3] for an indicative diagram and explanation) so as to remain TP. This is a convenient way to generate TP matrices of arbitrary size. Much more subtly, it was shown in [4] that between any two consecutive lines (rows or columns) of a TP matrix, a line may be inserted so as to remain TP. This has proven very valuable in several ways, such as for analyzing TP completion problems. Since scaling of any line in a TP matrix leaves it TP, it follows readily that a line may be inserted with a specified positive value in any particular position (*singly constrained line insertion*).

Since [4], this has left open the very natural and subtle question of whether *Doubly Constrained TP Line Insertion* (specified entries in two arbitrary positions) is possible. Of course, the two specified entries must themselves be consistent with total positivity (i.e. the resulting partial matrix must be partial TP [3], which we assume). This is also of considerable interest in TP completion problems. Our purpose here is to show that doubly constrained TP line insertion is always possible. We will also give a few immediate applications. It should be noted at this point that *triply constrained TP line insertion* is not generally possible. As we will note, this is a consequence of existing work on the TP completability of single-unspecified-entry patterns.

In addition to generalizing line insertion as much as possible (settling a longstanding question), we show that the recent main result of [1] follows from our insertion result (Theorem 4.1).

Recall that a *partial matrix* is one in which some entries are specified, while the remaining, *unspecified entries* are free to be chosen. A *completion* of a partial matrix is a choice of values for the unspecified entries, resulting in a conventional matrix. A partial matrix is called *partial TP* (*partial TN*), if all its minors, consisting entirely of specified entries, are positive (nonnegative). The *TP (TN) completion problem* asks which partial TP (TN) matrices have a TP (TN) completion. For some patterns of the specified entries, all partial TP (TN)

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matrices have a TP (TN) completion; these are called *TP-completable* (*TN-completable*) *patterns*. It is a major unsolved problem to determine these patterns. Some TP-completable patterns are not TN completable.

2 Main Result

We now set the notation to prove our main result. Suppose that the m -by- n matrix $A = (a_{ij})$ is TP. We will show that we can insert a doubly constrained column between columns i and $i + 1$ where $1 \leq i < n$. Since transposition preserves total positivity, we only need consider the case in which we insert a column. We denote the result of inserting the column vector w by $A(w)$. Suppose the two constraints are in rows k and l where $1 \leq k < l \leq m$ and the remaining entries of the inserted vector are unspecified (free to be chosen). This means that we wish to complete the following partial TP matrix $A(w)$:

$$\begin{bmatrix} * & \cdots & * & ? & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ * & \cdots & * & ? & * & \cdots & * \\ * & \cdots & * & w_k & * & \cdots & * \\ * & \cdots & * & ? & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ * & \cdots & * & ? & * & \cdots & * \\ * & \cdots & * & w_l & * & \cdots & * \\ * & \cdots & * & ? & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ * & \cdots & * & ? & * & \cdots & * \end{bmatrix}$$

in which each “*” denotes the position of a specified entry.

We now state the main result.

Theorem 2.1. *Insertion of a line in any position of a TP matrix, with 2 entries given that are consistent with total positivity, may be carried out so that the result is TP.*

Proof. Since total positivity is invariant under positive scaling of rows or columns, we assume without loss of generality that every entry in row k is 1. Then, since the two rows k and l must be TP (the only constraints upon the specified entries of the inserted column), we have $a_{l,i} < w_l < a_{l,i+1}$, as well as $w_k = 1$. Denote columns i and $i + 1$ of A by C_i and C_{i+1} . Our partial TP matrix now looks like the following:

$$\begin{bmatrix}
 * & \cdots & * & ? & * & \cdots & * \\
 \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 * & \cdots & * & ? & * & \cdots & * \\
 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
 * & \cdots & * & ? & * & \cdots & * \\
 \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 * & \cdots & * & ? & * & \cdots & * \\
 * & \cdots & a_{l,i} & w_l & a_{l,i+1} & \cdots & * \\
 * & \cdots & * & ? & * & \cdots & * \\
 \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 * & \cdots & * & ? & * & \cdots & *
 \end{bmatrix}$$

If we inserted C_i or C_{i+1} between the columns i and $i+1$, the resulting matrix $A(C_i)$ or $A(C_{i+1})$ would clearly be TN. Since both the set of TN completions and the set of TP completions are convex as a consequence of the linearity of the determinant, we have that $A(tC_i + (1-t)C_{i+1})$ is TN for any $0 < t < 1$. Let $x_t = tC_i + (1-t)C_{i+1}$. Now the k -th entry of x_t is 1 and the l -th entry will be any number strictly between $a_{l,i}$ and $a_{l,i+1}$, depending upon t .

Now, by [4, Theorem 2.3], there exists a vector y so that the insertion $A(y)$ is TP. As above, we may assume $y_k = 1$. Then $a_{l,i} < y_l < a_{l,i+1}$, as rows k and l of $A(y)$ must form a TP matrix. If $y_l = w_l$, $A(y)$ is the desired insertion and we are done. Suppose instead that $y_l < w_l$ (the case where $w_l < y_l$ is similar and will be omitted). Now, choose t above so that $(x_t)_l > w_l$ and then choose s , $0 < s < 1$, so that $sy_l + (1-s)(x_t)_l = w_l$. Let $z = sy + (1-s)x_t$. We show that $A(z)$ is TP and that z is the desired doubly constrained TP line insertion that completes the proof.

Since $z_k = 1$ and $z_l = w_l$, z meets the two constraints. To see that $A(z)$ is TP, recall that we only need to check its initial minors ([3, Theorem 3.1.4]). A minor is initial if both its row and column indices are consecutive and at least one of the index sets begins with 1. Of course, we only need to check minors that include the column z . Since column z is positive, the only initial minors that need to be checked, must also include column C_i or C_{i+1} or both. The relevant cases are Case 1a: C_i and z ; Case 1b: z and C_{i+1} ; Case 2: C_i , z , and C_{i+1} . Since cases 1a and 1b are similar, we will only do 1a.

First consider an initial minor in case 2, which we will show must be positive. The determinant is preserved under the operation of adding a multiple of one row (column) to another row (column). Hence, we can perform this operation on a submatrix when analyzing the value of a minor, show that the determinant after the operation is positive, and then state that the original minor is positive. Remember that $z = sy + (1-s)x_t = sy + (1-s)(tC_i + (1-t)C_{i+1})$ where $0 < s, t < 1$, and add multiples of C_i and C_{i+1} to z in order to eliminate that portion of z that has a positive coefficient on either C_i or C_{i+1} . This leaves a minor that is a positive scaling of the corresponding minor in $A(y)$. The minor in $A(y)$ is positive, so that our minor in $A(z)$ is also positive.

Now, consider an initial minor in case 1a. By the same logic above, use C_i to eliminate that portion of z that has a positive coefficient on C_i . This leaves our inserted column that is a positive linear combination of y and C_{i+1} . By the linearity of the determinant, this is a positive linear combination of the corresponding minor in $A(y)$ and the corresponding minor in $A(C_{i+1})$. The former is positive and the latter is a minor of A (because column C_{i+1} is missing but a positive multiple of it appears in the inserted column). Thus, our initial minor is positive in this case as well, completing the proof. $\square$

3 Concluding Remarks

Next, we note that triply constrained TP line insertion is not possible in general. Indeed, consider a 4-by-4 TP matrix and insert a column with the first, second, and fourth entries specified between columns 2 and 3. The result is a 4-by-5 partial TP matrix in which the (3, 3) entry is the only unspecified one. According to [2], there is partial TP data for this pattern with no TP completion. So, triply constrained TP line insertion is not generally possible. Of course, some triply, or more, constrained line insertions do end up yielding completable patterns depending upon the relative position of the unspecified entries.

Doubly constrained TP line insertion has some nice consequences for TP completability theory. Call a pattern *column type 2* if every column is one of two types: either full (all entries specified) or a column with exactly 2 specified entries (and each such column has its specified (and so unspecified) entries in the same places). There is no restriction on the number of columns of each type or the order. A *row type 2* pattern is just the transpose of a column type 2 one. Call a pattern *type 2* if it is either column type 2 or row type 2. Now, we may show that every type 2 pattern is TP-completable.

Theorem 3.1. *Any partial TP matrix that has a type 2 pattern is TP-completable.*

Proof. Take a matrix A with a type 2 pattern and assume it has unspecified entries in more than one column (For purposes of proof, we assume A is column type 2 as total positivity is preserved by transposition). If there is only one column containing unspecified entries, then we are done by the main result. When completing one of the columns in A , all new entries will be in the same rows in which the other partial columns have unspecified entries. Hence, the completed entries are not included in any fully specified minors that include entries from the remaining partial columns. This means we can consider the partial matrix that contains only the column we are currently trying to complete, as well as the other fully specified columns. This is a line insertion that may be made, by the main result. Completing each partial column sequentially in this way will yield a TP completion of A . $\square$

Recall that a *border pattern* is one in which the first and last rows, as well as the first and last columns, are fully specified, and no other entries are specified. An immediate consequence of Theorem 3.1 is that the border patterns of [1] are TP-completable. Since a border pattern is type 2, we have the following corollary.

Corollary. *Each m -by- n border pattern is TP-completable.*

Thus, the main result of [1] is a special case of our result, Theorem 2.1.

The generalized border patterns of [1] are also TP-completable. It also follows from theorem 3.1 that any 3-by- n pattern, with just one unspecified entry, is TP-completable, and it is easily seen that “expansions” [5] of these and other type 2 patterns are also TP-completable. We suspect that our main result will be helpful in establishing other TP completion results beyond these as well, as singly constrained line insertion has been.

We also note that even the unconstrained TP line insertion result is not valid in partial TP matrices, although bordering (exterior line insertion) is. A simple example is that, while

$$\begin{bmatrix} * & ? & * \\ ? & * & * \\ * & * & * \end{bmatrix}$$

is not a TP-completable pattern [2], the pattern

$$\begin{bmatrix} * & * & ? & * \\ ? & * & * & * \\ * & * & * & * \end{bmatrix}$$

with an unconstrained column inserted between the original columns 1 and 2, is TP-completable. Thus, any partial TP data for the former pattern that does not permit a TP completion does not allow a column to be inserted so as to remain partial TP.

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