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Permutative universal realizability

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Abstract: A list of complex numbers Λ is said to be *realizable*, if it is the spectrum of a nonnegative matrix. In this paper we provide a new sufficient condition for a given list Λ to be *universally realizable* (UR), that is, realizable for each possible Jordan canonical form allowed by Λ. Furthermore, the resulting matrix (that is explicity provided) is permutative, meaning that each of its rows is a permutation of the first row. In particular, we show that a real Suleĭmanova spectrum, that is, a list of real numbers having exactly one positive element, is UR by a permutative matrix.

Keywords: Permutative matrix; Nonnegative matrix; Nonnegative inverse eigenvalue problem; Universal realizability

MSC: 15A18, 15A20, 15A29

1 Introduction

The *nonnegative inverse eigenvalue problem* (NIEP) is the problem of characterizing all possible spectra of entrywise nonnegative matrices. The NIEP remains unsolved. A solution is known only for $n \leq 4$, which shows the difficulty of the problem. For an elaborate exposition on the history of NIEP we refer the reader to [7]. A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers, is said to be *realizable*, if Λ is the spectrum of an $n \times n$ nonnegative matrix A, and A is said to be a *realizing matrix*. From the Perron-Frobenius Theorem one can easily conclude that if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of an $n \times n$ nonnegative matrix A, then the leading eigenvalue of A equals to the spectral radius of A, namely $\rho(A) =: \max_{1 \leq i \leq n} |\lambda_i|$. This eigenvalue is called the *Perron eigenvalue,* and we shall assume in this paper, that $\rho(A) = \lambda_1$. If Λ contains only real numbers, the problem is called *the real nonnegative inverse eigenvalue problem* (RNIEP), while if the realizing matrix is required to be symmetric, then the problem is called *the symmetric nonnegative inverse eigenvalue problem* (SNIEP) (see [19, 20] and the references therein).

A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers, is said to be *diagonalizably realizable* (DR), if there is a diagonalizable realizing matrix for Λ [2]. Moreover, Λ is said to be *universally realizable* (UR), if it is realizable for each possible Jordan canonical form (JCF) allowed by Λ. The problem of the universal realizability of spectra, is called the *universal realizability problem* (URP). The URP contains the NIEP, and both problems are equivalent if the given numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. In terms of n, both problems remain unsolved for $n \geq 5$. It is clear that if Λ is UR, then Λ is DR. The first known results on the URP are due to Minc [13, 14]. Minc showed that if a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers, is realizable by a positive diagonalizable matrix, then Λ is UR. In [8], it was proved that if Λ is ODP realizable, that is, realizable by an off-diagonally positive matrix, then Λ is UR. This result contains, as a particular case, the result by Minc in [13].

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A matrix $A = [a_{ij}]$, is said to have *constant row sums*, if each of its rows sums up to the same constant α. The set of all matrices, with constant row sums equal to α, will be denoted by $\mathcal{CS}_\alpha$. Then, any matrix in $\mathcal{CS}_\alpha$ has the eigenvector $\mathbf{e}^T = [1, 1, \ldots, 1]$, corresponding to the eigenvalue α. The real matrices, with constant row sums, are important because it is known that, the problem of finding a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, is equivalent to the problem of finding a nonnegative matrix in $\mathcal{CS}_{\lambda_1}$, with spectrum Λ (see [6]). We shall denote by $\mathbf{e}_k$, the n-dimensional vector, with one in the k-th position and zeros elsewhere. Since our interest is about nonnegative permutative matrices, we give the following definition:

Definition 1.1. *Let $\mathbf{x} \in \mathbb{R}^n$ and let $P_2, P_3, \ldots, P_n$ be $n \times n$ permutation matrices. A permutative matrix is any matrix of the form*

$$P = \begin{bmatrix} \mathbf{x}^T \\ (P_2\mathbf{x})^T \\ \vdots \\ (P_n\mathbf{x})^T \end{bmatrix}.$$

Then, an $n \times n$ permutative matrix, is a matrix in which every row is a permutation of its first row. It is clear that $P \in \mathcal{CS}_S$, where S is the sum of the entries of the vector $\mathbf{x}$. Permutative matrices were introduced and first studied in [5]. There, the authors give conditions under which, permutative matrices are rank deficient, and they pointed out that a Latin square is a permutative matrix whose transpose is also permutative, and that these type of permutative matrices have been studied in statistical experimental design, and combinatorics. The name *permutative* was introduced by Johnson [5].

The following results will be used throughout the paper. The first two, have been shown to be very useful, not only to derive sufficient conditions for realizability in both problems, the NIEP and the URP, but for constructing a realizing matrix, as well. The first result, by Brauer [1], shows how to modify one single eigenvalue of a matrix, via a rank-one perturbation, without changing any of its remaining eigenvalues. The second result, by Rado, and introduced by Perfect in [17], is an extension of Brauer's result and it shows how to change r eigenvalues of an $n \times n$ matrix ($r < n$), via a perturbation of rank r, without changing any of its remaining $n - r$ eigenvalues (see [8, 11, 17, 19], and the references therein, to see how Brauer and Rado results have been applied to the NIEP and to the URP).

Theorem 1.1. *(Brauer [1]). Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{v}^T = [v_1, \ldots, v_n]$ be an eigenvector of A corresponding to the eigenvalue λ_k, and let $\mathbf{q}$ be any n-dimensional vector. Then the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \ldots, \lambda_n$.*

Theorem 1.2. *(Rado [17]). Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $X = [\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_r]$ be such that $rank(X) = r$ and $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, \ldots, r$, $r \leq n$. Let C be an $r \times n$ matrix. Then $A + XC$ has eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + CX$ with $\Omega = diag\{\lambda_1, \ldots, \lambda_r\}$.*

The following result, in [20], is a symmetric version of Rado's result.

Theorem 1.3. *([20]). Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and for some $r \leq n$, let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ be an orthonormal set of eigenvectors of A spanning the invariant subspace associated with $\lambda_1, \lambda_2, \ldots, \lambda_r$. Let X be the $n \times r$ matrix with i-th column $\mathbf{x}_i$, let $\Omega = diag\{\lambda_1, \ldots, \lambda_r\}$, and let C be any $r \times r$ symmetric matrix. Then the symmetric matrix $A + XCX^T$ has eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + C$.*

Lemma 1.1. *([21]). Let $\mathbf{q}^T = [q_1, q_2, \ldots, q_n]$ be an arbitrary n-dimensional vector and $A \in \mathcal{CS}_{\lambda_1}$ an $n \times n$ matrix with JCF $J(A)$. Let $\lambda_1 + \sum_{i=1}^{n} q_i \neq \lambda_i$, $i = 2, \ldots, n$. Then the matrix $B = A + \mathbf{e}\mathbf{q}^T$ has JCF $J(A) + (\sum_{i=1}^{n} q_i)E_{11}$, where*

E_{11} is the matrix with 1 in position $(1,1)$ and zeros elsewhere. In particular, if $\sum_{i=1}^{n} q_i = 0$ then $J(B) = J(A)$ and A and B are similar.

Here, we study the *permutative universal realizability problem,* that is, the problem of determining the existence and construction of a nonnegative permutative matrix, with prescribed complex spectrum $\Lambda = \{\lambda_1, \dots, \lambda_n\}$, for each possible JCF allowed by Λ.

Definition 1.2. *(Spectra of Suleĭmanova type)*
i) A list of real numbers $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ is called a real Suleĭmanova spectrum, whenever

$$\sum_{i=1}^{n} \lambda_i \geq 0, \tag{1}$$

and Λ contains only one positive eigenvalue.
ii) A list of complex numbers $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ is called a complex Suleĭmanova spectrum, whenever it satisfies (1), $\lambda_1 > 0$ and

$$\operatorname{Re}\lambda_k \leq 0, \quad |\operatorname{Re}\lambda_k| \geq |\operatorname{Im}\lambda_k|, \quad k = 2, \dots, n.$$

In [15], Paparella proved that the permutative RNIEP has a solution, when the given spectrum is of real Suleĭmanova type [24]. Paparella [15], also showed that all realizable lists, with $n \leq 4$, are in particular, permutatively realizable. In [23], Soto extends results in [15] to more general lists, of real and complex numbers. In particular, by applying Brauer's Theorem, a very simple and short proof, that real Suleĭmanova spectra are permutatively realizable, was also given in [23]. It was also showed in [23], that a complex Suleĭmanova spectrum is in particular permutatively realizable. Loewy [12] gave a negative answer to the following question set by Paparella in [15]: can all realizable spectra of real numbers, be realized by a permutative matrix, or by a direct sum of permutative matrices? In [16], Paparella gives a solution to the RNIEP for a particular class of permutative matrices.

Outline of the paper: The paper is organized as follows: In Section 2, we recall the realizability problem for permutative matrices and we introduce some new sufficient conditions for the problem to have a solution. In Section 3, we study the universal realizability problem with permutative structure. Sufficient conditions for the existence and construction, of a permutative nonnegative matrix with a given spectrum $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, for each possible JCF allowed by Λ, are given. In particular, we show that a real spectrum of Suleĭmanova type is permutatively universally realizable. Examples are also shown to illustrate the results.

2 Permutative realizability

Since circulant matrices are permutative, the spectrum of a circulant matrix is also the spectrum of a permutative matrix. Let $\boldsymbol{\lambda}^T = [\lambda_1, \lambda_2, \dots, \lambda_n]$ be the vector of eigenvalues of a circulant matrix $\mathcal{C}$. Let $\mathbf{c} = [c_0, c_1, \dots, c_{n-1}] \in \mathbb{C}^n$. An $n \times n$ circulant matrix is of the form

$$\mathcal{C}(\mathbf{c}) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{bmatrix}.$$

Each row is a cycle shift of one position to the right of the row directly above. Then $\mathcal{C}(\mathbf{c})$ is fully specified by its first row, and the vector $\boldsymbol{\lambda}^T = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ above is said to be a conjugate-even vector, that is,

$$\lambda_1 \in \mathbb{R}, \quad \overline{\lambda}_j = \lambda_{n-j+2}, \quad j = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{2}$$

Thus, for this first approach to the permutative realizability of spectra, the problem will be to guarantee the nonnegativity of $\mathcal{C}(\mathbf{c})$. In [19], Soto and Rojo, give a necessary and sufficient condition, for a 5-dimensional conjugate-even spectrum to be realizable by a 5×5 symmetric circulant matrix $\mathcal{C}(\mathbf{c})$. For instance, the spectrum $\Lambda = \{6, 1, 1, -4, -4\}$ can be arranged in the form of a conjugate-even vector $\boldsymbol{\lambda}^T = [6, 1, -4, -4, 1]$, and then Λ is the spectrum of the nonnegative permutative (circulant) matrix

$$A = \begin{bmatrix} 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 \end{bmatrix}.$$

Moreover, a left circulant matrix, denoted by $\mathcal{C}_L(\mathbf{c})$, is a matrix in which each row is a cycle shift of one position to left of the row directly above, and therefore it is a Hankel matrix, that is, a square matrix with constant skew-diagonals. If $\mathbf{c} = [c_0, \ldots, c_{n-1}]$ has nonnegative entries, then $\mathcal{C}_L(\mathbf{c})$ is a real nonnegative symmetric matrix, and its eigenvalues are the real numbers

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{m+1}, -\lambda_{m+1}, \ldots, -\lambda_3, -\lambda_2, \tag{3}$$

if $n = 2m + 1$, or they are

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{m+1}, \lambda_{m+2}, -\lambda_{m+1}, \ldots, -\lambda_3, -\lambda_2, \tag{4}$$

if $n = 2m + 2$. An immediate consequence is that, if $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of an $n \times n$ real left circulant matrix, then they can be arranged as

$$\mu_j = -\mu_{n-j+2}, \quad j = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor.$$

If $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a realizable list of real numbers, of the form (3) or (4), then Λ is permutatively realizable. In [3, 18], the authors prove certain perturbation results, for spectra realizable by circulant matrices. In particular, from the results in [18], and for the spectrum $\{6, 1, 1, -4, -4\}$, $\Lambda_t = \{6 + 2t, 1 \pm t, -4, -4, 1 \pm t\}$, $t > 0$, is also permutatively realizable.

Remark 2.1. *If $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of a circulant matrix, then $\boldsymbol{\lambda}^T = [\lambda_1, \ldots, \lambda_n]$ must be a conjugate-even vector. Although circulant matrices are permutative, the conjugate-even condition is not necessary for the spectrum of a permutative matrix. Thus both problems, circulant realizability and permutative realizability are different. A lot is known about circulant matrices, and it is relatively easy to construct this type of matrices with a prescribed spectrum. Thus, the connection between circulant and permutative matrices is important, in the sense that in the following results, it will often be necessary to initially have permutative realizations of smaller size than the final realizing matrix, and often they can be obtained as circulant realizations.*

The next result extends to a complex spectrum, a similar result for a real spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ given in [23].

Theorem 2.1. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a spectrum of complex numbers. Suppose that:*
i) There exists a partition $\Lambda = \Lambda_0 \cup \underbrace{\Lambda_1 \cup \cdots \cup \Lambda_1}_{r\ times}$, where

$$\Lambda_0 = \{\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0r}\}, \quad \lambda_1 = \lambda_{01}, \quad \Lambda_1 = \{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1p}\},$$

such that $\Gamma_1 = \{\lambda\} \cup \Lambda_1$, $0 \le \lambda \le \lambda_1$, is permutatively (circulantly) realizable.
ii) There exists an $r \times r$ permutative (circulant) nonnegative matrix with spectrum Λ_0 and diagonal entries $\lambda, \lambda, \dots, \lambda$ (r times).
Then, Λ is permutatively realizable, with λ_1 being the Perron eigenvalue of the realizing permutative matrix.

Proof. The proof is analogous to the proof in [23, Theorem 2.4]. □

Example 2.1. *Consider the left half-plane spectrum*

$$\begin{aligned} \Lambda &= \{7, -1, -1+2i, -1-2i, -1+2i, -1-2i\}, \text{ with} \\ \Lambda_0 &= \{7, -1\},\ \Lambda_1 = \{-1+2i, -1-2i\}, \text{ and} \\ \Gamma_1 &= \{3, -1+2i, -1-2i\}. \end{aligned}$$

We apply Theorem 2.1. Then Γ_1 is the spectrum of the nonnegative permutative (circulant) matrix

$$A_1 = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{3}}{3} + \frac{4}{3} & \frac{4}{3} - \frac{2\sqrt{3}}{3} \\ \frac{4}{3} - \frac{2\sqrt{3}}{3} & \frac{1}{3} & \frac{2\sqrt{3}}{3} + \frac{4}{3} \\ \frac{2\sqrt{3}}{3} + \frac{4}{3} & \frac{4}{3} - \frac{2\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}.$$

In this case,

$$X^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix},\ B = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}.$$

Thus,

$$\begin{aligned} A &= A_1 \oplus A_1 + XCX^T \\ &= \begin{bmatrix} \frac{1}{3} & \frac{4+2\sqrt{3}}{3} & \frac{4-2\sqrt{3}}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{4-2\sqrt{3}}{3} & \frac{1}{3} & \frac{4+2\sqrt{3}}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{4+2\sqrt{3}}{3} & \frac{4-2\sqrt{3}}{3} & \frac{1}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{1}{3} & \frac{4+2\sqrt{3}}{3} & \frac{4-2\sqrt{3}}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4-2\sqrt{3}}{3} & \frac{1}{3} & \frac{4+2\sqrt{3}}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4+2\sqrt{3}}{3} & \frac{4-2\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

is nonnegative permutative with spectrum Λ.

Now, we explore a different approach, based on Theorem 1.3, with the initial matrix A not necessarily symmetric.

Theorem 2.2. *Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, n even, be a realizable list of complex numbers, with λ_1, λ_2 being real numbers. Suppose that Λ admits the associated partition*

$$\Lambda = \Lambda_1 \cup \Lambda_2,$$

where Λ_1 is permutatively realizable, $\Lambda_1 = \Lambda_2 = \{\mu, \alpha_2, \dots, \alpha_{\frac{n}{2}}\}$ with $\mu = \frac{\lambda_1+\lambda_2}{2}$, and $\alpha_i \in \{\lambda_3, \lambda_4, \dots, \lambda_n\}$, for $i = 2, 3, \dots, \frac{n}{2}$. Then, Λ is permutatively realizable.

Proof. Let A_1 be the permutatively realizing matrix for Λ_1. Then $A_1 \in \mathcal{CS}_\mu$, and the matrix

$$A = \begin{bmatrix} A_1 & \frac{2}{n}\beta \mathbf{e}\mathbf{e}^T \\ \frac{2}{n}\beta \mathbf{e}\mathbf{e}^T & A_1 \end{bmatrix} \tag{5}$$

is permutative, with spectrum Λ, where $\beta > 0$, and $\begin{bmatrix} \mu & \beta \\ \beta & \mu \end{bmatrix}$ has eigenvalues λ_1, λ_2. □

Example 2.2. *$\Lambda = \{10, -2, -2+3i, -2-3i, -2+3i, -2-3i\}$ is a realizable list in the left half-plane. Then, we take the associated spectrum $\Lambda_1 = \{4, -2+3i, -2-3i\}$, with the permutative (circulant) realizing matrix*

$$A_1 = \begin{bmatrix} 0 & 2-\sqrt{3} & \sqrt{3}+2 \\ \sqrt{3}+2 & 0 & 2-\sqrt{3} \\ 2-\sqrt{3} & \sqrt{3}+2 & 0 \end{bmatrix}.$$

Next, we compute $\beta = 6$, and from (5),

$$A = \begin{bmatrix} 0 & 2-\sqrt{3} & \sqrt{3}+2 & 2 & 2 & 2 \\ \sqrt{3}+2 & 0 & 2-\sqrt{3} & 2 & 2 & 2 \\ 2-\sqrt{3} & \sqrt{3}+2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2-\sqrt{3} & \sqrt{3}+2 \\ 2 & 2 & 2 & \sqrt{3}+2 & 0 & 2-\sqrt{3} \\ 2 & 2 & 2 & 2-\sqrt{3} & \sqrt{3}+2 & 0 \end{bmatrix}$$

is a permutative realization for Λ. Note that A is a diagonalizable ODP matrix, that is, a diagonalizable nonnegative matrix with positive off-diagonal entries. Therefore, from [8], Λ is UR (although it is not necessarily permutatively universally realizable).

The following lemma, will be useful to establish a sufficient condition, for real spectra to be permutatively realizable.

Lemma 2.1. *The matrix*

$$M = \begin{bmatrix} \omega & \omega-\lambda_2 & \omega-\lambda_3 & \cdots & \omega-\lambda_n \\ \omega-\lambda_2 & \omega & \omega-\lambda_3 & \cdots & \omega-\lambda_n \\ \omega-\lambda_3 & \omega-\lambda_2 & \ddots & \cdots & \omega-\lambda_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega-\lambda_n & \omega-\lambda_2 & \omega-\lambda_3 & \cdots & \omega \end{bmatrix}$$

has the spectrum $\lambda_1 = n\omega - \sum_{i=2}^{n} \lambda_i, \lambda_2, \ldots, \lambda_n$.

Proof. It is clear that $M \in CS_{\lambda_1}$, and $\det(M - \lambda I) = 0$, for $\lambda = \lambda_i$, $i = 2, 3, \ldots, n$. □

Then, the next proposition is immediate.

Proposition 2.3. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a spectrum of real numbers, with $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$, and $\frac{1}{n}\sum_{i=1}^{n} \lambda_i \geq \lambda_2$. Then, Λ is permutatively realizable.*

Remark 2.2. *Proposition 2.3 shows that a real Suleĭmanova spectrum, is in particular permutatively realizable.*

Example 2.3. *The Suleĭmanova spectrum*

$$\Lambda = \{45, -1, -2, -3, -4, -5, -6, -7, -8, -9\}$$

is permutatively realizable, as $\omega = 0 \geq -1$. The nonnegative spectrum

$$\Gamma = \{45, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$$

is also permutatively realizable, as $\omega = 9 \geq 9$.

3 Permutative universal realizability

In this section, we consider the problem of permutative universal realizability. Recall that a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of complex numbers is said to be universally realizable (UR), if it is realizable for each possible Jordan canonical form (JCF) allowed by Λ. There are spectra that are, in a natural way, permutatively realizable, and furthermore, they can be universally realizable. This is the case of *n-th* roots of unity, which as is well known, are the spectrum of the permutative matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $A + \alpha I$, $\alpha > 0$, is also permutative, then $\Lambda + \alpha = \{\lambda_1 + \alpha, \lambda_2 + \alpha, \dots, \lambda_n + \alpha\}$ is permutatively UR. Moreover, $A + \mathbf{e}\mathbf{q}^T$, with $\mathbf{q}^T = [\alpha, \alpha, \dots, \alpha]$, is also a permutative matrix, and its spectrum is UR. There are other spectra, such as the Suleĭmanova type spectra, that have shown to have a very good behavior, e.g., they are not only realizable, but also UR [21, 22]. Moreover they are symmetrically [4], persymmetrically [9], and centrosymmetrically [10] realizable. Now, we show that a list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of real numbers, with $\lambda_1 > 0 > \lambda_2 > \lambda_3 \geq \lambda_4 \geq \cdots \geq \lambda_n$, is permutatively universally realizable.

Theorem 3.1. *Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a list of real numbers, with*

$$\lambda_1 > 0 > \lambda_2 > \lambda_3 \geq \lambda_4 \geq \cdots \geq \lambda_n.$$

Then the following statements are equivalent

i) $\sum_{i=1}^{n} \lambda_i \geq 0$,

ii) *Λ is permutatively realizable,*

iii) *Λ is permutatively UR.*

Proof. The equivalence between *i)* and *ii)* has been proved in [23, Theorem 2.1]. It is clear that if Λ is permutatively UR, then Λ is permutatively realizable. Thus, *iii)* implies *ii)*. It remains to show that *i)* implies *iii)*. Let $\alpha = \sum_{i=1}^{n} \lambda_i \geq 0$ and let $\Lambda_\alpha = \{\lambda_1 - \alpha, \lambda_2, \dots, \lambda_n\}$. Then, given any JCF allowed by Λ_α, we construct an initial matrix $B \in CS_{\lambda_1 - \alpha}$ that is similar to this JCF. In order to understand the arguments and procedure of the proof, suppose for instance, that B has eigenvalues $\lambda_3, \lambda_4, \lambda_5$, with algebraic multiplicities $2, 2, 3$, respectively. That is,

$$B = \begin{bmatrix} \lambda_1 - \alpha & & & & & & & & \\ \lambda_1 - \alpha - \lambda_2 & \lambda_2 & & & & & & & \\ \lambda_1 - \alpha - \lambda_3 & & \lambda_3 & & & & & & \\ \lambda_1 - \alpha - \lambda_3 & & & \lambda_3 & & & & & \\ \lambda_1 - \alpha - \lambda_4 & & & & \lambda_4 & & & & \\ \lambda_1 - \alpha - \lambda_4 & & & & & \lambda_4 & & & \\ \lambda_1 - \alpha - \lambda_5 & & & & & & \lambda_5 & & \\ \lambda_1 - \alpha - \lambda_5 & & & & & & & \lambda_5 & \\ \lambda_1 - \alpha - \lambda_5 & & & & & & & & \lambda_5 \end{bmatrix} \in \mathcal{CS}_{\lambda_{1-\alpha}}.$$

Then for

$$\mathbf{q}^T = [\alpha - \lambda_1, -\lambda_2, -\lambda_3, -\lambda_3, -\lambda_4, -\lambda_4, -\lambda_5, -\lambda_5, -\lambda_5] + \frac{\alpha}{9}\mathbf{e}^T,$$

$B + \mathbf{eq}^T$ is nonnegative permutative, with diagonal JCF. To obtain a permutative realization $A = B + \mathbf{eq}^T$, with a JCF

$$J_A = diag\{J_1(\lambda_1), J_1(\lambda_2), J_2(\lambda_3), J_2(\lambda_4), J_3(\lambda_5)\},$$

where $J_{n_i}(\lambda_i)$ represents the Jordan block of size n_i associated to eigenvalue λ_i, we apply Brauer's Theorem as,

$$\begin{bmatrix} \lambda_1-\alpha & & & & & & & & \\ \lambda_1-\alpha-\lambda_2 & \lambda_2 & & & & & & & \\ \lambda_1-\alpha-\lambda_3 & -a & \lambda_3 & a & & & & & \\ \lambda_1-\alpha-\lambda_3 & & & \lambda_3 & & & & & \\ \lambda_1-\alpha-\lambda_4 & & & -b & \lambda_4 & b & & & \\ \lambda_1-\alpha-\lambda_4 & & & & & \lambda_4 & & & \\ \lambda_1-\alpha-\lambda_5 & & & & -c_1 & & \lambda_5 & c_1 & \\ \lambda_1-\alpha-\lambda_5 & & & & -c_2 & & & \lambda_5 & c_2 \\ \lambda_1-\alpha-\lambda_5 & & & & & & & & \lambda_5 \end{bmatrix} + \mathbf{eq}^T + \frac{\alpha}{9}\mathbf{e}^T, \tag{6}$$

where

$$a = \lambda_3 - \lambda_2, \quad b = \lambda_4 - \lambda_3, \quad c_1 = c_2 = \lambda_5 - \lambda_4$$

are located in suitable positions $(i, i+1)$, with the reciprocals in convenient positions to the left of (i, i). Then, we obtain

$$B + \mathbf{eq}^T = \begin{bmatrix} 0 & -\lambda_2 & -\lambda_3 & -\lambda_3 & -\lambda_4 & -\lambda_4 & -\lambda_5 & -\lambda_5 & -\lambda_5 \\ -\lambda_2 & 0 & -\lambda_3 & -\lambda_3 & -\lambda_4 & -\lambda_4 & -\lambda_5 & -\lambda_5 & -\lambda_5 \\ -\lambda_3 & -\lambda_3 & 0 & -\lambda_2 & -\lambda_4 & -\lambda_4 & -\lambda_5 & -\lambda_5 & -\lambda_5 \\ -\lambda_3 & -\lambda_2 & -\lambda_3 & 0 & -\lambda_4 & -\lambda_4 & -\lambda_5 & -\lambda_5 & -\lambda_5 \\ -\lambda_4 & -\lambda_2 & -\lambda_3 & -\lambda_4 & 0 & -\lambda_3 & -\lambda_5 & -\lambda_5 & -\lambda_5 \\ -\lambda_4 & -\lambda_2 & -\lambda_3 & -\lambda_3 & -\lambda_4 & 0 & -\lambda_5 & -\lambda_5 & -\lambda_5 \\ -\lambda_5 & -\lambda_2 & -\lambda_3 & -\lambda_3 & -\lambda_5 & -\lambda_4 & 0 & -\lambda_4 & -\lambda_5 \\ -\lambda_5 & -\lambda_2 & -\lambda_3 & -\lambda_3 & -\lambda_5 & -\lambda_4 & -\lambda_5 & 0 & -\lambda_4 \\ -\lambda_5 & -\lambda_2 & -\lambda_3 & -\lambda_3 & -\lambda_4 & -\lambda_4 & -\lambda_5 & -\lambda_5 & 0 \end{bmatrix},$$

which is nonnegative permutative with the desired JCF. To obtain Jordan blocks of smaller size, we make zero the entries a, b, or c_2, in (6), according what JCF we want to obtain. It is clear that this particular case can be generalized to

$$B = \begin{bmatrix} \lambda_1-\alpha & & & & & & & & \\ \lambda_1-\alpha-\lambda_2 & \lambda_2 & & & & & & & \\ \vdots & & \ddots & & & & & & \\ \lambda_1-\alpha-\lambda_k & & -a_k & \lambda_k & a_k & & & & \\ \lambda_1-\alpha-\lambda_k & & & & \lambda_k & & & & \\ \lambda_1-\alpha-\lambda_{k+1} & & & -b_{k+1} & & \lambda_{k+1} & b_{k+1} & & \\ \lambda_1-\alpha-\lambda_{k+1} & & & & & & \lambda_{k+1} & & \\ \vdots & & & & & & & \ddots & \\ \lambda_1-\alpha-\lambda_n & & & & & & & & \lambda_n \end{bmatrix} \in CS_{\lambda_1-\alpha},$$

with $B + \mathbf{eq}^T$ being nonnegative permutative with the desired JCF, where

$$\mathbf{q}^T = [\alpha - \lambda_1, -\lambda_2, \ldots, -\lambda_n] + \frac{\alpha}{n}\mathbf{e}^T.$$

Note that

$$\lambda_1 - \alpha + \sum_{i=1}^{n} q_i = -\sum_{i=2}^{n} \lambda_i + \alpha > 0$$

and so $\lambda_1 - \alpha + \sum_{i=1}^{n} q_i \neq \lambda_i$ for $i = 2, \ldots, n$. Therefore, by Lemma 1.1 $B + \mathbf{eq}^T$ has JCF

$$J(B + \mathbf{eq}^T) = J(B) + \alpha E_{11},$$

which is the desired JCF. Thus, the proof is complete. □

The following example illustrates Theorem 3.1. It shows how we may obtain, a permutative realization, for each possible JCF allowed by a given real list $\lambda_1 > 0 > \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_n$.

Example 3.1. *Let us consider the list*

$$\Lambda = \{30, -1, -5, -5, -5, -7, -7\}.$$

We start with

$$B = \begin{bmatrix} 30 & 0 & 0 & 0 & 0 & 0 & 0 \\ 31 & -1 & 0 & 0 & 0 & 0 & 0 \\ 35 & 4 & -5 & -4 & 0 & 0 & 0 \\ 35 & 4 & 0 & -5 & -4 & 0 & 0 \\ 35 & 0 & 0 & 0 & -5 & 0 & 0 \\ 31 & 6 & 2 & 0 & 0 & -7 & -2 \\ 31 & 6 & 0 & 0 & 0 & 0 & -7 \end{bmatrix}.$$

Then, for $\mathbf{q}^T = \begin{bmatrix} -30 & 1 & 5 & 5 & 5 & 7 & 7 \end{bmatrix}$, *we have that*

$$B + \mathbf{eq}^T = \begin{bmatrix} 0 & 1 & 5 & 5 & 5 & 7 & 7 \\ 1 & 0 & 5 & 5 & 5 & 7 & 7 \\ 5 & 5 & 0 & 1 & 5 & 7 & 7 \\ 5 & 5 & 5 & 0 & 1 & 7 & 7 \\ 5 & 1 & 5 & 5 & 0 & 7 & 7 \\ 1 & 7 & 7 & 5 & 5 & 0 & 5 \\ 1 & 7 & 5 & 5 & 5 & 7 & 0 \end{bmatrix}$$

is permutative with JCF $J(B + \mathbf{eq}^T) = diag\{J_1(30), J_1(-1), J_3(-5), J_2(-7)\}$. *For*

$$B + \mathbf{eq}^T = \begin{bmatrix} 30 & 0 & 0 & 0 & 0 & 0 & 0 \\ 31 & -1 & 0 & 0 & 0 & 0 & 0 \\ 35 & 4 & -5 & -4 & 0 & 0 & 0 \\ 35 & 0 & 0 & -5 & 0 & 0 & 0 \\ 35 & 0 & 0 & 0 & -5 & 0 & 0 \\ 31 & 6 & 2 & 0 & 0 & -7 & -2 \\ 31 & 6 & 0 & 0 & 0 & 0 & -7 \end{bmatrix} + \mathbf{eq}^T,$$

we obtain the JCF $J(B + \mathbf{eq}^T) = diag\{J_1(30), J_1(-1), J_2(-5), J_1(-5), J_2(-7)\}$, *and so on.*

From Rado's Theorem, we have the following result:

Theorem 3.2. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a realizable list of real numbers, where*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0 > \lambda_{p+1} \geq \lambda_{p+2} \geq \cdots \geq \lambda_n,$$

with $-\lambda_n \geq \lambda_2$, $n \geq 2p$ *for n even, and* $n \geq 2p + 1$ *for n odd* n, $p \geq 2$. *Suppose that:*
i) Λ *admits a partition* $\Lambda = \Lambda_0 \cup \underbrace{\Lambda_1 \cup \cdots \cup \Lambda_1}_{p\ times}$, *where*

$$\Lambda_0 = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}, \quad \Lambda_1 = \{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1r}\},$$

$$\lambda_{1k} \quad \in \quad \{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n\}, \quad k = 1, 2, \ldots, r,$$

such that $\Gamma_1 = \{\lambda\} \cup \Lambda_1$, $0 \leq \lambda \leq \lambda_1$, *is permutatively (circulantly) realizable.*
ii) There exists a $p \times p$ *permutative (circulant) nonnegative matrix with spectrum* Λ_0 *and diagonal entries* $\lambda, \lambda, \ldots, \lambda$ *(p times).*
Then, Λ *is permutatively universally realizable.*

Proof. Let A_1 be an $(r+1) \times (r+1)$ permutative realizing matrix for Γ_1. Then,

$$A = \begin{bmatrix} A_1 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_1 \end{bmatrix}$$

is a $p(r+1) \times p(r+1)$ nonnegative permutative matrix with spectrum $\Gamma_1 \cup \cdots \cup \Gamma_1$. From *ii*) let B be a $p \times p$ permutative nonnegative matrix, with spectrum Λ_0, and diagonal entries $\lambda, \lambda, \ldots, \lambda$ (p times). Let X be the $n \times p$ matrix of eigenvectors $\mathbf{x}_i$ of A. Since $A_1 \in CS_\lambda$, then columns of X are of the form

$$\mathbf{x}_1^T = [\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}], \; \mathbf{x}_2^T = [\mathbf{0}, \mathbf{1}, \ldots, \mathbf{0}], \ldots, \mathbf{x}_p^T = [\mathbf{0}, \ldots, \mathbf{0}, \mathbf{1}],$$

where $\mathbf{1}$ and $\mathbf{0}$ represent $\underbrace{1, 1, \ldots, 1}_{r+1 \text{ times}}$, and $\underbrace{0, 0, \ldots, 0}_{r+1 \text{ times}}$, respectively. Let $C' = B - \Omega$, where $\Omega = diag\underbrace{\{\lambda, \lambda, \ldots, \lambda\}}_{p \text{ times}}$. Then, from Rado's Theorem, $A + XC$, where C is C' with r zero columns interlaced between each column of C', is permutative with spectrum Λ and diagonal JCF. To obtain a possible nondiagonal JCF, with a Jordan block $J_k(\lambda_k)$ of size $k \geq 2$, we set appropriate real numbers on the free positions (zero positions) on the last row of the block $J_k(\lambda_k)$, under the main diagonal, in such a way that the modified matrix A', from A, preserves the spectrum Λ, $A' \in CS_\lambda$, and $A' + XC$ is nonnegative permutative with the desired JCF. Observe that from [19, Theorem 5] for $S = \left[X \mid Y\right]$ nonsingular with $S^{-1} = \left[\frac{U}{V}\right]$, we have

$$S^{-1}(A' + XC)S = \begin{bmatrix} B & CY + UA'Y \\ 0 & VA'Y \end{bmatrix}.$$

Moreover, from [22, Lemma 2.2], if B and $VA'Y$ have no common eigenvalues, then $J(A'+XC) = J(B) \oplus J(VA'Y)$. This is the case with the eigenvalues of Λ_0 and Λ_1, which allow us to obtain the desired JCF. As we can do this for each possible Jordan block, we may obtain all possible JCFs allowed by Λ. Thus, Λ is permutatively universally realizable. □

Example 3.2. *Consider the spectrum*

$$\begin{aligned} \Lambda &= \{4, 1, 1, -2, -2, -2\}, \text{ with} \\ \Lambda_0 &= \{4, 1, 1\}, \; \Gamma_1 = \{2, -2\}. \end{aligned}$$

Then,

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \text{ and } A_1' = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$

are permutative, realizing Λ_0 *and* Γ_1*, respectively. Then,*

$$A_1 = A_1' \oplus A_1' \oplus A_1' + XC = \begin{bmatrix} 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

is nonnegative permutative with diagonal JCF. Next, for

$$A'' = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 2 & 0 \end{bmatrix},$$

we obtain $A_2 = A'' + XC$, *nonnegative permutative, with JCF having one* 2×2 *Jordan block* $J_2(-2)$. *Next, for*

$$A''' = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix},$$

we obtain $A_3 = A''' + XC$, *nonnegative permutative, with JCF having a* 3×3 *Jordan block* $J_3(-2)$. *Thus,* $\Lambda = \{4, 1, 1, -2, -2, -2\}$ *is permutatively UR.*

The following example shows that in many cases, if λ_2 is non-simple in Theorem 3.1, it is still possible to obtain universal realizability for Λ, but this is difficult to predict.

Example 3.3. *Consider the spectrum*

$$\Lambda = \{8, -1, -1, -3, -3\}.$$

We start with

$$B = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 & 0 \\ 9 & 0 & -1 & 0 & 0 \\ 11 & 0 & 0 & -3 & 0 \\ 11 & 0 & 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{q}^T = \begin{bmatrix} -8 & 1 & 1 & 3 & 3 \end{bmatrix}.$$

Then

$$A_1 = B + \mathbf{e}\mathbf{q}^T = \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 0 & 1 & 3 & 3 \\ 1 & 1 & 0 & 3 & 3 \\ 3 & 1 & 1 & 0 & 3 \\ 3 & 1 & 1 & 3 & 0 \end{bmatrix}$$

is permutative with diagonal JCF. Next,

$$A_2 = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 & 0 \\ 9 & 0 & -1 & 0 & 0 \\ 11 & 0 & 0 & -3 & 0 \\ 11 & 0 & 2 & -2 & -3 \end{bmatrix} + \mathbf{e}\mathbf{q}^T = \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 0 & 1 & 3 & 3 \\ 1 & 1 & 0 & 3 & 3 \\ 3 & 1 & 1 & 0 & 3 \\ 3 & 1 & 3 & 1 & 0 \end{bmatrix}$$

has JCF, with a 2×2 *Jordan block* $J_2(-3)$, *corresponding to* $\lambda = -3$, *while other blocks are* 1×1. *Moreover,*

$$A_3 = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 11 & -3 & 0 & 0 & 0 \\ 11 & 0 & -3 & 0 & 0 \\ 9 & 0 & 0 & -1 & 0 \\ 9 & 0 & -2 & 2 & -1 \end{bmatrix} + \mathbf{e}\mathbf{r}^T = \begin{bmatrix} 0 & 3 & 3 & 1 & 1 \\ 3 & 0 & 3 & 1 & 1 \\ 3 & 3 & 0 & 1 & 1 \\ 1 & 3 & 3 & 0 & 1 \\ 1 & 3 & 1 & 3 & 0 \end{bmatrix},$$

where $\mathbf{r}^T = \begin{bmatrix} -8 & 3 & 3 & 1 & 1 \end{bmatrix}$, *has JCF with a* 2×2 *Jordan block* $J_2(-1)$, *while other blocks are* 1×1. *Finally,*

$$A_4 = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 11 & -3 & 0 & 0 & 0 \\ 9 & 0 & -1 & 0 & 0 \\ 11 & -2 & 2 & -3 & 0 \\ 9 & -2 & 2 & 0 & -1 \end{bmatrix} + \mathbf{e}\mathbf{s}^T = \begin{bmatrix} 0 & 3 & 1 & 3 & 1 \\ 3 & 0 & 1 & 3 & 1 \\ 1 & 3 & 0 & 3 & 1 \\ 3 & 1 & 3 & 0 & 1 \\ 1 & 1 & 3 & 3 & 0 \end{bmatrix},$$

where $\mathbf{s}^T = \begin{bmatrix} -8 & 3 & 1 & 3 & 1 \end{bmatrix}$, *has JCF with the Jordan blocks* $J_2(-3)$, $J_2(-1)$, $J_1(8)$.

Remark 3.1. *In [23] was shown that a complex Suleĭmanova spectrum is realizable by a permutative nonnegative matrix. It is an open question whether or not a complex Suleĭmanova spectrum is permutatively universally realizable.*

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