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Schrödinger's tridiagonal matrix

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Abstract: In the third part of his famous 1926 paper 'Quantisierung als Eigenwertproblem', Schrödinger came across a certain parametrized family of tridiagonal matrices whose eigenvalues he conjectured. A 1991 paper wrongly suggested that his conjecture is a direct consequence of an 1854 result put forth by Sylvester. Here we recount some of the arguments that led Schrödinger to consider this particular matrix and what might have led to the wrong suggestion. We then give a self-contained elementary (though computational) proof which would have been accessible to Schrödinger. It needs only partial fraction decomposition. We conclude this paper by giving an outline of the connection established in recent decades between orthogonal polynomial systems of the Hahn class and certain tridiagonal matrices with fractional entries. It also allows to prove Schrödinger's conjecture.

Keywords: tridiagonal matrix; eigenvalues; partial fraction decomposition; rational function identities; orthogonal polynomials; quantum theory; history

MSC: 15A15; 15B99; 47B36

1 Introduction

Erwin Schrödinger won the 1933 Nobel prize for physics mainly due to the paper 'Quantisierung als Eigenwertproblem' which appeared in 1926 as a series of four articles in the first of which [32, p. 362] a version of the Schrödinger equation is announced.

The article relevant for us is the third article, [33]. After the introduction it is split into two parts. Part I, consisting in Sections 1 and 2 (§1, §2), is titled 'Perturbation Theory'. It is a general study of relevant parts of that theory and does not make direct connection to physics. §1 commences by recalling the eigenvalue theory of a linear self-adjoint differential operator of order 2 in one variable, in particular Sturm-Liouville theory.

The author relies here often on the classic book by Courant and Hilbert [6] of which in those times only the first volume existed. Already in §2 Schrödinger seems to follow largely his own ideas and develops the theory, as far as possible 'by analogy', for linear partial differential operators and hence for the multivariable case.

These developments are made with the objective to explain in Part II, titled 'Applications to the Stark effect' (and consisting of §3,§4,§5), the splitting of the spectrum of light emitted in the presence of strong electric fields (Stark effect). These fields enter in the corresponding Schrödinger equation as perturbations which happen to split also multiple eigenvalues.

The perturbed Schrödinger equation is treated in §3 'by a method which corresponds to Epstein's' via separation of variables using spatial parabolic coordinates and then again in §5 by the 'method of Bohr' , 'as an example of the general perturbation theory of §2 [...] if one does *not* notice that in parabolic coordinates the perturbed equation is also exactly separable' .

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In §4 , titled ‘Tentative computation of intensities and polarisations of a splitting image’, the results of the computations are compared with experiments and remarks are made concerning the relation of his wave mechanics with Heisenberg’s matrix mechanics.

It is relevant to say at this place that Schrödinger’s ponderings or problems have been highly influential also in mathematics as is clear from newer editions of the Courant Hilbert book. The modern reader who wants to understand the context and reasoning of the Quantisierung paper and in particular find his way to the determinant we will study will find it vantageous to study the relevant parts of [7]. Thus e.g. [7] contains sections in chapters V (‘Oscillation and eigenvalue problems of Mathematical Physics’) and VI (‘Applications of the Calculus of Variations to eigenvalue problems’) explicitly dedicated to the eigenvalue problems raised by the Schrödinger equation because this equation leads ‘to a type of eigenvalue problems whose spectra have a very different structure than the previously treated [oscillation problems], namely a continuous and a discrete part’ .

In the multivariable case, according to [7, V §13.2] ‘the complication which arises in the case of a multiple eigenvalue is that the eigenfunctions are only determined up to an orthogonal transformation and in case of a perturbation a continuous continuation of the eigenfunctions can only be expected after a suitable system of eigenfunctions is found’ . [7] then gets a system of equations that, according to the present author’s understanding, corresponds (in other notation) very nearly to Schrödinger’s determinantal equation §2 (23).

It is in §5 [33, p. 480-1] using the more general method of Bohr which he had judged on p. 439 as ‘theoretically more beautiful because more generalizable but in wave mechanics more laborious’ , that Schrödinger derives from the determinant (23) he has found in §2, determinants he writes as

$$\begin{vmatrix} -\varepsilon & \varepsilon_{m+1,m} & & & & \\ \varepsilon_{m+1,m} & -\varepsilon & \varepsilon_{m+2,m} & & & \\ & \varepsilon_{m+2,m} & -\varepsilon & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & -\varepsilon & \varepsilon_{l-1,m} \\ & & & & \varepsilon_{l-1,m} & -\varepsilon \end{vmatrix},$$

defining $\varepsilon_{n,m} = -6lg\sqrt{\frac{(l^2-n^2)(n^2-m^2)}{4n^2-1}}$. Here g is a constant and l, m, n are positive integers, so that the matrix is $(l-m)\times(l-m)$ and n may assume the values $m, m+1, \dots, l-1$; m the values $1, \dots, l-1$.

Schrödinger then says: ‘If one divides each term by the common factor $6lg$ of the ε_{nm} and views $k^* = -\frac{\varepsilon}{6lg}$ as unknown, then the equation [determinant=0] has the roots $k^* = \pm(l-m-1), \pm(l-m-3), \pm(l-m-5), \dots$ which series stops with ±1 or with 0 (inclusively) depending on whether the degree $l-m$ is even or odd. The proof of this, unfortunately, you do *not* find in the mathematical appendix, since I did not succeed in finding it ...’ (author’s translation; emphasis by Schrödinger).

For ease of comparison, above we kept Schrödinger’s notation. The definitive notation we use is given in Section 3.

The 1991 paper [35, p. 342] claims that Schrödinger unwittingly considered in his famous paper determinants known as Sylvester-Kac determinants. This claim is at least twice repeated; see e.g. [17, p. 28], [4, p. 408]. The Sylvester-Kac determinant is given as

$$\begin{aligned} \mathrm{SK}_n(x) &= \begin{vmatrix} x & 1 & & & \\ n-1 & x & 2 & & \\ & n-2 & \ddots & \ddots & \\ & & \ddots & x & n-1 \\ & & & 1 & x \end{vmatrix} \\ &= (-n+1+x)(-n+3+x)\cdots(n-3+x)(n-1+x). \end{aligned}$$

Note that depending on whether n is odd or even, 0 is or is not a root; that is an eigenvalue of the underlying zero-axial matrix.

According to the history reported in Olga Taussky and John Todd [35], see also Sir Thomas Muir's monumental tome [23, pp. 425, 442], this formula was first stated without proof by James Joseph Sylvester in 1854 in a half page long paper [34], reprinted in [35], but a proof was provided only in 1866 by Francesco Mazza. Concerning a note on this see [16, p. 1207].

In 1947 Mark Kac [19] and in 1957 Péter Rósza [28] independently, not knowing about the earlier results found other proofs; all these are reviewed in [35]. But perhaps the best proof this author knows of can be found in the paper [10, p.18] by Alan Edelman and Eric Kostlan.

Although the Sylvester-Kac matrix plays indeed a rôle in our proof - this is just one more example where it inspired new results (others are [2, 8, 9, 13–16, 18, 25, 27] to name a few) - it seems not at all obvious how to deduce from it the value of the Schrödinger determinant. While we are at it, we may add that discrete versions of Schrödinger's equation and in fact Sturm Liouville problems play a part in new approaches to the problem of inverting tridiagonal matrices. See [11].

The claim in [35] seems due to a peculiar confusion about which we will not speculate too much here: But it is interesting that at about the same time Schrödinger wrote his famous series of papers, he also published together with Friedrich Wilhelm Kohlrausch a short paper [22] on Ludwig Boltzmann's H-theorem and in that paper indeed stumbled in the second section over a linear system whose matrix (never written down) would be essentially that of the Sylvester-Kac matrix. Now the 1968 appendix of Mark Kac's Chauvenet prize-winning paper [19] or [1], on Brownian motion to which [35] refers, contains in its references [22] but not [33], while [35] has in its references [33] but not [22]. Furthermore, as we saw, for the $n \times n$ Schrödinger determinants the values of the $n \times n$ Sylvester-Kac determinant are predicted. For example, we have after substituting $k^* = -\varepsilon/6lg$ by x and putting $\varepsilon_{nm} = \sqrt{\frac{(l^2-n^2)(n^2-m^2)}{4n^2-1}}$ for $(l, m) = (10, 5)$ and $(l, m) = (11, 6)$, respectively, the values $\varepsilon_{m+1,m} = 8/\sqrt{13}$ and $\varepsilon_{m+1,m} = 2\sqrt{6/5}$ and get the determinants

$$\begin{vmatrix} x & \frac{8}{\sqrt{13}} & & & \\ \frac{8}{\sqrt{13}} & x & 2\sqrt{\frac{102}{65}} & & \\ & 2\sqrt{\frac{102}{65}} & x & 6\sqrt{\frac{13}{85}} & \\ & & 6\sqrt{\frac{13}{85}} & x & 2\sqrt{\frac{14}{17}} \\ & & & 2\sqrt{\frac{14}{17}} & x \end{vmatrix}, \quad \begin{vmatrix} x & 2\sqrt{\frac{6}{5}} & & & \\ 2\sqrt{\frac{6}{5}} & x & 2\sqrt{\frac{133}{85}} & & \\ & 2\sqrt{\frac{133}{85}} & x & 30\sqrt{\frac{2}{323}} & \\ & & 30\sqrt{\frac{2}{323}} & x & \frac{8}{\sqrt{19}} \\ & & & \frac{8}{\sqrt{19}} & x \end{vmatrix},$$

respectively. Both these determinants are equal to the 5×5 Sylvester-Kac determinant which is

$$\begin{vmatrix} x & 1 & & & \\ 4 & x & 2 & & \\ & 3 & x & 3 & \\ & & 2 & x & 4 \\ & & & 1 & x \end{vmatrix} = (-4 + x)(-2 + x)(x)(2 + x)(4 + x).$$

Is this so in general?

If we define $n = l - m$, then we have $l^2 - (m + i)^2 = (n - i)(2m + n + i)$, and Schrödinger's $n \times n$ determinants can be written

$$\mathrm{Sch}_n(x) = \begin{vmatrix} x & \varepsilon_{1,2} & & & & \\ \varepsilon_{1,2} & x & \varepsilon_{2,3} & & & \\ & \varepsilon_{2,3} & x & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x & \varepsilon_{n-1,n} \\ & & & & \varepsilon_{n-1,n} & x \end{vmatrix},$$

where $i = 1, 2, \dots, n - 1$, $m \in \mathbb{Z}$, is a free parameter which we mostly will suppress in notation and the $\varepsilon_{i,1+i} = \varepsilon_{i,1+i}(m) \geq 0$ are now redefined by

$$\varepsilon_{i,1+i}^2 = \frac{i(n - i)(2m + i)(2m + n + i)}{(2m + 2i - 1)(2m + 2i + 1)}.$$

From this formula and an argument in Section 2 it is clear that as $m \to \infty$, we have $\varepsilon_{i,1+i} \to i(n-i)$ and so whatever the Schrödinger determinant is, it converges to the Sylvester-Kac determinant of the same size. Schödinger's conjecture $\mathrm{Sch}_n(x) = \mathrm{SK}_n(x)$ says of course much more: namely that the polynomial in x, $\mathrm{Sch}_n(x)$, has coefficients that do not depend on m. From observations on partial fraction decomposition it will follow in Section 3 that the coefficients of the polynomial $\mathrm{Sch}_n(x)$ must be $\mathbb{Q}$-linear combinations of the fractions $1/(2m+2i-1)$, $i = 1, 2, \ldots, n-1$. Hence an elementary possibility to show the conjecture is to show that the coefficients of these fractions are all 0. To prove this is the topic of sections 4 and 6. Section 5 proves some special cases, partly to cover limiting cases not transparently covered by the arguments for the typical case, partly to prepare the reader's mind for the somewhat technical section that follows. In particular an extended example contains all the essential ideas of the formal inductive proof. Section 7 tells how the modern theory of orthogonal polynomials is brought to bear on the problem. Paving the way via Krawtchouk polynomials to the more complicated Hahn class polynomials, a newcomer's effort (exploiting the valuable input of a referee) is made to turn relevant papers by Askey and Oste and van der Jeugt more accessible to a layman of the theory. That certain substitutions into the tridiagonal matrices pertaining to Hahn polynomials yield precisely Schrödinger's matrix, solving his problem and even yielding somewhat more, seems to have gone unnoticed by those authors.

As can be inferred from above words, the bulk of the present paper was found before we learned about the possibility to apply the theory of orthogonal polynomials. While the present proof is lengthy, it is elementary. By examining the messy calculations with a hidden beauty (see e.g. the treatment of the case $\mu = 5$ in Section 5), methods, perhaps independent of tridiagonal matrices, might be obtained for proving fractional identities like the one we show at the end of Section 6. Conversely it also transpires that tridiagonal matrices contain a potential for yielding interesting fractional identities.

2 Translation to a conjecture for coefficients

Chapter XIII of the treatise by Sir Thomas Muir and William Metzler, [24], is completely dedicated to 'continuants', nowadays better known as determinants of tridiagonal matrices. That book contains the determinants of many tridiagonal matrices that later where rediscovered. For example as well the determinant in Richard Askey [2, (3.25)] (after transposition) as the determinant in da Fonseca et al. [16, (2.1)] are special cases of the determinant considered in [24, §576]. The Sylvester determinant occurs for the 6×6 case in §578 (surprisingly without reference either to Sylvester or to Mazza). However, [24] is devoid of eigenvector statements for these matrices found by later writers.

In [24] the compact notation

$$K\begin{pmatrix} b_1\ b_1\ \cdots\ b_{n-1} \\ a_1\ a_2\ \cdots a_{n-1}\ a_n \\ c_1\ c_2\ \cdots\ c_{n-1} \end{pmatrix} := \begin{vmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & b_3 & & \\ & & c_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & b_{n-1} \\ & & & & c_{n-1} & a_n \end{vmatrix},$$

is used, which, if context allows is abbreviated to $K(1, n)$.

After some combinatorial reasoning [24, §545] concludes that the determinant $K(1, n)$ can be formed from $a_1 a_2 \ldots a_n$ by replacing in all possible ways 0,1,2, or more pairs of consecutive as by the signed product of bs and c that have the same index as the first of the consecutive as. In other words for each pair $a_r a_{1+r}$ replaced we have to write $-b_r c_r$.

Thus for example

$$K\begin{pmatrix} b_1\ b_2\ b_3\ b_4 \\ a_1\ a_2\ a_3\ a_4\ a_5 \\ c_1\ c_2\ c_3\ c_4 \end{pmatrix} = \begin{cases} a_1a_2a_3a_4a_5 + (-b_1c_1)a_3a_4a_5 + a_1(-b_2c_2)a_4a_5 \\ +a_1a_2(-b_3c_3)a_5 + a_1a_2a_3(-b_4c_4) + (-b_1c_1)(-b_3c_3)a_5 \\ +(-b_1c_1)a_3(-b_4c_4) + a_1(-b_2c_2)(-b_4c_4). \end{cases}$$

An important obvious consequence is

Lemma 1. *a. Assume* $b_1, c_1, \ldots, b_{n-1}, c_{n-1}$ *and* $b'_1, c'_1, \ldots, b'_{n-1}, c'_{n-1}$ *are complex numbers such that for* $i = 1, \ldots, n-1$ *there holds* $b_i c_i = b'_i c'_i$. *Then*

$$K\begin{pmatrix} & b_1\ b_2\ \cdots\ b_{n-1} \\ a_1\ a_2 & \cdots a_{n-1}\ a_n \\ & c_1\ c_2\ \cdots\ c_{n-1} \end{pmatrix} = K\begin{pmatrix} & b'_1\ b'_2\ \cdots\ b'_{n-1} \\ a_1\ a_2 & \cdots a_{n-1}\ a_n \\ & c'_1\ c'_2\ \cdots\ c'_{n-1} \end{pmatrix}$$

b. In particular the determinant of a tridiagonal matrix remains invariant under simultaneous changes of sign of all super and subdiagonal elements.

□

We write from now on

$$e_i = \varepsilon_{i,1+i}^2 \quad \text{and} \quad c_i = i(n-i).$$

By the previous lemma, the $n \times n$ Schrödinger determinant then is

$$\mathrm{Sch}_n(x) = K\begin{pmatrix} & e_1 & & e_2 & \ldots & e_{n-1} & \\ x & & x & & & & x \\ & 1 & & 1 & \ldots & 1 & \end{pmatrix},$$

while the $n \times n$ Sylvester-Kac determinant is

$$\mathrm{SK}_n(x) = K\begin{pmatrix} & c_1 & & c_2 & \ldots & c_{n-1} & \\ x & & x & & & & x \\ & 1 & & 1 & \ldots & 1 & \end{pmatrix}.$$

According to the previous arguments, a tridiagonal $n \times n$ matrix whose diagonal consists entirely of $x$es must have as determinant a monic polynomial in x of degree n with coefficients of $x^{n-1}, x^{n-3}, \ldots$ equal to 0. Hence we have for certain reals coef_ν and coef'_ν developments

$$\mathrm{Sch}_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \mathrm{coef}_{n-2i}\, x^{n-2i} \quad \text{and} \quad \mathrm{SK}_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \mathrm{coef}'_{n-2i}\, x^{n-2i}.$$

From this it follows that for n odd, $x|\mathrm{Sch}_n(x)$ so that 0 is an eigenvalue of the underlying zero axial matrix. To keep notation simple, Muir's rule suggests to define $a \prec b$ to say that $b - a \geq 2$. With this we can write the nonzero coefficients of $\mathrm{Sch}_n(x)$ and of $\mathrm{SK}_n(x)$ as follows.

$$\begin{array}{lcl lcl}
\mathrm{coef}_n & = & 1 & \mathrm{coef}'_n & = & 1 \\
\mathrm{coef}_{n-2} & = & -\sum\limits_{1\le i\le n-1} e_i & \mathrm{coef}'_{n-2} & = & -\sum\limits_{1\le i\le n-1} c_i \\
\mathrm{coef}_{n-4} & = & +\sum\limits_{1\le i\prec j\le n-1} e_i e_j & \mathrm{coef}'_{n-4} & = & +\sum\limits_{1\le i\prec j\le n-1} c_i c_j \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathrm{coef}_{n-2\mu} & = & (-1)^\mu \sum\limits_{1\le i_1\prec i_2\prec\ldots\prec i_\mu\le n-1} e_{i_1} e_{i_2} \cdots e_{i_\mu} & \mathrm{coef}'_{n-2\mu} & = & (-1)^\mu \sum\limits_{1\le i_1\prec i_2\prec\ldots\prec i_\mu\le n-1} c_{i_1} c_{i_2} \cdots c_{i_\mu} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Thus the conjecture is equivalent to:

Conjecture 2. *For all* $\mu = 0, 1, 2, \ldots, \lfloor n/2 \rfloor$ *there holds* $\mathrm{coef}_{n-2\mu} = \mathrm{coef}'_{n-2\mu}$, *that is,*

$$\sum_{1\le i_1\prec i_2\prec\ldots\prec i_\mu\le n-1} e_{i_1} e_{i_2} \cdots e_{i_\mu} = \sum_{1\le i_1\prec i_2\prec\ldots\prec i_\mu\le n-1} c_{i_1} c_{i_2} \cdots c_{i_\mu}.$$

After skimming over Proposition 3 below, the reader will have no difficulty to follow the proofs of some of these equalities, namely the cases $\mu = 0, 1$, and n even, $\mu = \lfloor n/2 \rfloor$ given in Section 5. For an example of how this conjecture looks explicitly, see the remark at the end of Section 6.

3 Notation and auxiliary results

Our proof of Schrödinger's conjecture unfortunately makes necessary a number of abbreviations in order to obtain manageable expressions. Also, we avoid repetitions giving to certain letters a fixed meaning.

Throughout the rest of the paper, the letters n, T, μ, E have the following meanings.

- n is a fixed integer ≥ 3; it indicates the size of the determinant.
- μ is a fixed integer $\in \{0, 1, \ldots, \lfloor n/2 \rfloor\}$; it refers to the μ in Conjecture 2.
- T is a fixed integer $\in \{1, \ldots, n\}$. Its significance is explained below.
- x will replace $2m$ as the parameter in e_i; x is not used anymore in the previous sense.

The symbols $c_i, d_i, t_i, e_i, e_{i_1,i_2,\ldots,i_\mu}$ (also written $e_{i_1 i_2 \ldots i_\mu}$) and E are defined as follows:

$$\begin{array}{rcll}
c_i & := & i(-i+n) & i \in \mathbb{Z} \\
d_i & := & 1/(x+2i-1) & i = 1, \ldots, n \\
t_i & := & -2^{-1} c_{-1+i} c_i d_i & i = 1, \ldots, n \\
e_i & := & c_i \dfrac{(x+i)(x+n+i)}{(x+2i-1)(x+2i+1)} & i = 1, \ldots, n-1 \\
e_{i_1,i_2,\ldots,i_\mu} & := & e_{i_1} e_{i_2} \cdots e_{i_\mu} & \\
E & := & \sum\limits_{1 \leq i_1 \prec i_2 \prec \ldots \prec i_\mu \leq n-1} e_{i_1} e_{i_2} \cdots e_{i_\mu} & \\
\mathrm{cf}(p) & : & \text{see after Corollary 4.} &
\end{array}$$

Proposition 3. *Among the c_i, d_i, t_i, and e_i there hold the following relations*

$$\begin{array}{rcll}
e_i & = & c_i + t_i - t_{1+i} & \\
 & = & c_i(x+i)(x+n+i) d_i d_{1+i} & \\
d_i d_j & = & 2^{-1}(j-i)^{-1}(d_i - d_j) & i \neq j \\
d_i t_j & = & -2^{-2}(j-i)^{-1} c_{-1+j} c_j (d_i - d_j) & i \neq j \\
t_i t_j & = & 2^{-3}(j-i)^{-1} c_{-1+i} c_i c_{-1+j} c_j (d_i - d_j) & i \neq j \\
0 & = & c_{-2+i} - 2c_i + c_{2+i} + 8 & i \in \mathbb{Z}
\end{array}$$

and for $i \neq j, j+1$:

$$d_i e_j = 2^{-2} c_j (4 d_i - c_{-1+j}(j-i)^{-1}(d_i - d_j) + c_{1+j}(1+j-i)^{-1}(d_i - d_{1+j})).$$

Proof. The identities claimed can all be verified by direct computation. For example, the computation

$$\begin{array}{rcl}
c_i + t_i - t_{1+i} & = & c_i - 2^{-1} c_{-1+i} c_i d_i + 2^{-1} c_i c_{1+i} d_{1+i} \\
 & = & c_i(1 - 2^{-1} c_{-1+i} d_i + 2^{-1} c_{1+i} d_{1+i}) \\
 & = & c_i\left(1 - \dfrac{2^{-1}(-1+i)(1-i+n)}{x+2i-1} + \dfrac{2^{-1}(1+i)(-1-i+n)}{x+2i+1}\right) \\
 & = & c_i \dfrac{(x+i)(x+n+i)}{(x+2i-1)(x+2i+1)} = c_i(x+i)(x+n+i) d_i d_{1+i}
\end{array}$$

proves the identity given for e_i. More rapidly than by hand, the c_i, d_i, etc. may be defined e.g. in Mathematica© as `c[i_] := i (-i + n); d[i_] := 1/(x + 2 i - 1)`, etc. and the identities then checked. □

Of particular importance in these formulas is that $d_i d_j$ (for $i \neq j$) is a linear combination of d_i and d_j obtained by partial fraction decomposition. The fractions $1, d_1, \ldots., d_n$ are as elements of the rational function field $\mathbb{Q}(x)$ evidently linearly independent over $\mathbb{Q}$. Since the t_i are rational multiples of the d_i we can and will often see a polynomial in which d_i and t_i appear tacitly as a polynomial in the d_i only. We have by iterative use of the multiplication formulae for $d_i d_j$ the following elementary though important fact.

Corollary 4. *Every polynomial in $\mathbb{Q}[d_1, \ldots., d_n, t_1, \ldots, t_n]$ in whose monomials the d_i and t_i occur in degrees 0 or 1 only and never with the same indices can be written uniquely as a $\mathbb{Q}$-linear combination of $1, d_1, \ldots., d_n$.*

Example. The polynomial $3d_1d_3 - 2d_1d_2d_3$ equals $\frac{1}{2}d_1 + \frac{1}{2}d_2 - d_3$. Indeed, check that

$$\frac{3}{(x+1)(x+5)} - \frac{2}{(x+1)(x+3)(x+5)} = \frac{1}{2(x+1)} + \frac{1}{2(x+3)} - \frac{1}{(x+5)}.$$

Consequently if $p \in \mathbb{Q}[d_1, \ldots., d_n, t_1, \ldots, t_n]$ is such a square free polynomial, then we can define $\mathrm{cf}(p)$ as the rational coefficient of d_T in the presentation of p as a $\mathbb{Q}$–linear combination of the $1, d_1, d_2, \ldots, d_n$. Also, by speaking of 'the coefficient of d_T in p' or 'the cf of p', we shall mean $\mathrm{cf}(p)$. Thus if p is the polynomial of the example and $T = 2$, then $\mathrm{cf}(p) = 1/2$; if $T = 3$, then $\mathrm{cf}(p) = -1$.

It is easy to see that $E \in \mathbb{Q}[t_1, \ldots, t_n]$ is a square free polynomial in the said sense.

Corollary 5. *If $T \neq j, j+1$, then*

$$\mathrm{cf}(d_Te_j) = 2^{-2}c_j(4 - c_{-1+j}(j-T)^{-1} + c_{1+j}(1+j-T)^{-1}).$$

Proof. This follows from the formula found for d_ie_j putting $i = T$ and extracting the coefficient of d_T. □

Lemma 6. *Assume p, q are polynomials in $d_1, \ldots, d_n$, in which each d_i has degree at most 1 and q is d_T-free, i.e. does not have t_T or d_T as a variable. Then*

$$\mathrm{cf}(pq) = \mathrm{cf}(p)\mathrm{cf}(d_Tq).$$

Proof. After developing p into a linear combination of the d_i, we can write $p = \mathrm{cf}(p)d_T + r$, where r is d_T-free. Then

$$\mathrm{cf}(qp) = \mathrm{cf}(q\,\mathrm{cf}(p)d_T + qr) = \mathrm{cf}(q\,\mathrm{cf}(p)d_T) + \mathrm{cf}(qr) = \mathrm{cf}(p)\mathrm{cf}(qd_T) + 0.$$

□

Conventions: Whenever we write a sum of the form $\sum e_{l_1,\ldots,l_\mu}$, where the l_i can be partially fixed (by context), it will always be assumed that $1 \leq l_1 \prec \ldots \prec l_\mu \leq -1+n$. Furthermore we allow notations of the form $\sum\{s_i : i \in I\}$ instead of $\sum_{i\in I} s_i$, whenever the description of the index set over which summations have to be done is complicated.

Indeed, later we have to consider sums of the form

$$\sum\{e_{i_1,\ldots,i_s,i_t,\ldots,i_\mu} : {}^{1\leq i_1\prec\ldots\prec i_{s-1}\prec i_s\leq a,}_{b\leq i_t\prec i_{1+t}\prec\ldots\prec i_\mu\leq -1+n}\}$$

which will often be abbreviated to

$$\sum\{e_{i_1,\ldots,i_s,i_t,\ldots,i_\mu} : {}^{i_s\leq a}_{b\leq i_t}\} \quad \text{or even to} \quad \sum\{e_{\ldots} : {}^{i_s\leq a}_{b\leq i_t}\}.$$

Another tool to shorten lines is to write $-^l$ in place of $(-1)^l$.

Example: Assume, say, $n = 15$, $T = 4$. Then the sum

$$\sum\{e_{i_1,T,i_3,i_4,i_5} : {}^{i_3\leq 8}_{11\leq i_4}\}$$

consists of the $2 \times 3 \times 3 = 18$ summands associated to the 5-uples $i_1, 4, i_3, i_4, i_5$ for which $i_1 \in \{1, 2\}$, $i_3 \in \{6, 7, 8\}$, $(i_4, i_5) \in \{(11, 13), (11, 14), (12, 14)\}$.

Sums whose conditions are unsatisfiable are of course to be considered 0. Thus if in the example above e.g. $T = 2$, then the sum would be 0.

4 Definition of E_{red}.

If we recall that $e_i = c_i + t_i - t_{1+i}$, then it is obvious that the polynomial at the left-hand side of Conjecture 2, when expanded, has as the subsum of its $t_{\ldots}$-free terms its right-hand side. Thus we need to prove that for an arbitrary $T \in \{1, 2, \ldots, n\}$, the coefficient of d_T in E is 0. Given that we consider T fixed we have to show simply that $\mathrm{cf}(E) = 0$. But E in expanded form has many terms that cannot contribute to the coefficient of d_T. So we begin by focusing on the important ones. The summation conventions made in Section 3 are full in place.

Proposition 7. *Assume $\mu \geq 2$. Define the polynomial E_{red}, by the following sum of $1 + 2(\mu - 2) + 1 = 2\mu - 2$ sums :*

$$\begin{aligned} E_{\mathrm{red}} = \sum e_{-1+T,1+T,i_3,\dots,i_\mu} &+ \\ \sum e_{i_1,-1+T,1+T,i_4,\dots,i_\mu} &+ \sum e_{-2+T,T,i_3,\dots,i_\mu} \\ \sum e_{i_1,i_2,-1+T,1+T,i_5,\dots,i_\mu} &+ \sum e_{i_1,-2+T,T,i_4,\dots,i_\mu} \\ \sum e_{i_1,i_2,i_3,-1+T,1+T,i_6,\dots,i_\mu} &+ \sum e_{i_1,i_2,-2+T,T,i_5,\dots,i_\mu} \\ &\vdots \\ \sum e_{i_1,i_2,\dots,i_{\mu-2},-1+T,1+T} &+ \sum e_{i_1,i_2,\dots,i_{\mu-3},-2+T,T,i_\mu} \\ &+ \sum e_{i_1,i_2,\dots,i_{\mu-3},i_{\mu-2},-2+T,T}. \end{aligned}$$

Then

$$\mathrm{cf}(E_{\mathrm{red}}) = \mathrm{cf}(E).$$

Proof. Evidently a necessary condition for a product $e_{i_1,i_2,\dots,i_\mu}$ occurring in E to contribute terms in which occurs d_T, is that t_T occurs in $e_{i_1,i_2,\dots,i_\mu}$. Since by definition $e_{i_1 i_2 \dots i_\mu} = \prod_{\nu=1}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$, for the occurrence it is necessary that $T \in \{i_1, 1+i_1, \dots, i_\mu, 1+i_\mu\}$. By the definition of '$\prec$' , it can happen only for at most one $\nu \in \{1, 2, \dots, \mu\}$, that $T \in \{i_\nu, 1+i_\nu\}$, i.e. $i_\nu = T$ or $i_\nu = T - 1$. Hence cf(E) equals the cf of

$$\sum e_{-1+T,i_2,\dots,i_\mu} + \sum e_{T,i_2,\dots,i_\mu} + \sum e_{i_1,-1+T,i_3,\dots,i_\mu} + \sum e_{i_1,T,i_3,\dots,i_\mu}$$
$$+ \dots + \sum e_{i_1,i_2,i_3,\dots,i_{\mu-1},-1+T} + \sum e_{i_1,i_2,i_3,\dots,i_{\mu-1},T}.$$

We shrink this sum further. We look first at the case $T \in \{i_1, 1+i_1\}$ and compare

$$e_{-1+T,i_2,\dots,i_\mu} = (c_{-1+T} + t_{-1+T} - t_T)\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$$

with

$$e_{T,i_2,\dots,i_\mu} = (c_T + t_T - t_{1+T})\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu}).$$

Note that, whenever $T \prec i_2$ then also $-1+T \prec i_2$ and then the cfs of $e_{-1+T,i_2,\dots,i_\mu}$ and $e_{T,i_2,\dots,i_\mu}$ are simply the cfs of $-t_T\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$ and $t_T\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$, respectively , and hence cancel each other. Therefore the coefficient of d_T in $\sum e_{-1+T,i_2,\dots,i_\mu} + \sum e_{T,i_2,\dots,i_\mu}$ equals this coefficient in the surviving terms containing t_T and these pertain to the $i_2 i_3 \dots i_\mu$ for which $i_2 = 1+T$. So the coefficient of d_T of the sum of the two sums equals $\mathrm{cf}(\sum e_{-1+T,1+T,i_3,\dots,i_\mu})$.

Next we compare, supposing $2 \leq l \leq \mu - 1$,

$$e_{i_1,\dots,i_{l-1},-1+T,i_{l+1},\dots,i_\mu} = (c_{-1+T} + t_{-1+T} - t_T)\cdot\prod_{\nu\neq l}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$$

with

$$e_{i_1,\dots,i_{l-1},T,i_{l+1},\dots,i_\mu} = (c_T + t_T - t_{1+T})\cdot\prod_{\nu\neq l}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu}).$$

Whenever $i_{l-1} \prec -1+T$ and $T \prec i_{1+l}$, then both of above es occur in the sum E but the coefficients of their respective d_T have opposed signs and so cancel. The strings of indices subordinated to es whose coefficients of d_T do not cancel are those with $i_{l-1} = -2+T$ and those with $i_{l+1} = 1+T$. So we see that the coefficient of d_T in $\sum e_{i_1,\dots,i_{l-1},-1+T,i_{l+1},\dots,i_\mu} + \sum e_{i_1,\dots,i_{l-1},T,i_{l+1},\dots,i_\mu}$ equals the coefficient of d_T in $\sum e_{i_1,\dots,i_{l-2},-2+T,T,i_{l+1},\dots,i_\mu} + \sum e_{i_1,\dots,i_{l-1},-1+T,1+T,i_{l+1},\dots,i_\mu}$.

Finally we look at the coefficient of d_T in $\sum e_{i_1,\dots,i_{\mu-1},-1+T} + \sum e_{i_1,\dots,i_{\mu-1},T}$. If $i_{\mu-1} \prec -1+T$, then $i_{\mu-1} \prec T$. By analogous reasoning as in the cases before, it follows that the coefficient of d_T of the present sum is that of $\sum e_{i_1,\dots,i_{\mu-2},-2+T,T}$. This proves the proposition. □

Lemma 8. *We have the equalities (a) and (b) below and define by means of them reals C_* as indicated.*
a. $\mathrm{cf}(d_T e_{-1-i+T}) = \mathrm{cf}(d_T e_{i+T}) =: C_i\ (\,i \neq -1, 0)$
b. $\mathrm{cf}(e_{-1+T,1+T}) = -\mathrm{cf}(e_{-2+T,T}) = -2^{-4} c_{-2+T} c_{-1+T} c_T c_{1+T} =: -C_0$

Proof: a. We have by Corollary 5 the equalities

$$\begin{aligned}\mathrm{cf}(d_T e_{-1-i+T}) &= c_{-1-i+T}(1 + 2^{-2}(1+i)^{-1} c_{-2-i+T} - 2^{-2} i^{-1} c_{-i+T})\\ \mathrm{cf}(d_T e_{i+T}) &= c_{i+T}(1 - 2^{-2} i^{-1} c_{-1+i+T} + 2^{-2}(1+i)^{-1} c_{1+i+T}).\end{aligned}$$

One now proves by a lengthy but straightforward computation that the quantities at the right-hand side are equal.

b. Since e_{1+T} has t_T not as a variable, we find by Lemma 6 that $\mathrm{cf}(e_{-1+T}e_{1+T}) = \mathrm{cf}(e_{-1+T})\mathrm{cf}(d_T e_{1+T})$. Now $\mathrm{cf}(e_{-1+T}) = 2^{-1} c_{-1+T} c_T$, while by Corollary 4 and Proposition 3, $\mathrm{cf}(d_T e_{1+T}) = 2^{-3} c_{1+T}(8 - 2c_T + c_{2+T}) = -2^{-3} c_{-2+T} c_{1+T}$. Hence follows part of the claim concerning $\mathrm{cf}(e_{-1+T,1+T})$. The other part follows by very similar computations. □

Corollary 9. *Assume $\mu \geq 2$. With the definitions*

$$\begin{aligned} C_0 &:= 2^{-4} c_{-2+T} c_{-1+T} c_T c_{1+T},\\ I(\nu) &:= \sum \left\{ e_{i_1,\dots,i_{\nu-1},i_{\nu+2},\dots,i_\mu} : {}^{i_{\nu-1} \leq -3+T}_{3+T \leq i_{\nu+2}} \right\}, \qquad (\nu = 1, \dots, -1+\mu)\\ II(\nu) &:= \sum \left\{ e_{i_1,\dots,i_{\nu-1},i_{\nu+2},\dots,i_\mu} : {}^{i_{\nu-1} \leq -4+T}_{2+T \leq i_{\nu+2}} \right\}, \qquad (\nu = 1, \dots, -1+\mu)\end{aligned}$$

there holds

$$\mathrm{cf}(E_{\mathrm{red}}) = C_0 \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(d_T(-I(\nu) + II(\nu))).$$

Observation: In the cases $\nu = 1, \nu = -1+\mu$ occur in $I(\nu), II(\nu)$ formally conditions involving i_0 or $i_{1+\mu}$. These should be discarded.

Proof. Recall that a typical sum occurring in the left column in the definition of E_{red} in Proposition 7 has the implicit requirement $i_{\nu-1} \prec -1+T \prec 1+T \prec i_{\nu+2}$. This is taken care of via the specifications introduced in the definition of $I(\nu)$. A similar remark holds for the sums of the right column in E_{red} and $II(\nu)$. By the multiplicative definition of $e_{\dots}$ we can put in the left column of the proposition the products $e_{-1+T,1+T}$ into evidence and in the right column the products $e_{-2+T,T}$. So we get

$$E_{\mathrm{red}} = \sum_{\nu=1}^{-1+\mu} e_{-1+T,1+T} I(\nu) + \sum_{\nu=1}^{-1+\mu} e_{-2+T,T} II(\nu).$$

Using that $I(\nu), II(\nu)$ are d_T-free polynomials, lemmas 6 and 8 yield

$$\begin{aligned}\mathrm{cf}(E_{\mathrm{red}}) &= \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-1+T,1+T} I(\nu)) + \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-2+T,T} II(\nu)),\\ &= \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-1+T,1+T})\mathrm{cf}(d_T I(\nu)) + \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-2+T,T})\mathrm{cf}(d_T II(\nu))\\ &= \sum_{\nu=1}^{-1+\mu} -C_0 \mathrm{cf}(d_T I(\nu)) + \sum_{\nu=1}^{-1+\mu} C_0(\mathrm{cf}(d_T II(\nu))\\ &= C_0 \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(d_T(-I(\nu) + II(\nu)).\end{aligned}$$

This analysis refers to the 'typical' case that $2 \le \nu \le -2+\mu$ holds. The general formulation bears the vestiges of this case. If $\nu = 1$ the natural translation for E_{red} given in Proposition 7 to the formula for E_{red} given above requires evidently the interpretation $I(1) := \sum \left\{ e_{i_3,\ldots,i_\mu} : 3+T \le i_3 \right\}$. So either one discards the condition $i_0 \le -3+T$ or, equivalently, assumes it automatically satisfied. Similar remarks hold of course for $I(-1+\mu)$. □

5 The equation of Conjecture 2 for $\mu = 0, 1, 2, 3, 5, n/2$.

We prove the equation of Conjecture 2 for the cases $\mu = 1$ and n even, $\mu = n/2$ in a direct manner using for these simple proofs very little from the general developments above. The cases $\mu = 2, 3, 5$ are also treated here. Their study will ease the digestion of the proof for general μ in the next section since they rely on showing $\mathrm{cf}(E_{\text{red}}) = 0$.

CASE $\mu = 0$. The equation of Conjecture 2 says for $\mu = 0$ the triviality that the leading term of $\mathrm{Sch}_n(x)$ as well as that of $\mathrm{SK}_n(x)$ is x^n.

CASE $\mu = 1$. Then we have to prove

$$\sum_{1 \le i \le -1+n} e_i = \sum_{1 \le i \le -1+n} c_i.$$

Since $e_i = c_i + t_i - t_{i+1}$, the sum at the left yields

$$\sum_{1 \le i \le -1+n} c_i + \sum_{1 \le i \le -1+n} (t_i - t_{i+1}) = \sum_{1 \le i \le \mu} c_i + (t_1 - t_n),$$

by telescoping. But $t_1 = -2^{-1} c_0 c_1 d_1$ and $t_n = -2^{-1} c_{-1+n} c_n d_n$ and $c_0 = c_n = 0$. Done. □

CASE $\mu = 2$. Then

$$E = \sum_{1 \le i \prec j \le -1+n} e_{ij} \quad \text{and} \quad E_{\text{red}} = e_{-1+T,1+T} + e_{-2+T,T}.$$

Thus $\mathrm{cf}(E_{\text{red}}) = \mathrm{cf}(e_{-1+T,1+T}) + \mathrm{cf}(e_{-2+T,T}) = 0$ by Lemma 8. □

CASE $\mu = 3$. Then

$$E = \sum_{1 \le i \prec j \prec k \le -1+n} e_{ijk},$$

and

$$\begin{aligned} E_{\text{red}} &= \sum e_{-1+T,1+T,k} + \sum e_{-2+T,T,k} + \sum e_{i,-1+T,1+T} + \sum e_{i,-2+T,T} \\ &= e_{-1+T,1+T}\left(\sum\{e_k : 3+T \le k\} + \sum\{e_i : i \le -3+T\}\right) \\ &\quad + e_{-2+T,T}\left(\sum\{e_k : 2+T \le k\} + \sum\{e_i : i \le -4+T\}\right) \\ &= e_{-1+T,1+T}(I(1) + I(2)) + e_{-2+T,T}(II(1) + II(2)). \end{aligned}$$

Again, by Lemma 8b, we have that $\mathrm{cf}(e_{-1+T,1+T}) = -C_0$, while $\mathrm{cf}(e_{-2+T,T}) = C_0$. Expressions $I(.)$ and $II(.)$ are d_T-free polynomials. Also note that the set of k for which $3+T \le k$ differs from the set of k for which $2+T \le k$ by the single element $2+T$. Hence $-I(1) + II(1) = -\sum\{e_k : 3+T \le k\} + \sum\{e_k : 2+T \le k\} = e_{2+T}$. Similarly one shows $-I(2) + II(2) = -e_{-3+T}$ and we note that similar considerations underlie many of the equations we later write down. Therefore, by lemmas 6 and 8a, putting there $i = 2$, we find

$$\begin{aligned} \mathrm{cf}(E_{\text{red}}) &= -C_0 \mathrm{cf}(d_T(I(1) + I(2)) + C_0 \mathrm{cf}(d_T(II(1) + II(2)) \\ &= C_0 \mathrm{cf}(d_T(-I(1) + II(1)) + C_0 \mathrm{cf}(d_T(-I(2) + II(2)). \\ &= C_0 \mathrm{cf}(d_T\, e_{2+T}) + C_0 \mathrm{cf}(d_T \cdot -e_{-3+T}) \\ &= C_0 C_2 - C_0 C_2 \\ &= 0 \end{aligned}$$

□

We now look at the case $\mu = 5$ to show the pattern of the reasoning that will be used in Section 6 in the general case.

Case $\mu = 5$. Here

$$E = \sum\{e_{ijklm} : 1 \le i \prec j \prec k \prec l \prec m \le n-1\}$$

and hence

$$\begin{aligned} E_{\text{red}} &= \sum e_{-1+T,1+T,k,l,m} + \sum e_{-2+T,T,k,l,m} + \sum e_{i,-1+T,1+T,l,m} + \sum e_{i,-2+T,T,l,m} \\ &\quad + \sum e_{i,j,-1+T,1+T,m} + \sum e_{i,j,-2+T,T,m} + \sum e_{i,j,k,-1+T,1+T} + \sum e_{i,j,k,-2+T,T} \\ &= e_{-1+T,1+T}(I(1) + I(2) + I(3) + I(4)) + e_{-2+T,T}(II(1) + II(2) + II(3) + II(4)); \end{aligned}$$

and so, for similar reasons as in the case explained before, or by Corollary 9,

$$\text{cf}(E_{\text{red}}) = C_0\text{cf}(d_T(-I(1) + II(1) - I(2) + II(2) - I(3) + II(3) - I(4) + II(4)).$$

Hence we can write

$$\begin{aligned} \text{cf}(E_{\text{red}}) &\overset{*1}{=} C_0\text{cf}(d_T\times \\ &\qquad (-\sum\{e_{klm} : 3+T\le k\} + \sum\{e_{klm} : 2+T\le k\} \\ &\qquad -\sum\left\{e_{ilm} : {i\le -3+T \atop 3+T\le l}\right\} + \sum\left\{e_{ilm} : {i\le -4+T \atop 2+T\le l}\right\} \\ &\qquad -\sum\left\{e_{ijm} : {j\le -3+T \atop 3+T\le m}\right\} + \sum\left\{e_{ijm} : {j\le -4+T \atop 2+T\le m}\right\} \\ &\qquad -\sum\{e_{ijk} : k\le -3+T\} + \sum\{e_{ijk} : k\le -4+T\})) \\ &\overset{*2}{=} C_0\text{cf}(d_T\times \\ &\qquad (\qquad\qquad\qquad e_{2+T}\sum\{e_{lm} : 4+T\le l\} \\ &\qquad -e_{-3+T}\sum\{e_{lm} : 3+T\le l\} + e_{2+T}\sum\left\{e_{im} : {i\le -4+T \atop 4+T\le m}\right\} \\ &\qquad -e_{-3+T}\sum\left\{e_{im} : {i\le -5+T \atop 3+T\le m}\right\} + e_{2+T}\sum\{e_{ij} : j\le -4+T\} \\ &\qquad -e_{-3+T}\sum\{e_{ij} : j\le -5+T\} \qquad\qquad)) \\ &\overset{*3}{=} C_0C_2\text{cf}(d_T\times \\ &\qquad (\sum\{e_{lm} : 4+T\le l\} - \sum\{e_{lm} : 3+T\le l\} \\ &\qquad +\sum\left\{e_{im} : {i\le -4+T \atop 4+T\le m}\right\} - \sum\left\{e_{im} : {i\le -5+T \atop 3+T\le m}\right\} \\ &\qquad +\sum\{e_{ij} : j\le -4+T\} - \sum\{e_{ij} : j\le -5+T\})) \\ &\overset{*4}{=} C_0C_2\text{cf}(d_T\times \\ &\qquad (-e_{3+T}\sum\{e_m : 5+T\le m\} \\ &\qquad -e_{3+T}\sum\{e_i : i\le -5+T\} + e_{-4+T}\sum\{e_m : 4+T\le m\} \\ &\qquad\qquad\qquad e_{-4+T}\sum\{e_i : i\le -6+T\})) \\ &\overset{*5}{=} C_0C_2C_3\text{cf}(d_T\times \\ &\qquad (-\sum\{e_m : 5+T\le m\} + \sum\{e_m : 4+T\le m\} \\ &\qquad -\sum\{e_i : i\le -5+T\} + \sum\{e_i : i\le -6+T\})) \\ &\overset{*6}{=} C_0C_2C_3\text{cf}(d_T\times \\ &\qquad (\qquad\qquad e_{4+T} \\ &\qquad -e_{-5+T} \qquad\qquad)) \\ &\overset{*7}{=} C_0C_2C_3(C_4 - C_4)) \\ &= 0 \end{aligned}$$

Here in '$\overset{*1}{=}$' we used the definitions of $I(v)$, $II(v)$ for the case $\mu = 5$ and transcribed them to the notation $ijklm$ in place of $i_1i_2i_3i_4i_5$. In '$\overset{*2}{=}$' we carried through line for line the additions of the lines of the previous block. Consider for example

$$-I(2) + II(2) = -\sum \left\{ e_{ilm} : {}^{i\le -3+T}_{3+T\le l} \right\} + \sum \left\{ e_{ilm} : {}^{i\le -4+T}_{2+T\le l} \right\}$$

Whenever ilm is so that $i \le -4 + T$ and $3 + T \le l$ (supposing as always $i \prec l \prec m$) then e_{ilm} will occur in both sums and hence cancel. There remain thus the triples ilm with $i = -3 + T$ in the left sum and those with $l = 2 + T$ in the right sum. Putting e_{-3+T} and e_{2+T} into evidence, the net result for $-I(2) + II(2)$ is given as indicated by the second line of the second block as $-e_{-3+T} \sum\{e_{lm} : 3 + T \le l\} + e_{2+T} \sum \left\{ e_{im} : {}^{i\le -4+T}_{4+T\le m} \right\}$. To obtain '$\overset{*_3}{=}$' note that the block after the '×' in '$\overset{*_2}{=}$' represents a d_T-free polynomial which comes as a sum in the form $-e_{-3+T}q_1 + e_{2+T}q_2$ (with evident q_1, q_2). So, using Lemma 6 with the pairs $(p, q) = (d_T \cdot -e_{-3+T}, q_1)$ and $(d_T \cdot e_{2+T}, q_2)$ and Lemma 8a we get $\mathrm{cf}(d_T(-e_{-3+T}q_1 + e_{2+T}q_2)) = \mathrm{cf}(d_T \cdot -e_{-3+T})\mathrm{cf}(d_Tq_1) + \mathrm{cf}(d_Te_{2+T})\mathrm{cf}(d_Tq_2) = C_2\mathrm{cf}(d_Tq_2) - C_2\mathrm{cf}(d_Tq_1) = C_2(\mathrm{cf}(d_T(q_2 - q_1))$. This yields the block introduced by $*_3$. The block $*_4$ is obtained from block $*_3$ similarly as block $*_2$ is obtained from $*_1$. Block $*_5$ comes from block $*_4$ similarly as $*_3$ from $*_2$ using that by Lemma 8 $\mathrm{cf}(d_Te_{3+T}) = \mathrm{cf}(d_Te_{-4+T}) = C_3$. In detail: the block $*_4$ is of type $-e_{3+T}q_1 + e_{-4+T}q_2$ with evident q_1, q_2, which are d_T-free. So using Lemma 6, this time with pairs $(p, q) = (d_T \cdot -e_{3+T}, q_1)$ and $(d_T \cdot e_{-4+T}, q_2)$ and Lemma 8a, we get $\mathrm{cf}(d_T(-e_{3+T}q_1 + e_{-4+T}q_2)) = \mathrm{cf}(d_T \cdot -e_{3+T})\mathrm{cf}(d_Tq_1) + \mathrm{cf}(d_Te_{-4+T})\mathrm{cf}(d_Tq_2) = -C_3\mathrm{cf}(d_Tq_1) + C_3\mathrm{cf}(d_Tq_2) = C_3(\mathrm{cf}(d_T(q_2 - q_1))$. Block $*_6$ comes from block $*_5$ just as block $*_4$ is obtained from block $*_3$. Finally block $*_7$ comes from $*_6$ using $\mathrm{cf}(d_Te_{4+T}) = \mathrm{cf}(d_Te_{-5+T}) = C_4$. ☐

Remark: Note that in case it happens that one of the $C_i = 0$, then the desired fact $\mathrm{cf}(E_{\mathrm{red}}) = 0$ follows at an earlier instance in these arguments: this is because then $C_0C_2C_3C_4 = 0$. ☐

Case n even and $\mu = n/2$. In this case the equation boils down to the claim $e_1e_3 \cdots e_{n-3}e_{n-1} = c_1c_3 \cdots c_{n-1}$. Now working directly with the definitions of the e_i we find

$$e_1e_3 \cdots e_{n-3}e_{n-1} = c_1c_3 \cdots c_{n-3}c_{n-1} \cdot \prod_{\substack{i=1 \\ i \equiv_2 1}}^{-1+n} \frac{(x+i)(x+n+i)}{(x+2i-1)(x+2i+1)}$$

Evidently the first of the products at the right is independent of x. The second product can be rewritten by substituting the admitted product index i by $2i + 1$ and ranging with i from 0 to $\frac{n}{2} - 1$. We then get that this second product equals

$$\prod_{i=0}^{(n/2)-1} \frac{(x+2i+1)(x+n+2i+1)}{(x+4i+1)(x+4i+3)}.$$

The product of the first factors in the numerator equals $(x+1)(x+3)\cdots(x+n-1)$ while the product of the second factors equals $(x+n+1)(x+n+3)\cdots(x+2n-1)$. The product of these products is easily seen to be the product of the denominators. So the last product displayed is equal to 1. ☐

6 Proof of Conjecture 2

According to the introduction to Section 4 and Proposition 7 we have to show $\mathrm{cf}(E_{\mathrm{red}}) = 0$. We start from the equation for $\mathrm{cf}(E_{\mathrm{red}})$ found in Corollary 9:

$$\mathrm{cf}(E_{\mathrm{red}}) = C_0 \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(d_T(-I(\nu) + II(\nu))).$$

We have

$$\begin{aligned} &-I(\nu) + II(\nu) \\ &= -\sum \left\{ e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-1}\le -3+T}_{3+T\le i_{\nu+2}} \right\} + \sum \left\{ e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-1}\le -4+T}_{2+T\le i_{\nu+2}} \right\} \end{aligned}$$

$$= \quad -e_{-3+T}\sum\{e_{i_1,\ldots,i_{\nu-2},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-2}\le -5+T}_{3+T\le i_{\nu+2}}\} + e_{2+T}\sum\{e_{i_1,\ldots,i_{\nu-1},i_{\nu+3},\ldots,i_\mu} : {}^{i_{\nu-1}\le -4+T}_{4+T\le i_{\nu+3}}\}.$$

The explanation is a generalization of one given in Section 5, for the case $\mu = 5$. Whenever $i_1, \ldots, i_{\nu-1}, i_{\nu+2}, \ldots, i_\mu$ is an uple so that $i_{\nu-1} \le -4 + T$ and $3 + T \le i_{\nu+2}$ holds, then $e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu}$ occurs in both the sums of the first line but with opposite signs and therefore cancels. Noting $\mathbb{Z}_{\le -3+T} = \{-3 + T\} \uplus \mathbb{Z}_{\le -4+T}$ and $\mathbb{Z}_{\le 3+T} \uplus \{2 + T\} = \mathbb{Z}_{\le 2+T}$, one sees that the signed $e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu}$ which do not cancel are those of the form $-e_{i_1,\ldots,i_{\nu-2},-3+T,,i_{\nu+2},\ldots,i_\mu}$ and those of the form $e_{i_1,\ldots,i_{\nu-1},2+T,i_{\nu+3}\ldots,i_\mu}$. Putting e_{-3+T} and e_{2+T} into evidence, this and the abbreviation conventions introduced earlier explain the second equation above.

Now by Lemma 8 we find $\mathrm{cf}(d_T e_{-3+T}) = \mathrm{cf}(d_T e_{2+T}) = C_2$, and by Lemma 6 and using that the inner sums in $*_1$ below are d_T-free, we find $\mathrm{cf}(E_{\mathrm{red}}) = 0$ iff the following holds:

$$*_1 : \mathrm{cf}(d_T(\sum_{\nu=1}^{-1+\mu}(-\sum\{e_{i_1,\ldots,i_{\nu-2},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-2}\le -5+T}_{3+T\le i_{\nu+2}}\} + \sum\{e_{i_1,\ldots,i_{\nu-1},i_{\nu+3},\ldots,i_\mu} : {}^{i_{\nu-1}\le -4+T}_{4+T\le i_{\nu+3}}\}))) = 0.$$

This is the beginning of an inductive procedure in which the outer sum $\sum_{\nu=1}^{\cdots}$ extends over less and less summands till it vanishes. We define more generally the claim $*_l$ as follows and show how to deduce $*_{1+l}$:

$$*_l : \mathrm{cf}(d_T\left(\sum_{\nu=1}^{-l+\mu} -^l \underbrace{\sum\{e\ldots : {}^{i_{\nu-2}\le -4-l+T}_{2+l+T\le i_{\nu+l+1}}\}}_{q_\nu} + -^{1+l}\underbrace{\sum\{e\ldots : {}^{i_{\nu-1}\le -3-l+T}_{3+l+T\le i_{\nu+l+2}}\}}_{q'_\nu}\right)) = 0$$

The first step is to write $\sum_{\nu=1}^{-l+\mu}\ldots$ as

$$\begin{aligned}\sum_{\nu=1}^{-l+\mu}(-^l q_\nu + -^{1+l}q'_\nu) &= -^l q_1 + \sum_{\nu=1}^{-1-l+\mu}(-^{1+l}q'_\nu + -^l q_{1+\nu}) + -^{1+l}q'_{-l+\mu}\\ &= -^l\sum_{\nu=1}^{-1-l+\mu}(-q'_\nu + q_{1+\nu}),\end{aligned}$$

where we used that $q_1 = 0$ for it involves an undefined i_{-1} in its conditions, and that $q'_{-l+\mu} = 0$, since it involves $i_{\mu+2}$ in its conditions.

Now

$$\begin{aligned}-q'_\nu + q_{1+\nu} &= -\sum\{e_{i_1\ldots i_{\nu-1},i_{\nu+l+2}\ldots i_\mu} : {}^{i_{\nu-1}\le -l-3+T}_{3+l+T\le i_{\nu+l+2}}\} + \\ &\quad + \sum\{e_{i_1\ldots i_{\nu-1},i_{\nu+l+2}\ldots i_\mu} : {}^{i_{\nu-1}\le -l-4+T}_{2+l+T\le i_{\nu+l+2}}\}\\ &= -\sum\{e_{i_1\ldots i_{\nu-2},-3-l+T,i_{\nu+l+2}\ldots i_\mu} : {}^{i_{\nu-2}\le -l-5+T}_{3+l+T\le i_{\nu+l+2}}\} + \\ &\quad + \sum\{e_{i_1\ldots i_{\nu-1},2+l+T,i_{\nu+l+3}\ldots i_\mu} : {}^{i_{\nu-1}\le -l-4+T}_{4+l+T\le i_{\nu+l+3}}\}\\ &= -e_{-3-l+T}\underbrace{\sum\{e\ldots : {}^{i_{\nu-2}\le -l-5+T}_{3+l+T\le i_{\nu+l+2}}\}}_{\tilde q'_\nu} + e_{2+l+T}\underbrace{\sum\{e\ldots : {}^{i_{\nu-1}\le -l-4+T}_{4+l+T\le i_{\nu+l+3}}\}}_{\tilde q_{1+\nu}}.\end{aligned}$$

Here we used that the es with indices $i_1\ldots i_{\nu-1}, i_{\nu+l+2}\ldots i_\mu$ for which the conjunction $(i_{\nu-1} \le -l-4+T)\,\&\,(3+l+T \le i_{\nu+l+2})$ holds occur in the two sums with opposite signs and thus cancel. The es that do not cancel are those with $i_{\nu-1} = -3-l+T$ in the left-hand sum and those with $i_{\nu+l+2} = 2+l+T$ in the right-hand sum. Thus the sum of the two sums can be rewritten as in the second equality which yields in turn the third equality. Note that the subindices of the es present 'words' which at the beginning have length $\nu-1+\mu-(\nu+l+2)+1 = \mu-l-2$ but in the last two sums have length $\mu-l-3$.

So the left-hand side of claim $*_l$ can be processed as follows:

$$\mathrm{lhs}(*_l) \quad = \quad \mathrm{cf}(d_T \cdot -^l \sum_{\nu=1}^{-1-l+\mu}(-q'_\nu + q_{1+\nu}))$$

$$
\begin{aligned}
&= -^{l} \sum_{\nu=1}^{-1-l+\mu} \mathrm{cf}(d_T(-e_{-3-l+T} \cdot \tilde{q}'_\nu + e_{2+l+T} \cdot \tilde{q}_{1+\nu})) \\
&= -^{l} \sum_{\nu=1}^{-1-l+\mu} (\mathrm{cf}(d_T \cdot -e_{-3-l+T})\mathrm{cf}(d_T\tilde{q}'_\nu) + \mathrm{cf}(d_T e_{2+l+T})\mathrm{cf}(d_T\tilde{q}_{1+\nu})) \\
&= -^{l} \sum_{\nu=1}^{-1-l+\mu} (-C_{2+l})\mathrm{cf}(d_T\tilde{q}'_\nu) + C_{2+l}\mathrm{cf}(d_T\tilde{q}_{1+\nu})) \\
&= C_{2+l} \sum_{\nu=1}^{-1-l+\mu} (-^{1+l}\mathrm{cf}(d_T\tilde{q}'_\nu) + -^{l}\mathrm{cf}(d_T\tilde{q}_{1+\nu})) \\
&= C_{2+l}\mathrm{cf}(d_T \cdot \sum_{\nu=1}^{-1-l+\mu} (-^{1+l}\tilde{q}'_\nu + -^{l}\tilde{q}_{1+\nu})).
\end{aligned}
$$

Therefore if $C_{2+l} \neq 0$, then by the definitions of $\tilde{q}'_\nu$, $\tilde{q}_\nu$ we can conclude

$$
\mathrm{cf}(d_T(\sum_{\nu=1}^{-1-l+\mu} (-^{1+l} \sum\{e_{\dots} : {}^{i_{\nu-2} \le -l-5+T}_{3+l+T \le i_{\nu+l+2}}\} + -^{l} \sum\{e_{\dots} : {}^{i_{\nu-1} \le -l-4+T}_{4+l+T \le i_{\nu+l+2}}\}))) = 0.
$$

In other words we have proved that if $C_{2+l} \neq 0$, then $*_l$ is equivalent to $*_{1+l}$. If $C_{2+l} = 0$, then we are done: $\mathrm{cf}(E_{\mathrm{red}}) = 0$ for similar reasons as we have given in the remark after the example $\mu = 5$ in the previous section. (This present remark is made to save us from transporting accumulated C_i to the next step.)

We finally infer the validity of these claims by verifying claim $*_{\mu-3}$. Indeed for $l = \mu - 3$ the sum $\sum_{\nu=1}^{-1-l+\mu}(-q'_\nu + q_{1+\nu})$ is

$$
\begin{aligned}
&\sum_{\nu=1}^{2}(-q'_\nu + q_{1+\nu}) = -q'_1 + q_2 - q'_2 + q_3 \\
&\quad = -\sum\{e_{\dots} : {}^{i_0 \le -\mu+T}_{\mu+T \le i_\mu}\} + \sum\{e_{\dots} : {}^{i_0 \le -1-\mu+T}_{-1+\mu+T \le i_\mu}\} - \sum\{e_{\dots} : {}^{i_1 \le -\mu+T}_{\mu+T \le i_{\mu+1}}\} + \sum\{e_{\dots} : {}^{i_1 \le -1-\mu+T}_{1+\mu+T \le i_{\mu+1}}\} \\
&\quad = -\sum\{e_{i_\mu} : \mu+T \le i_\mu\} + \sum\{e_{i_\mu} : -1+\mu+T \le i_\mu\} - \sum\{e_{i_1} : i_1 \le -\mu+T\} + \sum\{e_{i_1} : i_1 \le -1-\mu+T\} \\
&\quad = e_{-1+\mu+T} - e_{-\mu+T}.
\end{aligned}
$$

Since $\mathrm{cf}(d_T e_{-1+\mu+T}) = \mathrm{cf}(d_T e_{-\mu+T}) = C_{\mu-1}$ claim $*_{\mu-3}$ follows and Schrödinger's conjecture is proved. □

Remark If we choose in Conjecture 2 $\mu = 2$, do the routine evaluation of the right-hand side and write the left-hand side explicitly we get results like this one:

$$
\sum_{1 \le i \prec j \le n-1} \frac{i(n-i)j(n-j)(x+i)(x+n+i)(x+j)(x+n+j)}{(x+2i-1)(x+2i+1)(x+2j-1)(x+2j+1)} = \binom{n+1}{5}\frac{7+5n}{3}.
$$

A research program might center around the question to find alternative definitions for c_i, t_i, e_i such that relations like those of Conjecture 2 take place.

7 Connection with the theory of Orthogonal Polynomials

The connection between orthogonal polynomial sequences $\{p_n(x)\}_{n\ge0}$, where $\deg p_n = n$, and linear relations between three consecutive of the p_n and hence to tridiagonal matrices is known since long as Favard's Theorem; see [5]. In 2005 Richard Askey [2] noted that in particular the well studied sequences of Krawtchouk polynomials have a close connection to the Sylvester determinant. Among others, Askey studies in his paper also the relation of the sequence of Hahn polynomials to tridiagonal matrices and it turns out that these have a connection to Schrödinger's matrix. We report briefly on this approach.

We use the Pochhammer symbol $(u)_m = u(u+1)\cdots(u+m-1)$, $m = 0,1,2,\ldots$, and define for abstract variables n, x, p, N the hypergeometric series

$$ {}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix} ;\frac{1}{p}\right) = \sum_{k=0}^{\infty} \frac{(-n)_k(-x)_k}{(-N)_k k!}\left(\frac{1}{p}\right)^k. \qquad (*_1)$$

With the short-hand notation ${}_2F_1(n-) = {}_2F_1\left(\begin{matrix}-(n-1),-x\\ -N\end{matrix} ;\frac{1}{p}\right)$, a similar definition for ${}_2F_1(n+)$, and writing ${}_2F_1(n)$ for the original ${}_2F_1$ above it is not difficult to verify the following three-term contiguous relation for hypergeometric functions

$$(-x+(pN-2pn+n)){}_2F_1(n) = p(N-n){}_2F_1(n+) + n(1-p){}_2F_1(n-). \qquad (*_2)$$

For some information about contiguous relations between hypergeometric functions of type ${}_2F_1, {}_3F_2$, and ${}_4F_3$ and their connection to orthogonal polynomials see Wilson [37]; also compare this with Rainville [30, Section 48] or [29].

Now let N be a positive integer and define the Krawtchouk polynomials $K_n(x) = K_n(x;p,N)$, $n \in \{0,1,\ldots,N\}$ from $(*_1)$ by terminating the above series after $k = n$. Note that for $k = n+1$ the sum $(*_1)$ has a zero term; in fact the infinite series is interrupted by zero terms until a cancellation occurs with a denominator zero. One gets that the Krawtchouk polynomials satisfy the modified relation $(*_2)$ obtained by replacing ${}_2F_1(n-), {}_2F_1(n), {}_2F_1(n+)$ by $K_{n-1}(x), K_n(x), K_{n+1}(x)$, respectively.

When $n = N \in \mathbb{Z}_{\geq 1}$, the modified relation $(*_2)$ contains the apparently ill-defined expression $K_{N+1}(x)$ as one ends up with a zero in the denominator in the definition $(*_1)$. However we can use the identity $(*_2)$ for $n = N$ by seeing the expression $p(N-n)K_{n+1}(x)$ as a whole. One finds using $(-n-1)_{n+1} = (-1)^{n+1}(n+1)!$ and $(-N)_{n+1} = -(N-n)(-N)_n$ that

$$p(N-n)K_{n+1}(x) = \frac{(-1)^n(-x)_{n+1}}{(-N)_n p^n} + p(N-n)\sum_{k=0}^{n}\frac{(-n-1)_k(-x)_k}{(-N)_k k!}\left(\frac{1}{p}\right)^k. \qquad (*_3)$$

So

$$\lim_{n\to N} p(N-n)K_{n+1}(x) = \frac{(-x)_{N+1}}{N!p^N}.$$

For illustration of the connection to matrices take the case $N = 5$. For those values of x for which the limit is 0, that is for $x = 0,1,2,3,4,5$, the relation $(*_2)$ needs not mention K_6. Also note $K_{-1}(x) = 0$, $K_0(x) = 1$, and Lemma 1b. So $(*_2)$ can be written as

$$\begin{bmatrix} -x+0+5p & 5p & & & & \\ 1-p & -x+1+3p & 4p & & & \\ & 2(1-p) & -x+2+p & 3p & & \\ & & 3(1-p) & -x+3-p & 2p & \\ & & & 4(1-p) & -x+4-3p & p \\ & & & & 5(1-p) & -x+5-5p \end{bmatrix} \begin{bmatrix} K_0(x) \\ K_1(x) \\ K_2(x) \\ K_3(x) \\ K_4(x) \\ K_5(x) \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix}.$$

We have here a matrix which maps for $x = 0,1,2,3,4,5$ a nontrivial vector to the zero vector. Its determinant will evidently be a monic polynomial in x of degree 6; so it must be $(-x)_6$. In general an $(N+1)\times(N+1)$ matrix following the obvious generalization of the pattern shown will have determinant $(-1)^{N+1}x(x-1)\cdots(x-N) = (-x)_{N+1}$.

In the case $p = 1/2$ we get $(pN-2pn+n) = N/2$, and therefore a matrix which has diagonal entries all equal to $(-x+N/2)$. Multiplying the matrix with 2 and putting $y = -2x+N$, we get the matrix $SK_{N+1}(y)$ introduced in the introduction. From the above its determinant is $2^{N+1}(\frac{y}{2}-\frac{N}{2})_{N+1} = 2^{N+1}(y/2-N/2)(y/2-N/2+1)(y/2-N/2+2)\cdots(y/2-N/2+N) = (y-N)(y-N+2)\cdots(y+N)$ which is precisely Sylvester's claim.

As said, for Schrödinger's matrix the orthogonal sequence of Hahn polynomials is of relevance. The Hahn polynomials are extracted from the hypergeometric series

$$ {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x\\ \alpha+1,\ -N\end{matrix} ;1\right) = \sum_{k=0}^{\infty}\frac{(-n)_k(n+\alpha+\beta+1)_k(-x)_k}{(\alpha+1)_k(-N)_k k!}$$

in a manner similar as the Krawtchouk polynomials from the hypergeometric series shown earlier. Again, contiguous relations for such series permit to establish for the Hahn polynomials $Q_n(x) = Q_n(x; \alpha, \beta, N)$ a three-term recurrence

$$-xQ_n(x) = a_n Q_{n+1}(x) - (a_n + c_n)Q_n(x) + c_n Q_{n-1}(x)$$

with quite complicated values for a_n, c_n, and being $b_n = -a_n - c_n$, namely

$$a_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \; c_n = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \; b_n = -a_n - c_n.$$

One can show similarly as before a limit relation which this time reads

$$\lim_{n\to N} a_n Q_{n+1}(x, \alpha, \beta, N) = \frac{(-x)_{N+1}(N+\alpha+\beta+1)_N}{(\alpha+1)_N N!}.$$

Again let us look at the case $N = 5$. For the case $N = 5$ and the values of x which make the limit vanish, that is again $x = 0, 1, 2, 3, 4, 5$ the equations can be put into the matrix form

$$\begin{bmatrix} x+b_0 & a_0 & & & & \\ c_1 & x+b_1 & a_1 & & & \\ & c_2 & x+b_2 & a_2 & & \\ & & c_3 & x+b_3 & a_3 & \\ & & & c_4 & x+b_4 & a_4 \\ & & & & c_5 & x+b_5 \end{bmatrix} \begin{bmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ Q_4(x) \\ Q_5(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and so the matrix will have determinant $(-x)_6$; the analogous matrix for general N will have determinant $(-1)^{N+1}(-x)_{N+1}$.

In the case $\alpha = \beta$ the expressions for a_n, b_n, c_n simplify to

$$a_n = \frac{(n+2\alpha+1)(N-n)}{2(2n+2\alpha+1)}, \quad c_n = \frac{n(n+2\alpha+N+1)}{2(2n+2\alpha+1)}, \quad b_n = -N/2.$$

To get the connection with Schrödinger's conjecture, we do the replacements $n \to i-1$, $\alpha \to m$, $N \to n-1$, suggested to me by Mikhail Tyaglov [36]. Then one gets $a_{i-1}c_i = \frac{1}{4}\varepsilon_{i,i+1}^2$, where

$$\varepsilon_{i,1+i}^2 = \frac{i(n-i)(2m+i)(2m+n+i)}{(2m+2i-1)(2m+2i+1)}$$

is the quantity introduced in Section 1.

Recall that in the determinant $\mathrm{Sch}_{N+1}(x)$ the products of corresponding super- and subdiagonal elements are $\varepsilon_{i,i+1}^2$. Again recall Lemma 1. Now multiplying the underlying matrix above with 2 and then replacing the new diagonal elements $2x - N$ by y we get as its determinant a polynomial in y which by computations similarly to the above yields indeed the value of $\mathrm{Sch}_{N+1}(y)$ we proved in the previous sections. So we have here another proof for our main result. This result is also found as a consequence to results in Oste and Van der Jeugt in [26] as [27, Proposition 3], again using Hahn polynomials. Finally it should be noted that the approach outlined in this section yields in the cases $p = 1/2$ for Krawtchouk polynomials and $\alpha = \beta$ for Hahn polynomials actually also the eigenvectors of the corresponding matrices.

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References

[1] J. Abbot (Ed.), The Chauvenet papers, Vol. I., 1978, MAA (Math. Assoc. Amer.).
[2] R. Askey, Evaluation of Sylvester type determinants using orthogonal polynomials, In:[3] 1-16.
[3] H. G. W. Begehr et al. (Eds.), Advances in analysis. *Proceedings of the 4th international ISAAC Congress*. Paper presented at the conference (Toronto, Canada), 2003, August 11–16, 2003. Hackensack, NJ: World Scientific (ISBN 981-256-398-9/hbk).
[4] T. Bóros, P. Rósza, An explicit formula for the singular values of the Sylvester-Kac matrix, *Linear Algebra Appl.* 421 (2007), 407-416.
[5] T. Chihara, An Introduction to Orthogonal Polynomials. Reprint of a 1978 Gordon and Breach publication, Dover 2011.
[6] R. Courant, D. Hilbert, Methoden der Mathematischen Physik, Springer 1924.
[7] Third (and probably later and also english) editions of [6], Springer 1968.
[8] W. Chu, Spectrum and eigenvectors for a class of tridiagonal matrices, *Linear Algebra Appl.* 582 (2019) 499-516.
[9] W. Chu, X. Wang, Eigenvectors of tridiagonal matrices of Sylvester type, *Calcolo* 45 (2008), 217-233.
[10] A. Edelman, E. Kostlan, How many zeros of a random polynomial are real?, *Bull. Amer. Math. Soc.* 32 (1) (1995) 1- 34.
[11] A.M. Encinas, M.J. Jiménez, Explicit inverse of nonsingular Jacobi matrices, *Discrete Appl. Math.* 263 (2019) 130-139.
[12] A. Erdelyi, M. Weber, On the finite difference Analogue of Rodrigues' Formula, *Amer. Math. Monthly* 59 (1952), 163-168.
[13] C.M. da Fonseca, A short note on the Determinant of a Sylvester-Kac type matrix, *Int. J. Nonlinear Sci. Numer. Simul.* 21 (2020), 361-362.
[14] C.M. da Fonseca, E. Kılıç, A new type of Sylvester-Kac matrix and its spectrum, *Linear Multilinear Algebra* 2019. DOI: 10.1080/03081087.2019.1620673.
[15] C.M. da Fonseca, E. Kılıç, An observation on the determinant of a Sylvester-Kac type matrix, *An. Univ. Ştiinţ "Ovidius" Constanţa* 28 (2020), 111-115.
[16] C. M. da Fonseca, D. A. Mazilu, I. Mazilu, H. Thomas Williams, The eigenpairs of a Sylvester-Kac type matrix associated with a simple model for one-dimensional deposition and evaporation, *Appl. Math. Lett.* 26(12) (2013), 1206-1211.
[17] P. Hanlon, To the Latimer-Macduffee theorem and beyond, *Linear Algebra Appl.* 280 (1998), 21-37.
[18] O. Holtz, Evaluation of Sylvester type determinants using block-triangularization, In: [3] 395-405.
[19] M. Kac, Random Walks and Brownian Motion, *Amer. Math. Monthly* 54 (1947), 369-391. Reprinted with an appendix in [1].
[20] S. Karlin, J. L. McGregor, Hahn Polynomials, Formulas and an Application, *Scripta Math.* 26, 1(1961), 33-46.
[21] R. Koekoek, R. Swartouw, The Askey Scheme of hypergeometrical orthogonal polynomials and its q-analogues, `http://aw.twi.tudelft.nl/~koekoek/askey/` .
[22] K. W. F. Kohlrausch, E. Schrödinger, Das Ehrenfestsche Modell der H-Kurve, *Physikalische Zeitschrift* 27 (1926), 306-313.
[23] T. Muir, The Theory of Determinants in Historical Order of Development, vol. 2:1841-1860. Four-volume reprint in two volumes. Dover 1960.
[24] T. Muir, A Treatise on the Theory of Determinants, revised and enlarged by W. Metzler, Reprint of a 1933 Longman, Greens and Co. publication. Dover 1960.
[25] R. Oste, J. Van der Jeugt, Tridiagonal test matrices for eigenvalue computations: two-parameter extensions of the Clement matrix, *J. Comput. Appl. Math.* 314 (2017), 30-39.
[26] R. Oste; J. Van der Jeugt, Doubling (dual) Hahn polynomials: classification and applications, SIGMA *Symmetry, Integrability and Geometry: Methods and Applications* 12 (2016) 003.
[27] R. Oste, J. van der Jeugt, A finite quantum oscillator model related to special sets of Racah-polynomials, *Phys. Atomic Nucl.* 80 (2017), 786.
[28] P. Rósza, Bemerkungen über die Spektralzerlegung einer stochastischen Matrix, *Magyar. Tud. Akad. Mat. Fiz. Oszt. Közl.* 7 (1957): 199-206.
[29] E. D. Rainville, The contiguous function relations for ${}_pF_q$ with application to Bateman's $J_n^{u,v}$ and Rice's $H_n(\zeta, p, v)$, *Bull. Amer. Math. Soc. 51* (1945), 714-723.
[30] E. D. Rainville, Special Functions, Chelsea 1960.
[31] E. B. Saff, R. S. Varga (Eds.), Padé and Rational Approximation, Academic Press 1977.
[32] E. Schrödinger, Quantisierung als Eigenwertproblem: Erste Mitteilung, *Ann. Phys.* 79 (1926), 361-376.
[33] E. Schrödinger , Quantisierung als Eigenwertproblem: Dritte Mitteilung: Störungstheorie mit Anwendungen auf den Starkeffekt der Balmerlinien, *Ann. Phys.* 80 (1926), 437-490.
[34] J. J. Sylvester, Theoreme sur les determinants. *Nouvelles Annales de Math.* 13 (1854), 305.
[35] O. Taussky, J. Todd, Another look at a matrix of Mark Kac, *Linear Algebra Appl.* 150 (1991), 341-360.
[36] M. Tyaglov, Private communication, May 2020.
[37] J. A. Wilson, Three term contiguous relations and some new orthogonal Polynomials, In: [31], 227-232.