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Further extensions of Hartfiel's determinant inequality to multiple matrices

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Abstract: Following the recent work of Zheng et al., in this paper, we first present a new extension Hartfiel's determinant inequality to multiple positive definite matrices, and then we extend the result to a larger class of matrices, namely, matrices whose numerical ranges are contained in a sector. Our result complements that of Mao.

Keywords: determinant inequality, positive definite matrix, numerical range, sector matrix

MSC: 15A45, 15A60

1 Introduction

Throughout the paper, we denote by $\mathbb{M}_n$ the set of $n \times n$ complex matrices. Recall that the numerical range (see, e.g., [3]) of $A \in \mathbb{M}_n$ is defined as the set on the complex plane

$$W(A) = \{v^*Av : v \in \mathbb{C}^n,\ v^*v = 1\}.$$

For a fixed $\theta \in [0, \pi/2)$, obviously the set on the complex plane

$$S_\theta = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \le (\Re z)\tan\theta\}$$

is a sector excluding the vertex. We shall mainly consider matrices whose numerical range is contained in S_θ, the so called sector matrices [11]. For any $A \in \mathbb{M}_n$, its real (or Hermitian) part is denoted by $\Re A := (A + A^*)/2$. Clearly, if $W(A) \subset S_\theta$, then $\Re A$ is positive definite.

A fundamental determinant inequality states that if $A, B \in \mathbb{M}_n$ are positive definite, then

$$\det(A + B) \ge \det A + \det B. \tag{1}$$

In [5], Haynsworth proved the following improvement of (1).

Theorem 1.1. *[5, Theorem 3] Suppose $A, B \in \mathbb{M}_n$ are positive definite. Let A_k and B_k, $k = 1, \dots, n-1$, denote the kth leading principal submatrices of A and B, respectively. Then*

$$\det(A + B) \ge \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right)\det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right)\det B.$$

Using a clever argument, Hartfiel [4] refined Haynsworth's result by adding a nonnegative term on the right side of the inequality.

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Theorem 1.2. *[4] Under the same condition as in Theorem 1.1,*

$$\det(A+B) \geq \left(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det A_k}\right)\det A + \left(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det B_k}\right)\det B + (2^n-2n)\sqrt{\det AB}. \tag{2}$$

Hartfiel's inequality (2) has been extended to sector matrices by a number of authors; see [8, 10, 14, 17]. In [8], Hou and Dong extended Hartfiel's inequality to a triple of matrices. By making use of Hou and Dong's result, Zheng et al. [17] improved and extended the main result in [10], moreover, they obtained the following two theorems.

Theorem 1.3. *[17, Theorem 2.6] Suppose $A, B \in \mathbb{M}_n$ such that $W(A), W(B) \subset S_\theta$. Let A_k and B_k, $k = 1, \dots, n-1$, denote the kth leading principal submatrices of A and B, respectively. Then*

$$\sec^{2n-1}(\theta)|\det(A+B)| \geq \left(1+\sum_{k=1}^{n-1}\left|\frac{\det B_k}{\det A_k}\right|\right)|\det A| + \left(1+\sum_{k=1}^{n-1}\left|\frac{\det A_k}{\det B_k}\right|\right)|\det B| +(2^n-2n)\sqrt{|\det AB|}.$$

Remark 1.4. *In [10], Lin proved a weaker result with the first coefficient $\sec^{3n-2}(\theta)$ instead of $\sec^{2n-1}(\theta)$.*

Theorem 1.5. *[17, Theorem 2.8] Let $M, N, L \in \mathbb{M}_n$ such that $W(M), W(N), W(L) \subset S_\theta$. Then it holds*

$$\begin{aligned}|\det(M+N+L)| \geq\ & \prod_{j=1}^{n}\left(\frac{|\det N_j|}{|\det N_{j-1}|}+\frac{|\det L_j|}{|\det L_{j-1}|}\right)\cos^j(\theta) \\ &+\prod_{j=1}^{n}\left(\frac{|\det L_j|}{|\det L_{j-1}|}+\frac{|\det M_j|}{|\det M_{j-1}|}\right)\cos^j(\theta) \\ &+\prod_{j=1}^{n}\left(\frac{|\det M_j|}{|\det M_{j-1}|}+\frac{|\det N_j|}{|\det N_{j-1}|}\right)\cos^j(\theta) \\ &-(|\det M|+|\det N|+|\det L|),\end{aligned}$$

where by convention, $\det M_0 = \det N_0 = \det L_0 = 1$.

Very recently, Mao in [14] extended Theorem 1.3 to any number of sector matrices. More precisely, she obtained the following result.

Theorem 1.6. *Let $A_j \in \mathbb{M}_n$ with $W(A_j) \subset S_\theta$, and let A_{jk}, $k = 1, \dots, n-1$, denote the kth leading principal submatrix of A_j, $j \in \mathcal{M} := \{1, \dots, m\}$. Then*

$$\sec^{2n-1}(\theta)\left|\det\left(\sum_{j=1}^{m}A_j\right)\right| \geq \sum_{j=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{i\in\mathcal{M}, i\neq j}|\det A_{ik}|}{|\det A_{jk}|}\right)|\det A_j| +(2^n-2n)\sum_{1\leq i<j\leq m}\sqrt{|\det A_iA_j|}.$$

The main goal of the present paper is to extend Theorem 1.5 to any number of sector matrices. To this end, we first present a relevant result for positive definite matrices. Some corollaries are included.

2 Main Results

Before stating our results, we need to present some lemmas that are useful in our proofs.

The first lemma is folklore in matrix analysis.

Lemma 2.1. *[5, Lemma 2] Let $A, B \in \mathbb{M}_n$ be positive definite and let A_j, B_j denote the kth leading principal submatrix of A, B, respectively. Then*

$$\frac{\det(A_j + B_j)}{\det(A_{j-1} + B_{j-1})} \geq \frac{\det A_j}{\det A_{j-1}} + \frac{\det B_j}{\det B_{j-1}}, \qquad j = 1, \ldots, n,$$

where by convention $\det A_0 = \det B_0 = 0$.

The second lemma is known as the Ostrowski-Taussky inequality (see [7, p. 510]).

Lemma 2.2. *Let $A \in \mathbb{M}_n$ with $\Re A$ positive definite. Then*

$$\det(\Re A) \leq |\det A|.$$

The third lemma gives a reverse of the Ostrowski-Taussky inequality.

Lemma 2.3. *[10, Lemma 2.6] Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\theta$. Then*

$$\det(\Re A) \geq \cos^n(\theta)|\det A|.$$

The fourth lemma pays a key role in our new extension of Hartfiel's inequality to multiple positive definite matrices.

Lemma 2.4. *(ref. [1, Corollary 4.4]) Suppose $A_j \in \mathbb{M}_n$, $j \in 1, \ldots, m$, are positive definite. Then*

$$\det\left(\sum_{j=1}^m A_j\right) + (m-2)\sum_{j=1}^m \det A_j \geq \sum_{1\leq i<j\leq m} \det(A_i + A_j).$$

It is worthy to mention that in [1, Corollary 4.4], the authors stated their result for the generalized matrix functions, which includes the determinant as a special case. The inequality in Lemma 2.4 is a direct extension of the main result in [13].

Below is our extension of Hartfiel's inequality to multiple positive definite matrices.

Proposition 2.5. *Let $A_j \in \mathbb{M}_n$, $j = 1, \ldots, m$, be positive definite, and let A_{jk}, $k = 1, \ldots, n-1$, denote the kth leading principal submatrix of A_j. Then*

$$\det\left(\sum_{j=1}^m A_j\right) \geq \sum_{1\leq i<j\leq m} \prod_{k=1}^n \left(\frac{\det A_{ik}}{\det A_{i(k-1)}} + \frac{\det A_{jk}}{\det A_{j(k-1)}}\right) - (m-2)\sum_{j=1}^m \det A_j,$$

where by convention $\det A_{j0} = 1$ *for all j.*

Proof. Applying Lemma 2.1 to A_i, A_j gives

$$\frac{\det(A_{ik} + A_{jk})}{\det(A_{i(k-1)} + A_{j(k-1)})} \geq \frac{\det A_{ik}}{\det A_{i(k-1)}} + \frac{\det A_{jk}}{\det A_{j(k-1)}}, \qquad j = 1, \ldots, n.$$

Taking products for k from 1 to n gives

$$\det(A_i + A_j) \geq \prod_{k=1}^n \left(\frac{\det A_{ik}}{\det A_{i(k-1)}} + \frac{\det A_{jk}}{\det A_{j(k-1)}}\right).$$

Now by Lemma 2.4, we have

$$\det\left(\sum_{j=1}^m A_j\right) \geq \sum_{1\leq i<j\leq m} \det(A_i + A_j) - (m-2)\sum_{j=1}^m \det A_j$$

$$\geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{n} \left(\frac{\det A_{ik}}{\det A_{i(k-1)}} + \frac{\det A_{jk}}{\det A_{j(k-1)}} \right) - (m-2) \sum_{j=1}^{m} \det A_j.$$

This completes the proof. □

The following result extends Theorem 1.5.

Theorem 2.6. *Let $A_j \in \mathbb{M}_n$, $j = 1, \ldots, m$, such that $W(A_j) \subset S_\theta$, and let A_{jk}, $k = 1, \ldots, n-1$, denote the kth leading principal submatrix of A_j. Then it holds*

$$\left|\det\left(\sum_{j=1}^{m} A_j\right)\right| \geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{n} \left(\frac{|\det A_{ik}|}{|\det A_{i(k-1)}|} + \frac{|\det A_{jk}|}{|\det A_{j(k-1)}|} \right) \cos^k\theta - (m-2) \sum_{j=1}^{m} |\det A_j|,$$

where by convention $\det A_{j0} = 1$ *for all j.*

Proof. First of all, since $W(\sum_{j=1}^{m} A_j) \subset S_\theta$, then by Lemma 2.2,

$$\begin{aligned}\left|\det\left(\sum_{j=1}^{m} A_j\right)\right| &\geq \det\left(\Re\sum_{j=1}^{m} A_j\right) \\ &= \det\left(\sum_{j=1}^{m} \Re A_j\right).\end{aligned}$$

As $\Re A_j$ are positive definite for all j, we can apply Proposition 2.5 to get

$$\begin{aligned}\left|\det\left(\sum_{j=1}^{m} A_j\right)\right| &\geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{n} \left(\frac{\det \Re A_{ik}}{\det \Re A_{i(k-1)}} + \frac{\det \Re A_{jk}}{\det \Re A_{j(k-1)}} \right) - (m-2) \sum_{j=1}^{m} \det \Re A_j \\ &\geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{n} \left(\frac{\det \Re A_{ik}}{|\det A_{i(k-1)}|} + \frac{\det \Re A_{jk}}{|\det A_{j(k-1)}|} \right) - (m-2) \sum_{j=1}^{m} |\det A_j| \\ &\geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{n} \left(\frac{|\det A_{ik}|}{|\det A_{i(k-1)}|} + \frac{|\det A_{jk}|}{|\det A_{j(k-1)}|} \right) \cos^k\theta - (m-2) \sum_{j=1}^{m} |\det A_j|,\end{aligned}$$

in which the second inequality is by Lemma 2.2 and the third inequality is by Lemma 2.3, respectively. This completes the proof. □

A matrix $A \in \mathbb{M}_n$ is called accretive-dissipative if both real part $\Re A$ and imaginary part $\Im A := (A - A^*)/2i$ (in the sense of Cartesian decomposition) are positive definite. This class of matrices has appeared in numerical linear algebra [2, 6, 12] and has been studied recently by a number of authors [9, 15, 16]. Note that M is accretive-dissipative if and only if $W(e^{-i\pi/4}M) \subset \mathbb{S}_{\pi/4}$. This observation enables us to state the following two corollaries.

Corollary 2.7. *Let $A_j \in \mathbb{M}_n$, $j = 1, \ldots, m$, be accretive-dissipative, and let A_{jk}, $k = 1, \ldots, n-1$, denote the kth leading principal submatrix of A_j. Then it holds*

$$\begin{aligned}\left|\det\left(\sum_{j=1}^{m} A_j\right)\right| \geq& \sum_{1\leq i<j\leq m} \prod_{k=1}^{n} \left(\frac{|\det A_{ik}|}{|\det A_{i(k-1)}|} + \frac{|\det A_{jk}|}{|\det A_{j(k-1)}|} \right) 2^{-k/2} \\ &-(m-2) \sum_{j=1}^{m} |\det A_j|,\end{aligned}$$

where by convention $\det A_{j0} = 1$ *for all j.*

Proof. Since $A_j \in \mathbb{M}_n$ are accretive-dissipative, we have $W(e^{-i\pi/4}A_j) \subset \mathcal{S}_{\pi/4}$ for all j. Then we apply Theorem 2.6 to the matrices $e^{-i\pi/4}A_j$ to get the result, because in this case $\cos^k\theta = 2^{-k/2}$. □

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