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An elementary proof of Chollet's permanent conjecture for 4×4 real matrices

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Abstract: A proof of the statement $per(A \circ B) \leq per(A)per(B)$ is given for 4×4 positive semidefinite real matrices. The proof uses only elementary linear algebra and a rather lengthy series of simple inequalities.

Keywords: Chollet Conjecture, Positive Semidefinite Matrices, Hadamard Product, Matrix Permanent

MSc: 15A15, 15B48

1 Introduction

In 1982 [1], John Chollet introduced a conjecture concerning the *permanent* of positive semidefinite (psd) matrices. Before we state this conjecture, we remind the reader of the definition of the matrix permanent:

Definition 1.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix with complex entries, and let S_n denote the symmetric group on n elements. The permanent of A, denoted $per(A)$, is the complex number*

$$per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

In its original form, Chollet's conjecture was given as:

Conjecture 1.2 ([1]). *Let A, B be two $n \times n$ positive semidefinite matrices. Then:*

$$per(A \circ B) \leq per(A)per(B)$$

where $A \circ B$ denotes the Hadamard (or "entry-wise") product of the two matrices.

The 1980s saw a bit of progress made on this conjecture (which is the permanental analogue to Oppenheim's famous determinant inequality from [2]). In particular, this inequality was shown to hold when $n \leq 3$ by Gregorac and Hentzel in [3]. More recently, a stronger conjecture made by Bapat and Sunder in [4] was disproven by Drury in [5]. Aside from these results, however, and despite the sizeable interest that this question has garnered (e.g. [6],[7],[8]), we have seen very little progress on Chollet's conjecture in recent years.

The purpose of this paper is to provide a proof that Conjecture 1.2 is true for 4×4 real matrices. We will make use of the following result from Chollet, which we have restricted to real matrices for our purposes.

Proposition 1.3 ([1]). *Conjecture 1.2 is valid for all $n \times n$, real, psd matrices if and only if*

$$per(A \circ A) \leq (per(A))^2$$

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for all $n \times n$ real psd matrices A.

In light of this equivalency, we will work towards a proof of the following result:

Theorem 1.4. *Let A be a 4×4 positive semidefinite real matrix. Then*

$$per(A \circ A) \leq (per(A))^2 \tag{1.1}$$

2 Simplifying the Problem

In this section, we will show that it suffices to prove our inequality for matrices that satisfy one of two sign patterns. We will need two lemmas. In what follows, $diag(d_1, d_2, \ldots d_n)$ denotes the diagonal matrix with diagonal elements d_i.

Lemma 2.1. *Let A be an $n \times n$ real matrix, and let D be any $n \times n$ real, invertible, diagonal matrix. Then Inequality (1.1) holds for A if and only if it holds for DAD.*

Proof. Let $D = diag(d_1, d_2, \ldots, d_n)$. The (i, j)-th entry of DAD is given by $(d_i a_{ij} d_j)$. Thus, it is easy to verify that

$$\begin{aligned} per\left((DAD) \circ (DAD)\right) &= \sum_{\sigma \in S_n} \prod_{i=1}^{n} (d_i a_{i,\sigma(i)} d_{\sigma(i)})^2 \\ &= (per(D))^4 \sum_{\sigma \in S_n} \prod_{i=1}^{n} (a_{i,\sigma(i)})^2 \\ &= (per(D))^4 per(A \circ A), \end{aligned}$$

Likewise,

$$\begin{aligned} \left(per(DAD)\right)^2 &= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} d_i a_{i,\sigma(i)} d_{\sigma(i)} \right)^2 \\ &= \left((per(D))^2 \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} \right)^2 \\ &= (per(D))^4 \left(per(A)\right)^2 . \end{aligned}$$

whence we can see that

$$per\left((DAD) \circ (DAD)\right) \leq (per(DAD))^2 \quad \text{if and only if} \quad per(A \circ A) \leq (per(A))^2 .$$

□

Lemma 2.2. *Let A be an $n \times n$ psd, real matrix, and let P be an $n \times n$ permutation matrix. Then $P^T AP$ is a positive semidefinite matrix satisfying $per(P^T AP) = per(A)$.*

Proof. Let $x \in \mathbb{R}^n$. Positive semidefiniteness of P^TAP follows immediately from the fact that $x^T(P^TAP)x = (Px)^TA(Px) \geq 0$, as A is positive semidefinite.

Applying the transformation $A \mapsto P^TAP$ just amounts to applying a permutation to the rows and columns of A. The fact that the permanent is invariant under permutation of rows and columns is well-known. Indeed, permuting rows and columns will merely rearrange the summands in Definition 1.1, producing the same result. □

Corollary 2.3. *Let A be an $n \times n$ psd, real matrix, and let P be an $n \times n$ permutation matrix. Then Inequality (1.1) holds for A if and only if it holds for P^TAP.*

Proof. By Lemma 2.2, we see that:

$$(per(A))^2 = (per(P^TAP))^2, \text{ and } per(A \circ A) = per(P^T(A \circ A)P) = per(P^TAP \circ P^TAP).$$

The result follows. □

Since a positive semidefinite matrix must be symmetric, there are $2^6 = 64$ possible sign patterns for a 4×4 positive semidefinite matrix, represented by the following:

$$\begin{pmatrix} + & +/- & +/- & +/- \\ * & + & +/- & +/- \\ * & * & + & +/- \\ * & * & * & + \end{pmatrix}$$

where "+" represents a non-negative entry, "−" represents a non-positive entry, and $*$ represents an entry that is determined by the symmetry of the matrix.

We now endeavour to simplify our problem, so that we need only consider three of these 64 sign patterns. We will need a bit of terminology first. In what follows, recall that an $n \times n$ matrix with sign pattern S is called a *realization* of S.

Definition 2.4. *Let S_1, S_2 be two $n \times n$ sign patterns. We say that S_1 and S_2 are signature similar if there exists an $n \times n$ diagonal matrix $D = diag(1, \pm1, \pm1, \pm1, \ldots, \pm1)$, such that for all realizations A of S_1, $B = DAD$ is a realization of S_2.*

Remark 2.5. *It is easy to see that signature similarity is an equivalence relation on the set of $n \times n$ sign patterns. For the 64 4×4 sign patterns mentioned above, each equivalence class will consist of exactly 8 members, corresponding to the 8 choices for $D = diag(1, \pm1, \pm1, \pm1)$.*

Proposition 2.6. *If Inequality (1.1) holds for all 4×4 real, positive definite matrices with sign pattern:*

$$\begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix}, \begin{pmatrix} + & - & + & + \\ - & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix} \text{ or } \begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}$$

then the inequality holds for all 4×4 real, positive definite matrices.

Proof. One can easily verify that none of the following eight sign patterns are signature similar:

$$\underset{(1)}{\begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix}} \underset{(2)}{\begin{pmatrix} + & - & + & + \\ - & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix}} \underset{(3)}{\begin{pmatrix} + & + & - & + \\ + & + & + & + \\ - & + & + & + \\ + & + & + & + \end{pmatrix}} \underset{(4)}{\begin{pmatrix} + & + & + & - \\ + & + & + & + \\ + & + & + & + \\ - & + & + & + \end{pmatrix}}$$

$$\begin{pmatrix} + & + & + & + \\ + & + & - & + \\ + & - & + & + \\ + & + & + & + \end{pmatrix} \begin{pmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & - & + & + \end{pmatrix} \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & - \\ + & + & - & + \end{pmatrix} \begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}$$

(5) (6) (7) (8)

Thus, each of these sign patterns occupies a different equivalence class. By the remark preceding this proof, these 8 different equivalence classes must contain all 64 of the possible sign patterns.

Let A be a 4×4 positive semidefinite matrix. By the above argument, there necessarily exists a diagonal matrix $D = diag(1, \pm 1, \pm 1, \pm 1)$ such that DAD realizes one of the sign patterns (1)-(8). By Lemma 2.1, Inequality (1.1) holds for A if and only if it holds for DAD. We conclude that it suffices to prove Inequality (1.1) for all matrices that realize (at least) one of the sign patterns (1)-(8).

Now consider any matrix A that is a realization of one of sign patterns (3) through (7). There is necessarily a permutation matrix P such that $P^T AP$ is a realization of sign pattern 2. By Corollary 2.3, Inequality (1.1) holds for A if and only if it holds for $P^T AP$. In other words, Inequality (1.1) holds for all matrices that realize (at least) one of sign pattern (3)-(7) if and only it holds for all matrices that realize sign pattern (2). Our result follows. □

We conclude this section by noting that the first of these three sign patterns satisfies the inequality trivially.

Proposition 2.7. *Let A be an $n \times n$ matrix with exclusively non-negative real entries (i.e. a matrix with sign pattern (1)). Then $per(A \circ A) \leq (per(A))^2$.*

Proof. We must show that $(per(A))^2 - per(A \circ A) \geq 0$. Indeed:

$$\begin{aligned} \left(per(A)\right)^2 - per(A \circ A) &= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} \right)^2 - \sum_{\sigma \in S_n} \prod_{i=1}^{n} (a_{i,\sigma(i)})^2 \\ &= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} \right) \left(\sum_{\tau \in S_n} \prod_{j=1}^{n} a_{j,\tau(j)} \right) - \sum_{\sigma \in S_n} \prod_{i=1}^{n} (a_{i,\sigma(i)})^2 \\ &= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i,\sigma(i)} \left(\sum_{\substack{\tau \in S_n \\ \tau \neq \sigma}} \prod_{j=1}^{n} a_{j,\tau(j)} \right) \right), \end{aligned}$$

which is the sum of non-negative terms. □

In order to prove Theorem 1.4, it remains to show that Inequality (1.1) holds for all matrices that realize one of the remaining two sign patterns:

$$\begin{pmatrix} + & - & + & + \\ - & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix} \text{ and } \begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}.$$

This will be the focus of our next two sections.

3 Matrices with two non-positive entries

In this section, we will consider the first of the remaining two sign patterns. We begin by noting that it will suffice to restrict our discussion to matrices of a more particular form.

Proposition 3.1. *Inquality (1.1) holds for all positive semidefinite matrices with sign pattern*

$$\begin{pmatrix} + & - & + & + \\ - & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix}$$

if and only if it holds for all positive semidefinite matrices of the form

$$A = \begin{pmatrix} a & -1 & b & c \\ -1 & 1 & f & g \\ b & f & 1 & h \\ c & g & h & 1 \end{pmatrix} (\dagger)$$

where $a, b, c, f, g, h \geq 0$.

Proof. Let B be a positive semidefinite matrix with the above sign pattern. We first note that if any of the diagonal entries of B are 0, then B must have a zero row (as it is positive semidefinite). Thus, Inequality (1.1) holds trivially, as $per(B) = per(B \circ B) = 0$.

Now suppose that all diagonal entries of B are nonzero. We consider two cases: If $b_{12} = 0$, then B has all non-negative entries, and Inequality (1.1) necessarily holds by Proposition 2.7. If, however, $b_{12} \neq 0$, then we define the diagonal matrix $D := diag\left(\frac{\sqrt{b_{22}}}{|b_{12}|}, \frac{1}{\sqrt{b_{22}}}, \frac{1}{\sqrt{b_{33}}}, \frac{1}{\sqrt{b_{44}}}\right)$. It is easy to verify that $A := DBD$ has form (†), and we appeal to Lemma 2.1 to complete the proof. □

Thus we restrict our attention to positive semidefinite matrices of the form (†). There is nothing particularly special about matrices of this form, but while other authors have used a similar method to reduce the problem to that of correlation matrices, it is the author's opinion that form (†) actually makes the remaining proofs easiest to follow.

Given a matrix A of the form (†), we see that

$$per(A) = a + af^2 + ag^2 + ah^2 + 2afgh + 1 + h^2 + b^2 + b^2g^2 + 2bcfg + c^2f^2 + c^2 + 2bch - 2cfh - 2bgh - 2bf - 2cg,$$

whence we obtain:

$$\begin{aligned}
\frac{1}{2}((per(A))^2 - per(A \circ A)) &= a^2f^2 + a^2g^2 + a^2h^2 + 2a^2fgh + a + ah^2 + ab^2 + ab^2g^2 + 2abcfg + ac^2f^2 + ac^2 + 2abch \\
&+ a^2f^2g^2 + a^2f^2h^2 + 2a^2f^3gh + af^2 + af^2h^2 + ab^2f^2 + ab^2f^2g^2 + 2abcf^3g + ac^2f^4 + ac^2f^2 + 2abcf^2h \\
&+ a^2g^2h^2 + 2a^2fg^3h + ag^2 + ag^2h^2 + ab^2g^2 + ab^2g^4 + 2abcfg^3 + ac^2f^2g^2 + ac^2g^2 + 2abcg^2h \\
&+ 2a^2fgh^3 + ah^2 + ah^4 + ab^2h^2 + ab^2g^2h^2 + 2abcfgh^2 + ac^2f^2h^2 + ac^2h^2 + 2abch^3 \\
&+ a^2f^2g^2h^2 + 2afgh + 2afgh^3 + 2ab^2fgh + 2ab^2fg^3h + 4abcf^2g^2h + 2ac^2f^3gh + 2ac^2fgh + 4abcfgh^2 \\
&+ h^2 + b^2 + b^2g^2 + 2bcfg + c^2f^2 + c^2 + 2bch \\
&+ b^2h^2 + b^2g^2h^2 + 2bcfgh^2 + c^2f^2h^2 + c^2h^2 + 2bch^3 \\
&+ b^4g^2 + 2b^3cfg + b^2c^2f^2 + b^2c^2 + 2b^3ch \\
&+ 2b^3cfg^3 + b^2c^2f^2g^2 + b^2c^2g^2 + 2b^3cg^2h \\
&+ b^2c^2f^2g^2 + 2bc^3f^3g + 2bc^3fg + 4b^2c^2fgh \\
&+ c^4f^2 + 2bc^3f^2h \\
&+ 2bc^3h \\
&+ b^2c^2h^2 \\
&+ c^2f^2h^2 + 4bcfgh^2 + 4bcf^2h + 4c^2fgh \\
&+ b^2g^2h^2 + 4b^2fgh + 4bcg^2h \\
&+ b^2f^2 + 4bcfg \\
&+ c^2g^2 \\
&- 2acfh - 2abgh - 2abf - 2acg
\end{aligned}$$

$$
\begin{aligned}
&- \quad 2acf^3h - 2abf^2gh - 2abf^3 - 2acf^2g \\
&- \quad 2acfg^2h - 2abg^3h - 2abfg^2 - 2acg^3 \\
&- \quad 2acfh^3 - 2abgh^3 - 2abfh^2 - 2acgh^2 \\
&- \quad 4acf^2gh^2 - 4abfg^2h^2 - 4abf^2gh - 4acfg^2h \\
&- \quad 2cfh - 2bgh - 2bf - 2cg \\
&- \quad 2cfh^3 - 2bgh^3 - 2bfh^2 - 2cgh^2 \\
&- \quad 2b^2cfh - 2b^3gh - 2b^3f - 2b^2cg \\
&- \quad 2b^2cfg^2h - 2b^3g^3h - 2b^3fg^2 - 2b^2cg^3 \\
&- \quad 4bc^2f^2gh - 4b^2cfg^2h - 4b^2cf^2g - 4bc^2fg^2 \\
&- \quad 2c^3f^3h - 2bc^2f^2gh - 2bc^2f^3 - 2c^3f^2g \\
&- \quad 2c^3fh - 2bc^2gh - 2bc^2f - 2c^3g \\
&- \quad 4bc^2fh^2 - 4b^2cgh^2 - 4b^2cfh - 4bc^2gh.
\end{aligned}
$$

We now endeavour to show that the non-negative terms “outweigh" the non-positive terms in the above expansion. We can do this using only basic algebra and the following easy matrix inequalities.

Proposition 3.2. *Let A be a positive semidefinite matrix of the form (†). Then the following inequalities hold:*

$$
\begin{aligned}
a + b^2g^2 + c^2f^2 + h^2 + 2bch + 2bgh + 2cfh + 2afgh &\geq 2bf + 2cg + af^2 + ag^2 + ah^2 + b^2 + c^2 + 2bcfg + 1 && (3.2)\\
a &\geq 1 + b^2 + af^2 + 2bf && (3.3)\\
a + 2bch &\geq b^2 + c^2 + ah^2 && (3.4)\\
a &\geq 1 + c^2 + ag^2 + 2cg && (3.5)\\
1 + 2fgh &\geq f^2 + g^2 + h^2 && (3.6)\\
1 &\geq f, g, h && (3.7)
\end{aligned}
$$

Proof. These follow immediately from the non-negativity of the principal minors of the positive semidefinite matrix A. □

We will assume, without loss of generality, that $f \geq g$. Indeed, if $f < g$, we simply apply the transformation $A \mapsto P^TAP$, where $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. This transforms A into a matrix of the form (†) that satisfies $f \geq g$, while not affecting the veracity of Inequality (1.1) (by Corollary 2.3).

We now show that Inequality (1.1) holds for all matrices of this form:

Proposition 3.3. *Let A be a 4×4, positive semidefinite matrix of the form (†). Then*

$$per(A \circ A) \leq (per(A))^2.$$

Proof. We have expanded $\frac{1}{2}((per(A))^2 - per(A \circ A)) \geq 0$ on the previous page. It remains to show that this expansion is non-negative. Indeed, we have the following inequalities, the justification for which can be found in Appendix A.1:

$$
\begin{aligned}
a^2h^2 + c^2f^2 &\geq 2acfh && (3.8)\\
a^2g^2h^2 + ab^2 + ah^2 + 2bch^3 + b^2g^2h^2 + b^4g^2 + c^4f^2 &\geq 2abgh + 2b^3g^3h + 2abfh^2 + 2b^3f + 2b^3gh + 2c^3fh && (3.9)\\
a + 2afgh + c^2h^2 + c^2 + b^2g^2 + a^2f^2 + ac^2f^2g^2 + 2ab^2fgh &\geq 2cfh + 2cg + 2bgh + 2abf + 2acf^2g + 2bf + 4abf^2gh && (3.10)\\
a^2g^2 + ac^2 + b^2f^2 + ag^2 + 2a^2fg^3h + ab^2g^2 + ac^2h^2 &\geq 2acg + 2bc^2f + 2acg^3 + 2c^3g + 2abfg^2 + 2acgh^2 && (3.11)\\
ac^2f^4 + af^2h^2 &\geq 2acf^3h && (3.12)\\
2a^2fgh + 2ac^2f^3gh + 2afgh^3 &\geq 4acf^2gh^2 + 4abfg^2h^2 + 2abf^2gh && (3.13)\\
af^2 + ab^2f^2 + ac^2f^2 + b^2h^2 &\geq 2bgh^3 + 2bc^2f^3 + 2abf^3 + 2c^3f^2g && (3.14)\\
ah^2 + 2abch^3 + h^2 + 2afgh^3 + c^2f^2h^2 + ab^2g^2h^2 &\geq 2bfh^2 + 2cgh^2 + 2abgh^3 + 2cfh^3 && (3.15)\\
ah^4 + ac^2f^2h^2 &\geq 2acfh^3 && (3.16)
\end{aligned}
$$

$$a^2f^2g^2 + 4c^2fgh \geq 4acfg^2h \tag{3.17}$$
$$a^2f^2g^2h^2 + c^2g^2 \geq 2acfg^2h \tag{3.18}$$
$$ag^2h^2 + ab^2g^4 \geq 2abg^3h \tag{3.19}$$
$$ab^2h^2 + b^2c^2f^2 \geq 2b^2cfh \tag{3.20}$$
$$b^2 + b^2c^2g^2 \geq 2b^2cg \tag{3.21}$$
$$ab^2g^2 + ab^2f^2g^2 + b^2c^2h^2 + c^2f^2h^2 \geq 2b^3fg^2 + 2bc^2f^2gh + 2b^2cg^3 + 2b^2cfg^2h \tag{3.22}$$
$$4bcfgh^2 + 2bc^3f^3g + 2b^3cfg^3 \geq 4bc^2f^2gh + 4b^2cfg^2h \tag{3.23}$$
$$2b^3cfg + 2bc^3fg + 4bcfg \geq 4b^2cf^2g + 4bc^2fg^2 \tag{3.24}$$
$$ac^2f^2 \geq 2c^3f^3h \tag{3.25}$$
$$ac^2g^2 + b^2c^2 \geq 2bc^2gh \tag{3.26}$$
$$4bcf^2h + 4bcg^2h + 2bc^3h \geq 4bc^2fh^2 + 4bc^2gh \tag{3.27}$$
$$2b^3cg^2h + 2abch \geq 4b^2cgh^2 + 4b^2cfh \tag{3.28}$$

The left-hand side of these inequalities consist of non-negative terms from our expansion of $\frac{1}{2}((per(A))^2 - per(A \circ A))$, while the right-hand side exhausts all non-positive terms from this expansion. Combining the above inequalities, we see that $\frac{1}{2}((per(A))^2 - per(A \circ A)) \geq 0$. □

The main result of this section follows immediately:

Corollary 3.4. *Let A be a* 4×4 *positive semidefinite matrix with sign pattern*

$$\begin{pmatrix} + & - & + & + \\ - & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix}.$$

Then $per(A \circ A) \leq (per(A))^2$.

Proof. Combine Proposition 3.3 and Proposition 3.1. □

4 Matrices with uniformly non-positive off-diagonals

We now consider matrices of the form:

$$\begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}$$

As in the previous section, we can restrict our focus to matrices that possess a certain structure:

Proposition 4.1. *Inquality (1.1) holds for all positive semidefinite matrices with sign pattern*

$$\begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}$$

if and only if it holds for all positive semidefinite matrices of the form

$$A = \begin{pmatrix} a & -1 & -b & -c \\ -1 & 1 & -f & -g \\ -b & -f & 1 & -h \\ -c & -g & -h & 1 \end{pmatrix} (\dagger\dagger)$$

where $a, b, c, f, g, h \geq 0$.

Proof. Let B be a positive semidefinite matrix with the above sign pattern. As in the proof of Proposition 3.1, Inequality (1.1) holds trivially if any diagonal entry of B is 0. Further, if B is a diagonal matrix, then Inequality (1.1) also holds trivially. Let us then assume that B has all non-zero diagonal entries, and at least one non-zero off-diagonal entry, $b_{ij} \neq 0$. We define $C := P^T BP$, where P is the appropriate permutation matrix to ensure that $c_{12} = b_{ij}$. By Corollary 2.3, Inequality (1.1) holds for B if and only if it holds for C.

We then define the diagonal matrix $D = diag\left(\frac{\sqrt{c_{22}}}{|c_{12}|}, \frac{1}{\sqrt{c_{22}}}, \frac{1}{\sqrt{c_{33}}}, \frac{1}{\sqrt{c_{44}}}\right)$. It is easy to verify that $A := DCD$ has the desired form, and we appeal to Lemma 2.1 to complete the proof. □

For matrices A of the form (††), one can verify that:

$$per(A) = a + af^2 + ag^2 + ah^2 + 1 + h^2 + 2cfh + 2bgh + b^2 + b^2g^2 + 2bcfg + c^2f^2 + c^2 - 2afgh - 2bf - 2cg - 2bch,$$

whence we obtain:

$$\begin{aligned}
\frac{1}{2}(per(A))^2 - per(A \circ A) \quad = \quad & a^2f^2 + a^2g^2 + a^2h^2 + a + ah^2 + 2acfh + 2abgh + ab^2 + ab^2g^2 + 2abcfg + ac^2f^2 + ac^2 \\
+ \quad & a^2f^2g^2 + a^2f^2h^2 + af^2 + af^2h^2 + 2acf^3h + 2abf^2gh + ab^2f^2 + ab^2f^2g^2 + 2abcf^3g + ac^2f^4 + ac^2f^2 \\
+ \quad & a^2g^2h^2 + ag^2 + ag^2h^2 + 2acfg^2h + 2abg^3h + ab^2g^2 + ab^2g^4 + 2abcfg^3 + ac^2f^2g^2 + ac^2g^2 \\
+ \quad & ah^2 + ah^4 + 2acfh^3 + 2abgh^3 + ab^2h^2 + ab^2g^2h^2 + 2abcfgh^2 + ac^2f^2h^2 + ac^2h^2 \\
+ \quad & h^2 + 2cfh + 2bgh + b^2 + b^2g^2 + 2bcfg + c^2f^2 + c^2 \\
+ \quad & 2cfh^3 + 2bgh^3 + b^2h^2 + b^2g^2h^2 + 2bcfgh^2 + c^2f^2h^2 + c^2h^2 \\
+ \quad & c^2f^2h^2 + 4bcfgh^2 + 2b^2cfh + 2b^2cfg^2h + 4bc^2f^2gh + 2c^3f^3h + 2c^3fh \\
+ \quad & b^2g^2h^2 + 2b^3gh + 2b^3g^3h + 4b^2cfg^2h + 2bc^2f^2gh + 2bc^2gh \\
+ \quad & b^4g^2 + 2b^3cfg + b^2c^2f^2 + b^2c^2 \\
+ \quad & 2b^3cfg^3 + b^2c^2f^2g^2 + b^2c^2g^2 \\
+ \quad & b^2c^2f^2g^2 + 2bc^3f^3g + 2bc^3fg \\
+ \quad & c^4f^2 \\
+ \quad & a^2f^2g^2h^2 + 4abf^2gh + 4acfg^2h + 4abcfgh^2 \\
+ \quad & b^2f^2 + 4bcfg + 4b^2cfh \\
+ \quad & c^2g^2 + 4bc^2gh \\
+ \quad & b^2c^2h^2 \\
- \quad & 2a^2fgh - 2abf - 2acg - 2abch \\
- \quad & 2a^2f^3gh - 2abf^3 - 2acf^2g - 2abcf^2h \\
- \quad & 2a^2fg^3h - 2abfg^2 - 2acg^3 - 2abcg^2h \\
- \quad & 2a^2fgh^3 - 2abfh^2 - 2acgh^2 - 2abch^3 \\
- \quad & 2afgh - 2bf - 2cg - 2bch \\
- \quad & 2afgh^3 - 2bfh^2 - 2cgh^2 - 2bch^3 \\
- \quad & 4acf^2gh^2 - 4bcf^2h - 4c^2fgh - 4bc^2fh^2 \\
- \quad & 4abfg^2h^2 - 4b^2fgh - 4bcg^2h - 4b^2cgh^2 \\
- \quad & 2ab^2fgh - 2b^3f - 2b^2cg - 2b^3ch \\
- \quad & 2ab^2fg^3h - 2b^3fg^2 - 2b^2cg^3 - 2b^3cg^2h \\
- \quad & 4abcf^2g^2h - 4b^2cf^2g - 4bc^2fg^2 - 4b^2c^2fgh \\
- \quad & 2ac^2f^3gh - 2bc^2f^3 - 2c^3f^2g - 2bc^3f^2h \\
- \quad & 2ac^2fgh - 2bc^2f - 2c^3g - 2bc^3h.
\end{aligned}$$

We will continue as in Section 3, using the following inequalities:

Proposition 4.2. *Let A be a positive semidefinite matrix of the form* (††). *Then the following inequalities hold:*

$$a + b^2g^2 + c^2f^2 + h^2 \geq 2bf + 2cg + af^2 + ag^2 + ah^2 + b^2 + c^2 + 2bch + 2bgh + 2cfh + 2bcfg + 2afgh + 1 \quad (4.29)$$
$$a \geq 1 + b^2 + af^2 + 2bf \quad (4.30)$$
$$a \geq b^2 + c^2 + ah^2 + 2bch \quad (4.31)$$
$$a \geq 1 + c^2 + ag^2 + 2cg \quad (4.32)$$
$$1 \geq f^2 + g^2 + h^2 + 2fgh \quad (4.33)$$

Proof. These follow immediately from the non-negativity of the principal minors of the positive semidefinite matrix A. □

We break our proof into two cases: Matrices satisfying $ag \geq c$, and those that satisfy $ag < c$ Again, we will assume without loss of generality that $f \geq g$.

Proposition 4.3. *Let A be a 4×4, positive semidefinite matrix of the form* (††), *satisfying $ag \geq c$. Then $per(A \circ A) \leq (per(A))^2$.*

Proof. We will show that $\frac{1}{2}((per(A))^2 - per(A \circ A)) \geq 0$. We have the following inequalities, the justification for which can be found in Appendix A.2:

$$a^2g^2 + a^2f^2h^2 \geq 2a^2fgh \quad (4.34)$$
$$ab^2 + b^4g^2 + b^2c^2f^2 + b^2h^2 + a + a^2h^2 + ac^2f^2h^2 \geq 2b^3f + 2b^2cg + 2abf + 2abcg^2h + 2abch + 2b^3ch + 2b^3cg^2h + 2ab^2fgh \quad (4.35)$$
$$ac^2 + ag^2 \geq 2acg \quad (4.36)$$
$$a^2f^2 + ab^2f^2g^2 + ac^2f^4 + af^2h^2 + a^2g^2h^2 + c^2g^2 + b^2 + ac^2f^2 \geq 2abf^3 + 2acf^2g + 2acg^3 + 2a^2fg^3h + 2a^2fgh^3 + 2c^3g + 2abcf^2h + 2a^2f^3gh + 2bf \quad (4.37)$$
$$c^2 + af^2 + ab^2g^2 \geq 2cg + 2abfg^2 \quad (4.38)$$
$$ab^2f^2 + ah^4 \geq 2abfh^2 \quad (4.39)$$
$$2acfh \geq 2acgh^2 \quad (4.40)$$
$$b^2c^2 + ab^2h^2 + ac^2h^2 \geq 2bc^3h + 2abch^3 \quad (4.41)$$
$$a^2f^2g^2 + ah^2 + ag^2h^2 \geq 2afgh + 2afgh^3 + 2bfh^2 \quad (4.42)$$
$$2abgh \geq 2bch \quad (4.43)$$
$$ah^2 + h^2 \geq 2cgh^2 + 2bch^3 \quad (4.44)$$
$$2acfg^2h + 2acf^3h \geq 4acf^2gh^2 \quad (4.45)$$
$$4abf^2gh \geq 4bcf^2h \quad (4.46)$$
$$2cfh + 2c^3fh \geq 4c^2fgh \quad (4.47)$$
$$2cfh^3 + 2c^3f^3h + b^2c^2h^2 \geq 4bc^2fh^2 \quad (4.48)$$
$$2abf^2gh + 2abg^3h \geq 4abfg^2h^2 \quad (4.49)$$
$$2bgh + 2b^3gh \geq 4b^2fgh \quad (4.50)$$
$$2bcfg + 2bcfgh^2 \geq 4bcg^2h \quad (4.51)$$
$$4b^2cfh \geq 4b^2cgh^2 \quad (4.52)$$
$$ab^2g^4 + ab^2g^2h^2 + b^2c^2g^2 \geq 2b^2cg^3 + 2ab^2fg^3h \quad (4.53)$$
$$ab^2g^2 \geq 2b^3fg^2 \quad (4.54)$$
$$2abcfg^3 + 2abcfgh^2 \geq 4abcf^2g^2h \quad (4.55)$$
$$2abcf^3g + 4bcfg \geq 4b^2cf^2g + 4bc^2fg^2 \quad (4.56)$$
$$2b^3cfg + 2bc^3fg \geq 4b^2c^2fgh \quad (4.57)$$
$$a^2f^2g^2h^2 + c^4f^2 \geq 2ac^2f^3gh \quad (4.58)$$
$$ac^2f^2 + c^2f^2 + ac^2g^2 \geq 2bc^2f^3 + 2c^3f^2g + 2ac^2fgh + 2bc^3f^2h \quad (4.59)$$
$$2abcfg \geq 2bc^2f \quad (4.60)$$

The left-hand sides of these inequalities consist of non-negative terms from our expansion of $\frac{1}{2}((per(A))^2 - per(A \circ A))$, while the right-hand sides exhausts all non-positive terms from this expansion. Thus, $\frac{1}{2}((per(A))^2 - per(A \circ A)) \geq 0$. □

We proceed similarily for the case where $ag < c$. In this case we can actually obtain a strict inequality, stronger than Inequality (1.1):

Proposition 4.4. *Let A be a 4×4, positive semidefinite matrix of the form (††), satisfying $ag < c$. Then $per(A \circ A) < (per(A))^2$.*

Proof. We will show that $\frac{1}{2}((per(A))^2 - per(A \circ A)) > 0$. We have the following inequalities, the justification for which can be found in Appendix A.3:

$$2acfh > 2a^2fgh \tag{4.61}$$

$$ab^2 + b^4g^2 + b^2c^2f^2 + b^2h^2 + a + ac^2f^4 + a^2h^2 + ac^2h^2 + 2abf^2gh \geq 2b^3f + 2b^2cg + 2abf + 2abcf^2h + 2abch + 2b^3ch + 4b^2fgh + 2ab^2fgh + 2bch \tag{4.62}$$

$$c^2 + a^2g^2 \geq 2acg \tag{4.63}$$

$$2acf^3h > 2a^2f^3gh \tag{4.64}$$

$$b^2 + af^2 + ab^2f^2 + h^2 + ah^2 + a^2f^2h^2 + ac^2g^2 \geq 2bf + 2bfh^2 + 2abf^3 + 2cgh^2 + 2abfh^2 + 2acgh^2 + 2bch^3 \tag{4.65}$$

$$a^2f^2 + ac^2f^2g^2 \geq 2acf^2g \tag{4.66}$$

$$2acfg^2h > 2a^2fg^3h \tag{4.67}$$

$$2abcfg + 2b^3cfg^3 + 2bc^3f^3g + 2bcfgh^2 > 4b^2cf^2g + 4bc^2fg^2 + 4b^2c^2fgh + 4abcf^2g^2h + 2abfg^2 \tag{4.68}$$

$$ac^2 + b^2c^2g^2 + c^4f^2 + c^2h^2 + ab^2h^2 + b^2c^2 + 2c^2f^2h^2 + 2c^3fh + ag^2 > 2bc^2f + 2c^3g + 2acg^3 + 2abch^3 + 4bc^2fh^2 + 2bc^3h + 4c^2fgh + 2ac^2fgh + 2cg \tag{4.69}$$

$$ab^2g^2 + ac^2f^2h^2 \geq 2abcg^2h \tag{4.70}$$

$$2acfh^3 > 2a^2fgh^3 \tag{4.71}$$

$$2cfh > 2afgh \tag{4.72}$$

$$2cfh^3 > 2afgh^3 \tag{4.73}$$

$$ac^2f^2 + ah^2 + 4acfg^2h \geq 4acf^2gh^2 + 4bcf^2h \tag{4.74}$$

$$4bcfgh^2 > 4abfg^2h^2 \tag{4.75}$$

$$2abgh + 2b^3gh \geq 4b^2cgh^2 + 4bcg^2h \tag{4.76}$$

$$ab^2g^2 + b^2g^2 \geq 2b^3fg^2 + 2b^2cg^3 + 2b^3cg^2h \tag{4.77}$$

$$2b^2cfg^2h > 2ab^2fg^3h \tag{4.78}$$

$$2c^3f^3h > 2ac^2f^3gh \tag{4.79}$$

$$ac^2f^2 + c^2f^2 \geq 2bc^2f^3 + 2c^3f^2g + 2bc^3f^2h \tag{4.80}$$

The left-hand side of these inequalities consist of non-negative terms from our expansion of $\frac{1}{2}((per(A))^2 - per(A \circ A))$, while the right-hand side exhausts all non-positive terms from this expansion. We conclude that $\frac{1}{2}((per(A))^2 - per(A \circ A)) > 0$. □

Our result follows:

Corollary 4.5. *Let A be a 4×4 positive definite matrix with sign pattern*

$$\begin{pmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{pmatrix}.$$

Then $per(A \circ A) \leq (per(A))^2$.

Proof. This follows from Propositions 4.1, 4.3, and 4.4. □

5 Conclusion

We now have all the results necessary to prove Theorem 1.4, which we restate here:

Theorem (Theorem 1.4)**.** *Let A be a 4×4 positive semidefinite, real matrix. Then*

$$per(A \circ A) \leq (per(A))^2.$$

Proof. Combine Proposition 2.6, Proposition 2.7, Corollary 3.4, and Corollary 4.5. □

We conclude by giving the result in the form that Chollet originally stated his conjecture:

Corollary 5.1. *Let A, B be two 4×4 positive semidefinite, real matrices. Then*

$$per(A \circ B) \leq per(A)per(B).$$

Proof. This follows from Theorem 1.4 and Proposition 1.3. □

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References

[1] J. Chollet. Is there a permanental analogue to Oppenheim's inequality? *The Amer. Math. Monthly*, 89(1):57–58, 1982.
[2] A. Oppenheim. Inequalities connected with definite Hermitian forms. *J. Lond. Math. Soc.*, 1(2): 114–119, 1930
[3] R. J. Gregorac and I.R. Hentzel. A note on the analogue of Oppenheim's inequality for permanents. *Linear Algebra Appl.*, 94: 109–112, 1987.
[4] R. B. Bapat and V. S. Sunder On majorization and Schur products. *Linear Algebra Appl.*, 72: 107–117, 1985.
[5] S. Drury. A counterexample to a question of Bapat and Sunder. *Electron. J. Linear Algebra*, 31: 69–70, 2016.
[6] F. Zhang. An update on a few permanent conjectures. *Spec. Matrices*, 4: 305–316, 2016.
[7] M. Marcus and M. Sandy. Bessel's inequality in tensor space. *Linear Multilinear Algebra*, 23(3): 233–249, 1988.
[8] F. Zhang. An analytic approach to a permanent conjecture. *Linear Algebra Appl.*, 438: 1570–1579, 2013.

Appendices

We now provide justification for the inequalities that appear in the proofs of Proposition 3.3, Proposition 4.3, and Proposition 4.4. In doing so, we will be making frequent use of the easy inequality $x^2 + y^2 \geq 2xy$ (by non-negativity of $(x-y)^2$). We will call this the "Squared Difference Inequality", abbreviated SDI.

A.1

We begin with the inequalities appearing in the proof of Proposition 3.3. Recall that we assume, without loss of generality, that $f \geq g$. In what follows, the inequality that we are justifying appears in **bold**.

(3.8): By the SDI,

$$a^2h^2 + c^2f^2 \geq 2acfh.$$

(3.9): We apply Inequality (3.3) to ab^2 and apply Inequality (3.4) to $(ah^2 + 2bch^3)$ to obtain:

$$\begin{aligned}
a^2g^2h^2 + ab^2 + (ah^2 + 2bch^3) + b^2g^2h^2 + b^4g^2 + c^4f^2 &\geq a^2g^2h^2 + (b^2 + b^4 + ab^2f^2 + 2b^3f) + (b^2h^2 + c^2h^2 + ah^4) \\
&\quad + b^2g^2h^2 + b^4g^2 + c^4f^2 \\
\text{Rearranging...} \quad &= (a^2g^2h^2 + b^2) + (b^4 + b^2g^2h^2) + (ab^2f^2 + ah^4) + 2b^3f \\
&\quad + (b^2h^2 + b^4g^2) + (c^2h^2 + c^4f^2) \\
\text{Factoring...} \quad &= (a^2g^2h^2 + b^2) + b^2(b^2 + g^2h^2) + a(b^2f^2 + h^4) + 2b^3f \\
&\quad + b^2(h^2 + b^2g^2) + c^2(h^2 + c^2f^2) \\
\text{Applying the SDI...} \quad &\geq 2abgh + 2b^3gh + 2abfh^2 + 2b^3f + 2b^3gh + 2c^3fh
\end{aligned}$$

(3.10): We apply Inequality (3.3) to both a and $2afgh$ to obtain:

$$\begin{aligned}
a + 2afgh + c^2h^2 + c^2 + b^2g^2 + a^2f^2 + ac^2f^2g^2 + 2ab^2fgh &\geq (1 + b^2 + af^2 + 2bf) + (2fgh + 2b^2fgh + 2af^3gh + 4bf^2gh) \\
&\quad + c^2h^2 + c^2 + b^2g^2 + a^2f^2 + ac^2f^2g^2 + 2ab^2fgh \\
\text{Rearranging...} \quad &= (1 + 2fgh) + b^2 + af^2 + 2bf + 2b^2fgh + 2af^3gh + 4bf^2gh \\
&\quad + c^2h^2 + c^2 + b^2g^2 + a^2f^2 + ac^2f^2g^2 + 2ab^2fgh \\
\text{Applying Inequality (3.6) to } (1 + 2fgh)\ldots \quad &\geq f^2 + g^2 + h^2 + b^2 + af^2 + 2bf + 2b^2fgh + 2af^3gh + 4bf^2gh \\
&\quad + c^2h^2 + c^2 + b^2g^2 + a^2f^2 + ac^2f^2g^2 + 2ab^2fgh \\
\text{Rearranging, and removing unwanted terms} \quad &\geq (f^2 + c^2h^2) + (g^2 + c^2) + (h^2 + b^2g^2) + (b^2 + a^2f^2) \\
&\quad + (af^2 + ac^2f^2g^2) + 2bf + (2af^3gh + 2ab^2fgh) \\
\text{Factoring...} \quad &= (f^2 + c^2h^2) + (g^2 + c^2) + (h^2 + b^2g^2) + (b^2 + a^2f^2) \\
&\quad + af^2(1 + c^2g^2) + 2bf + 2afgh(f^2 + b^2) \\
\text{Applying the SDI...} \quad &\geq 2cfh + 2cg + 2bgh + 2abf + 2acf^2g + 2bf + 4abf^2gh
\end{aligned}$$

(3.11): Applying Inequality (3.5) to ac^2, and applying $a \geq 1$ (by, for example, Inequality (3.3)) to $2a^2fg^3h$, we obtain:

$$\begin{aligned}
a^2g^2 + ac^2 + b^2f^2 + ag^2 + 2a^2fg^3h + ab^2g^2 + ac^2h^2 &\geq a^2g^2 + (c^2 + c^4 + ac^2g^2 + 2c^3g) + b^2f^2 + ag^2 + 2afg^3h \\
&\quad + ab^2g^2 + ac^2h^2 \\
\text{Applying Inequlaity 3.6 to } ag^2 + 2afg^3h\ldots \quad &\geq a^2g^2 + (c^2 + c^4 + ac^2g^2 + 2c^3g) + b^2f^2 + (af^2g^2 + ag^4 \\
&\quad + ag^2h^2) + ab^2g^2 + ac^2h^2 \\
\text{Rearranging...} \quad &= (a^2g^2 + c^2) + (c^4 + b^2f^2) + (ac^2g^2 + ag^4) + 2c^3g \\
&\quad + (af^2g^2 + ab^2g^2) + (ag^2h^2 + ac^2h^2) \\
\text{Factoring...} \quad &= (a^2g^2 + c^2) + (c^4 + b^2f^2) + ag^2(c^2 + g^2) + 2c^3g \\
&\quad + ag^2(f^2 + b^2) + ah^2(g^2 + c^2) \\
\text{Applying the SDI...} \quad &\geq 2acg + 2bc^2f + 2acg^3 + 2c^3g + 2abfg^2 + 2acgh^2
\end{aligned}$$

(3.12): By the SDI:

$$ac^2f^4 + af^2h^2 = af^2(c^2f^2 + h^2) \geq af^2(2cfh) = 2acf^3h$$

(3.13): Applying Inequality (3.3) to $2a^2fgh$ and applying $a \geq 1$ (by, for example, Inequality (3.3)) to $2a^2fgh^3$, we obtain:

$$2a^2fgh + 2ac^2f^3gh + 2a^2fgh^3 \geq (2afgh + 2ab^2fgh + 2a^2f^3gh + 4abf^2gh) + 2ac^2f^3gh + 2afgh^3$$

Rearranging, and removing unwanted terms... $\geq (2afgh + 2ac^2f^3gh) + (2ab^2fgh + 2afgh^3) + 2abf^2gh$

Factoring... $= 2afgh(1 + c^2f^2) + 2afgh((b^2 + h^2)) + 2abf^2gh$

By the SDI... $\geq 4acf^2gh + 4abfgh^2 + 2abf^2gh$

Since $g, h \leq 1$ (by Inequality (3.7))... $\geq 4acf^2gh^2 + 4abfg^2h^2 + 2abf^2gh$

(3.14): We apply Inequality (3.3) to af^2, and Inequality (3.5) to ac^2f^2 to obtain:

$$af^2 + ab^2f^2 + ac^2f^2 + b^2h^2 \geq (f^2 + b^2f^2 + af^4 + 2bf^3) + ab^2f^2 + (c^2f^2 + c^4f^2 + ac^2f^2g^2 + 2c^3f^2g) + b^2h^2$$

Rearranging, and removing unwanted terms... $\geq (f^2 + b^2h^2) + (b^2f^2 + c^4f^2) + (af^4 + ab^2f^2) + 2c^3f^2g$

By the SDI... $\geq 2bfh + 2bc^2f^2 + 2abf^3 + 2c^3f^2g$

Since $f \geq g$, and $f, h \leq 1$ (by Inequality (3.7))... $\geq 2bgh^3 + 2bc^2f^3 + 2abf^3 + 2c^3f^2g$

(3.15): Since $a \geq 1$ (by, for example, Inequality (3.3)), we see that $2abch^3 \geq 2bch^3$. Further, we apply Inequality (3.6) to $(h^2 + 2afgh^3)$ to obtain:

$$(ah^2 + 2abch^3) + (h^2 + 2afgh^3) + c^2f^2h^2 + ab^2g^2h^2 \geq (ah^2 + 2bch^3) + (f^2h^2 + g^2h^2 + h^4) + c^2f^2h^2 + ab^2g^2h^2$$

Applying Inequality (3.4) to $(ah^2 + 2bch^3)$... $\geq (b^2h^2 + c^2h^2 + ah^4) + (f^2h^2 + g^2h^2 + h^4) + c^2f^2h^2 + ab^2g^2h^2$

Rearranging... $= (b^2h^2 + f^2h^2) + (c^2h^2 + g^2h^2) + (ah^4 + ab^2g^2h^2) + (h^4 + c^2f^2h^2)$

Factoring... $= h^2(b^2 + f^2) + h^2(c^2 + g^2) + ah^2(h^2 + b^2g^2) + h^2(h^2 + c^2f^2)$

By the SDI... $\geq 2bfh^2 + 2cgh^2 + 2abgh^3 + 2cfh^3$

(3.16): By the SDI:

$$ah^4 + ac^2f^2h^2 = ah^2(h^2 + c^2f^2) \geq ah^2(2cfh) = 2acfh^3$$

(3.17): Since $h \leq 1$ (by Inequality (3.7)), and $f \geq g$:

$$a^2f^2g^2 + 4c^2fgh \geq a^2f^2g^2h + 4c^2g^2h$$

Factoring... $= g^2h(a^2f^2 + 4c^2)$

By the SDI... $\geq 4acfg^2h$

(3.18): By the SDI:

$$a^2f^2g^2h^2 + c^2g^2 = g^2(a^2f^2h^2 + c^2) \geq g^2(2acfh) = 2acfg^2h$$

(3.19): By the SDI,

$$ag^2h^2 + ab^2g^4 = ag^2(h^2 + b^2g^2) \geq ag^2(2bgh) = 2abg^3h$$

(3.20): As $a \geq 1$ (by, for example, Inequality (3.3)):

$$ab^2h^2 + b^2c^2f^2 \geq b^2h^2 + b^2c^2f^2$$

Factoring... $= b^2(h^2 + c^2f^2)$

By the SDI... $\geq 2b^2cfh$

(3.21): By the SDI:

$$b^2 + b^2c^2g^2 = b^2(1 + c^2g^2) \geq 2b^2cg$$

(3.22): Applying Inequality (3.5) to ab^2g^2, and Inequality (3.3) to $ab^2f^2g^2$, we obtain...

$$ab^2g^2 + ab^2f^2g^2 + b^2c^2h^2 + c^2f^2h^2 \geq (b^2g^2 + b^2c^2g^2 + ab^2g^4 + 2b^2cg^3) + (b^2f^2g^2 + b^4f^2g^2 + ab^2f^4g^2 + 2b^3f^3g^2) + b^2c^2h^2 + c^2f^2h^2$$

Rearranging, and removing unwanted terms... $\geq (b^2g^2 + b^4f^2g^2) + (b^2c^2g^2 + c^2f^2h^2) + 2b^2cg^3 + (b^2f^2g^2 + b^2c^2h^2)$

Factoring... $= b^2g^2(1 + b^2f^2) + c^2(b^2g^2 + f^2h^2) + 2b^2cg^3 + b^2(f^2g^2 + c^2h^2)$

By the SDI... $\geq 2b^3fg^2 + 2bc^2fgh + 2b^2cg^3 + 2b^2cfgh$

Since $f, g \leq 1$ (by Inequality (3.7))... $\geq 2b^3fg^2 + 2bc^2f^2gh + 2b^2cg^3 + 2b^2cfg^2h$

(3.23): By the SDI:

$$4bcfgh^2 + 2bc^3f^3g + 2b^3cfg^3 = 2bcfg(2h^2 + c^2f^2 + b^2g^2) = 2bcfg(h^2 + c^2f^2) + 2bcfg(h^2 + b^2g^2) \geq 4bc^2f^2gh + 4b^2cfg^2h$$

(3.24): Factoring, we obtain:

$$
\begin{aligned}
2b^3cfg + 2bc^3fg + 4bcfg &= 2bcfg(b^2 + c^2 + 2) \\
\text{Rearranging} \quad &= 2bcfg(b^2 + 1) + 2bcfg(c^2 + 1) \\
\text{By the SDI..} \quad &\geq 4b^2cfg + 4bc^2fg \\
\text{Since } f, g \leq 1 \text{ (by Inequality (3.7))...} \quad &\geq 4b^2cf^2g + 4bc^2fg^2
\end{aligned}
$$

(3.25): Applying Inequality (3.5) to ac^2f^2, we obtain:

$$
\begin{aligned}
ac^2f^2 &\geq (c^2f^2 + c^4f^2) + ac^2f^2g^2 + 2c^3f^2g \\
\text{Factoring, and removing unwanted terms...} \quad &\geq c^2f^2(1 + c^2) \\
\text{By the SDI...} \quad &\geq 2c^3f^2 \\
\text{Since } f, h \leq 1 \text{ (by Inequality (3.7))...} \quad &\geq 2c^3f^3h
\end{aligned}
$$

(3.26): Since $a \geq 1$, (by for example Inequality (3.3)), we obtain:

$$
\begin{aligned}
ac^2g^2 + b^2c^2 &\geq c^2g^2 + b^2c^2 \\
\text{Factoring...} \quad &= c^2(g^2 + b^2) \\
\text{By the SDI...} \quad &\geq 2bc^2g \\
\text{Since } h \leq 1 \text{ (by Inequality (3.7))...} \quad &\geq 2bc^2gh
\end{aligned}
$$

(3.27): Factoring, we obtain:

$$
\begin{aligned}
4bcf^2h + 4bcg^2h + 2bc^3h &= bch(4f^2 + 4g^2 + 2c^2) \\
\text{Rearranging...} \quad &= bch(4f^2 + c^2) + bch(4g^2 + c^2) \\
\text{By the SDI...} \quad &\geq 4bc^2fh + 4bc^2gh \\
\text{Since } h \leq 1 \text{ (by Inequality (3.7))...} \quad &\geq 4bc^2fh^2 + 4bc^2gh
\end{aligned}
$$

(3.28): Applying Inequality (3.3) to $abch$, we obtain:

$$
\begin{aligned}
2b^3cg^2h + 2abch &\geq 2b^3cg^2h + (2bch + 2b^3ch + 2abcf^2h + 4b^2cfh) \\
\text{Removing unwanted terms...} \quad &\geq (2b^3cg^2h + 2bch) + 4b^2cfh \\
\text{Factoring...} \quad &= 2bch(b^2g^2 + 1) + 4b^2cfh \\
\text{By the SDI...} \quad &\geq 4b^2cgh + 4b^2cfh \\
\text{Since } h \leq 1 \quad &\geq 4b^2cgh^2 + 4b^2cfh
\end{aligned}
$$

A.2

Continuing in the manner of Appendix A.1, we will now justify the inequalities appearing in the proof of Proposition 4.3. In what follows, recall that we are dealing with matrices where $ag \geq c$, and that we (again) assume without loss of generality that $f \geq g$.
(4.34): By the SDI:

$$a^2g^2 + a^2f^2h^2 = a^2(g^2 + f^2h^2) \geq 2a^2fgh$$

(4.35): We apply Inequality (4.29) to $ab^2 + b^4g^2 + b^2c^2f^2 + b^2h^2$ to obtain:

$$\begin{aligned}
&(ab^2 + b^4g^2 + b^2c^2f^2 + b^2h^2) + a + a^2h^2 + ac^2f^2h^2 \geq (2b^3f + 2b^2cg + ab^2f^2 + ab^2g^2 + ab^2h^2 + b^4 + b^2c^2\\
&\quad + 2b^3ch + 2b^3gh + 2b^2cfh + 2b^3cfg + 2ab^2fgh + b^2)\\
&\quad + a + a^2h^2 + ac^2f^2h^2\\
&\text{Rearranging, and removing unwanted terms...} \geq 2b^3f + 2b^2cg + (ab^2f^2 + a) + (ab^2g^2 + ac^2f^2h^2)\\
&\quad + (b^2c^2 + a^2h^2) + 2b^3ch + 2b^3cfg + 2ab^2fgh\\
&\text{Factoring...} = 2b^3f + 2b^2cg + a(b^2f^2 + 1) + a(b^2g^2 + c^2f^2h^2)\\
&\quad + (b^2c^2 + a^2h^2) + 2b^3ch + 2b^3cfg + 2ab^2fgh\\
&\text{By the SDI...} \geq 2b^3f + 2b^2cg + 2abf + 2abcfgh + 2abch + 2b^3ch\\
&\quad + 2b^3cfg + 2ab^2fgh\\
&\text{Since } f \geq g \text{ and } h \leq 1 \text{ (by Inequality (4.33))...} \geq 2b^3f + 2b^2cg + 2abf + 2abcg^2h + 2abch + 2b^3ch\\
&\quad + 2b^3cg^2h + 2ab^2fgh
\end{aligned}$$

(4.36): By the SDI:

$$ac^2 + ag^2 = a(c^2 + g^2) \geq 2acg$$

(4.37): Applying Inequality (4.29) to $(a^2f^2 + ab^2f^2g^2 + ac^2f^4 + af^2h^2)$, we obtain:

$$\begin{aligned}
&(a^2f^2 + ab^2f^2g^2 + ac^2f^4 + af^2h^2) + a^2g^2h^2 + c^2g^2 + b^2 + ac^2f^2 \geq (2abf^3 + 2acf^2g + a^2f^4 + a^2f^2g^2 + a^2f^2h^2 + ab^2f^2\\
&\quad + ac^2f^2 + 2abcf^2h + 2abf^2gh + 2acf^3h + 2abcf^3g\\
&\quad + 2a^2f^3gh + af^2) + a^2g^2h^2 + c^2g^2 + b^2 + ac^2f^2\\
&\text{Rearranging, and removing unwanted terms...} \geq 2abf^3 + 2acf^2g + (a^2f^4 + c^2g^2) + a^2f^2g^2 + a^2f^2h^2\\
&\quad + 2ac^2f^2 + 2abcf^2h + 2a^2f^3gh + af^2 + a^2g^2h^2 + b^2\\
&\text{Applying Inequality (4.33) to } a^2g^2h^2\ldots \geq 2abf^3 + 2acf^2g + (a^2f^4 + c^2g^2) + a^2f^2g^2 + a^2f^2h^2\\
&\quad + 2ac^2f^2 + 2abcf^2h + 2a^2f^3gh + af^2 + (a^2f^2g^2h^2\\
&\quad + a^2g^4h^2 + a^2g^2h^4 + 2a^2fg^3h^3) + b^2\\
&\text{Rearranging, and removing unwanted terms...} \geq 2abf^3 + 2acf^2g + (a^2f^4 + c^2g^2) + (a^2f^2g^2 + a^2g^4h^2)\\
&\quad + (a^2f^2h^2 + a^2g^2h^4) + 2ac^2f^2 + 2abcf^2h + 2a^2f^3gh\\
&\quad + (af^2 + b^2)\\
&\text{Applying } a \geq 1 \text{ (by, for example, Inequality (4.30)) to } af^2\ldots \geq 2abf^3 + 2acf^2g + (a^2f^4 + c^2g^2) + a^2g^2(f^2 + g^2h^2)\\
&\quad + a^2h^2(f^2 + g^2h^2) + 2ac^2f^2 + 2abcf^2h + 2a^2f^3gh\\
&\quad + (f^2 + b^2)\\
&\text{By the SDI...} \geq 2abf^3 + 2acf^2g + 2acf^2g + 2a^2fg^3h + 2a^2fgh^3\\
&\quad + 2ac^2f^2 + 2abcf^2h + 2a^2f^3gh + 2bf\\
&\text{Since } f \geq g\ldots \geq 2abf^3 + 2acf^2g + 2acg^3 + 2a^2fg^3h + 2a^2fgh^3\\
&\quad + 2ac^2g^2 + 2abcf^2h + 2a^2f^3gh + 2bf\\
&\text{Since } ag \geq c\ldots \geq 2abf^3 + 2acf^2g + 2acg^3 + 2a^2fg^3h + 2a^2fgh^3\\
&\quad + 2c^3g + 2abcf^2h + 2a^2f^3gh + 2bf
\end{aligned}$$

(4.38): Applying Inequality (4.30) to af^2, we obtain:

$$\begin{aligned}
&c^2 + af^2 + ab^2g^2 \geq c^2 + (f^2 + b^2f^2 + af^4 + 2bf^3) + ab^2g^2\\
&\text{Rearranging, and removing unwanted terms...} \geq (c^2 + f^2) + (af^4 + ab^2g^2)\\
&\text{Factoring...} = (c^2 + f^2) + a(f^4 + b^2g^2)\\
&\text{By the SDI...} \geq 2cf + 2abf^2g
\end{aligned}$$

Since $f \geq g$... $\geq 2cg + 2abfg^2$

(4.39): By the SDI:

$$ab^2f^2 + ah^4 = a(b^2f^2 + h^4) \geq 2abfh^2$$

(4.40): Since $f \geq g$, and $h \leq 1$ (by Inequality (4.33)), we obtain:

$$2acfh \geq 2acgh^2$$

.

(4.41): Applying Inequality (4.31) to ac^2h^2, we obtain...

$$b^2c^2 + ab^2h^2 + ac^2h^2 \geq b^2c^2 + ab^2h^2 + (b^2c^2h^2 + c^4h^2 + ac^2h^4 + 2bc^3h^3)$$

Rearranging, and removing unwanted terms... $\geq (b^2c^2 + c^4h^2) + (ab^2h^2 + ac^2h^4)$

Factoring... $= c^2(b^2 + c^2h^2) + ah^2(b^2 + c^2h^2)$

By the SDI... $\geq 2bc^3h + 2abch^3$

(4.42): Applying Inequality (4.30) to ah^2, we obtain:

$$a^2f^2g^2 + ah^2 + ag^2h^2 \geq a^2f^2g^2 + (h^2 + b^2h^2 + af^2h^2 + 2bfh^2) + ag^2h^2$$

Rearranging, and removing unwanted terms... $\geq (a^2f^2g^2 + h^2) + (af^2h^2 + ag^2h^2) + 2bfh^2$

Factoring... $= (a^2f^2g^2 + h^2) + ah^2(f^2 + g^2) + 2bfh^2$

By the SDI... $\geq 2afgh + 2afgh^2 + 2bfh^2$

Since $h \leq 1$ (by Inequality (4.33))... $\geq 2afgh + 2afgh^3 + 2bfh^2$

(4.43): Since $ag \geq c$, we obtain:

$$2abgh \geq 2bch$$

(4.44): Applying Inequality (4.31) to ah^2, we obtain:

$$ah^2 + h^2 \geq (b^2h^2 + c^2h^2 + ah^4 + 2bch^3) + h^2$$

Rearranging, and removing unwanted terms... $\geq (c^2h^2 + h^2) + 2bch^3$

Factoring... $= h^2(c^2 + 1) + 2bch^3$

By the SDI... $\geq 2ch^2 + 2bch^3$

Since $g \leq 1$ (by Inequality (4.33))... $\geq 2cgh^2 + 2bch^3$

(4.45): Factoring, we obtain:

$$2acfg^2h + 2acf^3h = 2acfh(g^2 + f^2)$$

By the SDI... $\geq 4acf^2gh$

Since $h \leq 1$ (by Inequality (4.33))... $\geq 4acf^2gh^2$.

(4.46): Since $ag \geq c$, we obtain:

$$4abf^2gh \geq 4bcf^2h.$$

(4.47): Factoring, we obtain:

$$2cfh + 2c^3fh = 2cfh(1 + c^2)$$

By the SDI... $\geq 4c^2fh$

Since $g \leq 1$ (by Inequality (4.33))... $\geq 4c^2fgh$

(4.48): Factoring...

$$2cfh^3 + 2c^3f^3h + b^2c^2h^2 = 2cfh(h^2 + c^2f^2) + b^2c^2h^2$$

By the SDI... $\geq 4c^2f^2h^2 + b^2c^2h^2$

Factoring... $= c^2h^2(4f^2 + b^2)$

By the SDI... $\geq 4bc^2fh^2$

(4.49): Factoring...

$$2abf^2gh + 2abg^3h = 2abgh(f^2 + g^2)$$

By the SDI... $\geq 4abfg^2h$

Since $h \leq 1$ (by Inequality (4.33))... $\geq 4abfg^2h^2$

(4.50): Factoring...

$$2bgh + 2b^3gh = 2bgh(1 + b^2)$$

By the SDI... $\geq 4b^2gh$

Since $f \leq 1$ (by Inequality (4.33))... $\geq 4b^2fgh$

(4.51): Factoring...

$$2bcfg + 2bcfgh^2 \geq 2bcfg(1 + h^2)$$

By the SDI... $\geq 4bcfgh$

Since $f \geq g$... $\geq 4bcg^2h$

(4.52): Since $f \geq g$, and $h \leq 1$ (by Inequality (4.33)), we obtain:

$$4b^2cfh \geq 4b^2cgh^2$$

(4.53): Applying Inequality (4.30) to ab^2g^4, we obtain:

$$ab^2g^4 + ab^2g^2h^2 + b^2c^2g^2 \geq (b^2g^4 + b^4g^4 + ab^2f^2g^4 + 2b^3fg^4) + ab^2g^2h^2 + b^2c^2g^2$$

Rearranging, and removing unwanted terms... $\geq (b^2g^4 + b^2c^2g^2) + (ab^2f^2g^4 + ab^2g^2h^2)$

$= b^2g^2(g^2 + c^2) + ab^2g^2(f^2g^2 + h^2)$

By the SDI... $\geq 2b^2cg^3 + 2ab^2fg^3h$

(4.54): Applying Inequality (4.30) to ab^2g^2, we obtain:

$$ab^2g^2 \geq b^2g^2 + b^4g^2 + ab^2f^2g^2 + 2b^3fg^2$$

Removing unwanted terms... $\geq 2b^3fg^2$

(4.55): Factoring...

$$2abcfg^3 + 2abcfgh^2 = 2abcfg(g^2 + h^2)$$

By the SDI... $\geq 4abcfg^2h$

Since $f \leq 1$ (by Inequality (4.33))... $\geq 4abcf^2g^2h$

(4.56): Applying Inequality (4.31) to $2abcf^3g$, we obtain:

$$2abcf^3g + 4bcfg \geq (2b^3cf^3g + 2bc^3f^3g + 2abcf^3gh^2 + 4b^2c^2f^3gh) + 4bcfg$$

Rearranging and removing unwanted terms... $\geq (2b^3cf^3g + 2bcfg) + (2bc^3f^3g + 2bcfg)$

Factoring... $= 2bcfg(b^2f^2 + 1) + 2bcfg(c^2f^2 + 1)$

By the SDI... $\geq 4b^2cf^2g + 4bc^2f^2g$

Since $f \geq g$... $\geq 4b^2cf^2g + 4bc^2fg^2$

(4.57): Factoring, we obtain:

$$2b^3cfg + 2bc^3fg \geq 2bcfg(b^2 + c^2)$$

By the SDI... $\geq 4b^2c^2fg$

Since $h \leq 1$ (by Inequality (4.33))... $\geq 4b^2c^2fgh$

(4.58): Factoring, we obtain:

$$a^2f^2g^2h^2 + c^4f^2 = f^2(a^2g^2h^2 + c^4)$$

By the SDI... $\geq 2ac^2f^2gh$

Since $f \leq 1$ (by Inequality (4.33))... $\geq 2ac^2f^3gh$

(4.59): Applying Inequality (4.31) to ac^2f^2, and Inequality (4.33) to c^2f^2, we obtain:

$$\begin{aligned}
ac^2f^2 + c^2f^2 + ac^2g^2 &\geq (b^2c^2f^2 + c^4f^2 + ac^2f^2h^2 + 2bc^3f^2h) + (c^2f^4 + c^2f^2g^2 \\
&\quad + c^2f^2h^2 + 2c^2f^3gh) + ac^2g^2 \\
\text{Rearranging, and removing unwanted terms...} \qquad &\geq (b^2c^2f^2 + c^2f^4) + (c^4f^2 + c^2f^2g^2) + (ac^2f^2h^2 + ac^2g^2) \\
&\quad + 2bc^3f^2h \\
\text{Factoring...} \qquad &= c^2f^2(b^2 + f^2) + c^2f^2(c^2 + g^2) + ac^2(f^2h^2 + g^2) + 2bc^3f^2h \\
\text{By the SDI...} \qquad &\geq 2bc^2f^3 + 2c^3f^2g + 2ac^2fgh + 2bc^3f^2h
\end{aligned}$$

(4.60): Since $ag \geq c$, it follows that:

$$2abcfg \geq 2bc^2f$$

A.3

Lastly, we justify the inequalities appearing in the proof of Proposition 4.4. We remind the reader that this result concerned matrices where $c > ag$, and that we (again) assumed without loss of generality that $f \geq g$.
(4.61): Since $c > ag$, we obtain

$$2acfh > 2a^2fgh$$

(4.62): Applying Inequality (4.29) to $ab^2 + b^4g^2 + b^2c^2f^2 + b^2h^2$, we obtain:

$$\begin{aligned}
(ab^2 + b^4g^2 + b^2c^2f^2 + b^2h^2) + a + ac^2f^4 + a^2h^2 + 2abf^2gh + ac^2h^2 &\geq (2b^3f + 2b^2cg + ab^2f^2 + ab^2g^2 + ab^2h^2 + b^4 + b^2c^2 \\
&\quad + 2b^3ch + 2b^3gh + 2b^2cfh + 2b^3cfg + 2ab^2fgh + b^2) \\
&\quad + a + ac^2f^4 + a^2h^2 + 2abf^2gh + ac^2h^2 \\
\text{Rearranging and removing unwanted terms...} \qquad &\geq 2b^3f + 2b^2cg + (ab^2f^2 + a) + (ab^2h^2 + ac^2f^4) \\
&\quad + (b^2c^2 + a^2h^2) + 2b^3ch + (2b^3gh + 2abf^2gh) \\
&\quad + 2ab^2fgh + (b^2 + ac^2h^2) \\
\text{Factoring...} \qquad &= 2b^3f + 2b^2cg + a(b^2f^2 + 1) + a(b^2h^2 + c^2f^4) \\
&\quad + (b^2c^2 + a^2h^2) + 2b^3ch + 2bgh(b^2 + af^2) + 2ab^2fgh \\
&\quad + (b^2 + ac^2h^2) \\
\text{Applying } a \geq 1 \text{ (by Inequality (4.30)) to } af^2 \text{ and } ac^2h^2\text{...} \qquad &\geq 2b^3f + 2b^2cg + a(b^2f^2 + 1) + a(b^2h^2 + c^2f^4) \\
&\quad + (b^2c^2 + a^2h^2) + 2b^3ch + 2bgh(b^2 + f^2) + 2ab^2fgh \\
&\quad + (b^2 + c^2h^2) \\
\text{By SDI...} \qquad &\geq 2b^3f + 2b^2cg + 2abf + 2abcf^2h + 2abch + 2b^3ch \\
&\quad + 4b^2fgh + 2ab^2fgh + 2bch
\end{aligned}$$

(4.63): By the SDI:

$$c^2 + a^2g^2 \geq 2acg$$

(4.64): Since $c > ag$, we obtain

$$2acf^3h > 2a^2f^3gh$$

(4.65): Applying Inequality (4.30) to af^2, Inequality (4.33) to h^2, and Inequality (4.31) to ah^2, we obtain:

$$\begin{aligned}
b^2 + af^2 + ab^2f^2 + h^2 + ah^2 + a^2f^2h^2 + ac^2g^2 &\geq b^2 + (f^2 + b^2f^2 + af^4 + 2bf^3) + ab^2f^2 + (f^2h^2 + g^2h^2 + h^4 \\
&\quad + 2fgh^3) + (b^2h^2 + c^2h^2 + ah^4 + 2bch^3) + a^2f^2h^2 + ac^2g^2 \\
\text{Rearranging, and removing unwanted terms...} \qquad &\geq (b^2 + f^2) + (b^2f^2 + h^4) + (af^4 + ab^2f^2) + (g^2h^2 + c^2h^2) \\
&\quad + (b^2h^2 + a^2f^2h^2) + (ah^4 + ac^2g^2) + 2bch^3 \\
\text{Factoring...} \qquad &\geq (b^2 + f^2) + (b^2f^2 + h^4) + af^2(f^2 + b^2) + h^2(g^2 + c^2) \\
&\quad + h^2(b^2 + a^2f^2) + a(h^4 + c^2g^2) + 2bch^3 \\
\text{By the SDI...} \qquad &\geq 2bf + 2bfh^2 + 2abf^3 + 2cgh^2 + 2abfh^2 + 2acgh^2 + 2bch^3
\end{aligned}$$

(4.66): Since $a \geq 1$ (by Inequality (4.30)), we see that...

$$a^2f^2 + ac^2f^2g^2 \geq af^2 + ac^2f^2g^2$$

Factoring... $= af^2(1 + c^2g^2)$
By the SDI... $\geq 2acf^2g$

(4.67): Since $c > ag$, we obtain

$$2acfg^2h > 2a^2fg^3h$$

(4.68): Applying Inequality (4.29), we obtain:

$$(2abcfg + 2b^3cfg^3 + 2bc^3f^3g + 2bcfgh^2) \geq 4b^2cf^2g + 4bc^2fg^2 + 2abcf^3g + 2abcfg^3 + 2abcfgh^2 + 2b^3cfg + 2bc^3fgh + 4b^2c^2fgh + 4b^2cfg^2h + 4bc^2f^2gh + 4b^2c^2f^2g^2 + 4abcf^2g^2h + 2bcfg$$

Removing unwanted terms... $\geq 4b^2cf^2g + 4bc^2fg^2 + 4b^2c^2fgh + 4abcf^2g^2h + 2bcfg$
Applying $c > ag$ to $2bcfg$... $> 4b^2cf^2g + 4bc^2fg^2 + 4b^2c^2fgh + 4abcf^2g^2h + 2abfg^2$

(4.69) Applying Inequality (4.29) to $ac^2 + b^2c^2g^2 + c^4f^2 + c^2h^2$, we obtain...

$$(ac^2 + b^2c^2g^2 + c^4f^2 + c^2h^2) + ab^2h^2 + b^2c^2 + 2c^2f^2h^2 + 2c^3fh + ag^2 \geq (2bc^2f + 2c^3g + ac^2f^2 + ac^2g^2 + ac^2h^2 + b^2c^2 + c^4 + 2bc^3h + 2bc^2gh + 2c^3fh + 2bc^3fg + 2ac^2fgh + c^2) + ab^2h^2 + b^2c^2 + 2c^2f^2h^2 + 2c^3fh + ag^2$$

Rearranging, and removing unwanted terms... $\geq 2bc^2f + 2c^3g + ac^2g^2 + (ac^2h^2 + ab^2h^2) + (2b^2c^2 + 2c^2f^2h^2) + 2bc^3h + 4c^3fh + 2ac^2fgh + c^2 + ag^2$

Applying Inequality (4.30) to ag^2... $\geq 2bc^2f + 2c^3g + ac^2g^2 + (ac^2h^2 + ab^2h^2) + (2b^2c^2 + 2c^2f^2h^2) + 2bc^3h + 4c^3fh + 2ac^2fgh + c^2 + (g^2 + b^2g^2 + af^2g^2 + 2bfg^2)$

Rearranging, and removing unwanted terms... $\geq 2bc^2f + 2c^3g + (ac^2g^2 + af^2g^2) + (ac^2h^2 + ab^2h^2) + (2b^2c^2 + 2c^2f^2h^2) + 2bc^3h + 4c^3fh + 2ac^2fgh + (c^2 + g^2)$

Factoring, and applying $c > ag$ to $4c^3fh$... $> 2bc^2f + 2c^3g + ag^2(c^2 + f^2) + ah^2(c^2 + b^2) + 2c^2(b^2 + f^2h^2) + 2bc^3h + 4ac^2fgh + 2ac^2fgh + (c^2 + g^2)$

By the SDI... $\geq 2bc^2f + 2c^3g + 2acfg^2 + 2abch^2 + 4bc^2fh + 2bc^3h + 4ac^2fgh + 2ac^2fgh + 2cg$

Since $f \geq g$, $h \leq 1$ (by Inequality (4.33)), and $a \geq 1$ (by Inequality (4.30))... $\geq 2bc^2f + 2c^3g + 2acg^3 + 2abch^3 + 4bc^2fh^2 + 2bc^3h + 4c^2fgh + 2ac^2fgh + 2cg$

(4.70): Factoring, we obtain:

$$ab^2g^2 + ac^2f^2h^2 = a(b^2g^2 + c^2f^2h^2)$$

By the SDI... $\geq 2abcfgh$
Since $f \geq g$ $\geq 2abcg^2h$

(4.71): Since $c > ag$, we obtain:

$$2acfh^3 > 2a^2fgh^3$$

(4.72): Since $c > ag$, we obtain:

$$2cfh > 2afgh$$

(4.73): Since $c > ag$, we obtain:

$$2cfh^3 > 2afgh^3$$

(4.74): Factoring...

$$ac^2f^2 + ah^2 + 4acfg^2h = a(c^2f^2 + h^2) + 4acfg^2h$$

By the SDI... $\geq 2acfh + 4acfg^2h$
Applying Inequality (4.30) to $acfh$... $\geq (2cfh + 2b^2cfh + 2acf^3h + 4bcf^2h) + 4acfg^2h$
Rearranging, and removing unwanted terms... $\geq (acf^3h + 4acfg^2h) + 4bcf^2h$

Factoring...	$= acfh(f^2 + 4g^2) + 4bcf^2h$
By the SDI...	$\geq 4acf^2gh + 4bcf^2h$
Since $h \leq 1$ (by Inequality (4.33))...	$\geq 4acf^2gh^2 + 4bcf^2h$

(4.75): Since $c > ag$, we obtain:

$$4bcfgh^2 > 4abfg^2h^2$$

(4.76): Applying Inequality (4.32) to $2abgh$, we obtain:

$$2abgh + 2b^3gh \geq (2bgh + 2bc^2gh + 2abg^3h + 4bcg^2h) + 2b^3gh$$

Rearranging, and removing unwanted terms...	$\geq (2bc^2gh + 2b^3gh) + 4bcg^2h$
Factoring...	$= 2bgh(c^2 + b^2) + 4bcg^2h$
By SDI, we obtain...	$\geq 4b^2cgh + 4bcg^2h$
Since $h \leq 1$ (by Inequality (4.33))...	$\geq 4b^2cgh^2 + 4bcg^2h$

(4.77): Applying Inequality (4.31) to ab^2g^2, and Inequality (4.33) to b^2g^2, we obtain:

$$ab^2g^2 + b^2g^2 \geq (b^4g^2 + b^2c^2g^2 + ab^2g^2h^2 + 2b^3cg^2h) + (b^2f^2g^2 + b^2g^4 + b^2g^2h^2 + 2b^2fg^3h)$$

Rearranging, and removing unwanted terms...	$\geq (b^4g^2 + b^2f^2g^2) + (b^2c^2g^2 + b^2g^4) + 2b^3cg^2h$
Factoring...	$= b^2g^2(b^2 + f^2) + b^2g^2(c^2 + g^2) + 2b^3cg^2h$
By the SDI...	$\geq 2b^3fg^2 + 2b^2cg^3 + 2b^3cg^2h$

(4.78): Since $c > ag$, we obtain

$$2b^2cfg^2h > 2ab^2fg^3h.$$

(4.79): Since $c > ag$, we obtain

$$2c^3f^3h > 2ac^2f^3gh$$

(4.80): Applying Inequality (4.31) to ac^2f^2, and Inequality (4.33) to c^2f^2, we obtain:

$$ac^2f^2 + c^2f^2 \geq (b^2c^2f^2 + c^4f^2 + ac^2f^2h^2 + 2bc^3f^2h) + (c^2f^4 + c^2f^2g^2 + c^2f^2h^2 + 2c^2f^3gh)$$

Rearranging, and removing unwanted terms...	$\geq (b^2c^2f^2 + c^2f^4) + (c^4f^2 + c^2f^2g^2) + 2bc^3f^2h$
Factoring...	$= c^2f^2(b^2 + f^2) + c^2f^2(c^2 + g^2) + 2bc^3f^2h$
By the SDI...	$\geq 2bc^2f^3 + 2c^3f^2g + 2bc^3f^2h$