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Further results on q -Lie groups, q -Lie algebras and q -homogeneous spaces

<https://doi.org/10.1515/spma-2020-0129>

Received October 28, 2020; accepted January 21, 2021

Abstract: We introduce most of the concepts for q -Lie algebras in a way independent of the base field K . Again it turns out that we can keep the same Lie algebra with a small modification. We use very similar definitions for all quantities, which means that the proofs are similar. In particular, the quantities solvable, nilpotent, semisimple q -Lie algebra, Weyl group and Weyl chamber are identical with the ordinary case $q = 1$. The computations of sample q -roots for certain well-known q -Lie groups contain an extra q -addition, and consequently, for most of the quantities which are q -deformed, we add a prefix q in the respective name. Important examples are the q -Cartan subalgebra and the q -Cartan Killing form. We introduce the concept q -homogeneous spaces in a formal way exemplified by the examples $\frac{SU_q(1,1)}{SO_q(2)}$ and $\frac{SO_q(3)}{SO_q(2)}$ with corresponding q -Lie groups and q -geodesics. By introducing a q -deformed semidirect product, we can define exact sequences of q -Lie groups and some other interesting q -homogeneous spaces. We give an example of the corresponding q -Iwasawa decomposition for $SL_q(2)$.

Keywords q -Lie algebra; q -Lie group; q -deformed semidirect product; q -root; q -homogeneous space; q -Iwasawa decomposition

MSC: Primary 33D15; 22E30 Secondary 22F30; 22E60

1 General introduction

The purpose of this article is to extend the theory of q -Lie algebras, and to a certain extent, the theory of q -Lie groups. In this paper we will deal with the subject in more depth; in the process we introduce such objects like q -hyperbolic space, q -sphere, etc. These objects have been mentioned before, but not in this form. We find that our q -additions fit naturally in the new context; one example is how the formulas for q -roots, q -Cartan Killing forms and q -Cartan subalgebras are transformed. However, we will not use the same symbol for q -addition between letters as for q -addition between matrix q -Lie algebras. In order to give a chronological summary, we refer to three papers:

In [3] we defined early versions of q -Lie groups, maximal q -tori for $SU_q(2)$, $SO_q(2)$ and $U_q(n)$, early versions of q -scalar product and q -determinants. This led to a formula for so-called q -Euler angles. In [6] we extended the theory for q -determinants and matrix q -Lie groups with two types of q -addition. Furthermore, we defined matrix q -Lie algebras, q -trace, stabilizer, kernel and q -Lie group morphism. In [3] the general real q -linear group was also introduced. Finally, in [7], we further extended the definition of q -Lie groups and gave many examples of it.

We decided to use the following terminology: center Z for q -Lie group, q -Lie subgroup, normal q -Lie subgroup, q -one parameter subgroup, q -torus, $GL_q(n, K)$, $\text{Ad} =$ adjoint representation (q -homomorphism for q -Lie groups), Aut , Der , ker , q -Lie bracket, q -ideal, radical, q -normalizer, q -factor algebra, q -center for q -Lie

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algebras, Weyl group, Weyl chamber, Weyl unitary trick, Dynkin diagram, q -Cartan Killing form, q -Cartan subalgebra (CSA), derived q -Lie algebra, q -root ($\{\alpha_i\}$), q -Lie subalgebra, commutator q -Lie subalgebra, commutator q -Lie algebra = $C(\mathfrak{g}_q)$, q -Lie algebra automorphism, solvable, nilpotent, semisimple (q -Lie algebra), ad = adjoint representation (q -homomorphism for q -Lie algebras), semisimple \mathfrak{g}_q -module, q -direct sum, exact sequence, splitting, q -homogeneous space, q -central subgroup, q -semidirect product, q -conjugation, q -homomorphism (for q -Lie groups), and $O_q(V)$ = group of linear q -isometries. The reason is that because the q -Lie algebras are almost the same, the concepts solvable, nilpotent, semisimple, Weyl group and Weyl chamber are the same. The concepts q -Cartan subalgebra, q -root system, exact sequence and splitting are similar and will be defined in each case.

The basic construction which replace the real numbers as function arguments in trigonometric functions etc. are the q -real numbers \mathbb{R}_q . The definition given for \mathbb{R}_q is not the most general, but it will do for the moment. A closer introduction to this q -umbral calculus is given in [4] and in [5]. We find that our objects have properties similar to manifolds.

This paper is organized as follows: In this section we give some fundamental definitions and theorems. In section 2 we start with a comparison with the similar Lie and topological groups. Several theorems from Lie groups have analogues for q -Lie groups, which is illustrated for $\text{SL}_q(2)$. Therefore, in subsection 2.1 we study the q -Lie group $\text{SL}_q(2)$ and prove the corresponding q -Iwasawa decomposition. Then we introduce the q -torus and the q -determinant from previous papers. The centers in subsection 2.2 have similar properties as in the ordinary case.

In section 3 a complete theory for q -Lie algebras is presented, which is very similar to Lie algebras. We start with the universal covering q -group in subsection 3.1. In subsection 3.2 we define the classical q -Lie algebras and in subsection 3.3 the important concepts solvable and nilpotent q -Lie algebras are defined. In subsection 3.4 some properties of q -Cartan Killing forms are discussed. The description of q -root systems in subsection 3.5 is very similar to the ordinary case. The Weyl group in subsection 3.6 is also very similar. Dynkin diagrams and a table of important q -Lie algebras are discussed in subsection 3.7. In subsection 3.8 we make some remarks on the relation between complex semisimple q -Lie groups and real compact q -Lie groups.

In section 4 we briefly describe q -homogeneous spaces. In subsection 4.1 we introduce q -differentials to prepare for the proof that q -homogeneous spaces are manifolds. In subsection 4.2 we study the q -Lie groups $\text{SU}_q(2)$ and $\text{SU}_q(1, 1)$ together with the corresponding q -homogeneous space $\frac{\text{SU}_q(1,1)}{\text{SO}_q(2)}$. In subsection 4.3 we study the q -Lie groups $\text{SO}_q(3)$ and $\text{SO}_q(2)$ together with the corresponding q -homogeneous space $\frac{\text{SO}_q(3)}{\text{SO}_q(2)}$. In subsection 4.4 we compute q -roots and q -Cartan subalgebras for the well-known q -Lie groups. We develop the irreducible representation of $\text{SO}_q(3)$ from [3].

In subsection 4.5 we present the important concept q -deformed semidirect product. In section 5 we make a short conclusion.

Something about the notation: Since real and complex q -Lie groups often can be treated simultaneously, we shall from now on use the letter K in order to denote either \mathbb{R} or \mathbb{C} and use the term K - q -Lie group in order to refer to real resp. complex q -Lie groups.

We denote direct sums of matrices by $\oplus_{q'}$ or \oplus_q . The notation $\oplus_{q'}$ denotes direct sum of two matrices in the context of q -Lie algebra or for sums like (44), and the notation \oplus_q denotes sums of commuting matrices.

Definition 1.1. Let $\alpha = [\alpha_{ij}]$ be an $m \times n$ matrix with matrix elements α_{ij} . The conjugate of α is the $m \times n$ matrix $\bar{\alpha} = [\bar{\alpha}_{ij}]$.

Let $0 < q < 1$. The conjugate transpose of α is the $n \times m$ matrix $\alpha^* \equiv [(\bar{\alpha}_{ij})]^T$. A square matrix H is called q -Hermitian if $H^* = H$.

Theorem 1.2. [8] We have $E_q(\cdot) - \infty, (1 - q)^{-1}(\cdot) =]0, \infty[$, and

$$E_q :] - \infty, (1 - q)^{-1}[\longrightarrow]0, \infty[$$

is an analytic isomorphism.

Because of this, we only consider values of x with $0 < x < (1 - q)^{-1}$ such that this function $E_q(x)$ converges. Other values of x , except the poles, would give other branches.

2 Introduction to q -Lie groups and q -Tori

The following introduction to Lie groups and topological groups is also applicable for q -Lie groups.

Definition 2.1. A topological group is a group G endowed with a Hausdorff topology such that both the group multiplication (group law)

$$\mu : G \times G \longrightarrow G, (a, b) \mapsto ab,$$

and the "inversion"

$$\iota : G \longrightarrow G, a \mapsto a^{-1}$$

are continuous maps.

Example 2.2. 1. The additive group $G \equiv \mathbb{R}$ endowed with its standard topology.

2. The group

$$G \equiv GL_n(\mathbb{R}) \subset \mathbb{R}^{n,n} \cong \mathbb{R}^{n^2},$$

endowed with the topology as a (n open) subset of \mathbb{R}^{n^2} .

Let us comment on the second example: First of all,

$$GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R}^*) \subset \mathbb{R}^{n,n}$$

is an open set of $\mathbb{R}^{n,n}$ as the inverse image of the open set $\mathbb{R}^* \equiv \mathbb{R} \setminus \{0\}$ (the punctured real line) with respect to a continuous map, the determinant

$$\det : \mathbb{R}^{n,n} \longrightarrow \mathbb{R}, A = (\alpha_{ij}) \mapsto \det(A) = \sum_{\pi \in \mathcal{S}_n} \text{sign}(\pi) \prod_{i=1}^n \alpha_{i,\pi(i)},$$

a polynomial in the entries α_{ij} of A . That matrix multiplication is continuous is immediate, while the continuity of the inversion $A \mapsto A^{-1}$ follows from the following formula

$$A^{-1} = (\gamma_{ij}) \text{ with } \gamma_{ij} = (-1)^{i+j} \frac{\det(A_{ji})}{\det(A)},$$

where $A_{k\ell} \in \mathbb{R}^{n-1,n-1}$ denotes the matrix obtained from $A \in \mathbb{R}^{n,n}$ by deleting the k -th row and the ℓ -th column. With other words, the entries γ_{ij} of A^{-1} are rational functions in the entries of A (with a non-vanishing denominator).

Example 2.3. The general linear group

$$GL_n(K) \equiv \{A \in K^{n,n}; \det(A) \neq 0\}$$

is a K -Lie group. Note that $GL_1(K)$ is nothing but the multiplicative group $K^* \equiv K \setminus \{0\}$ of K .

In the following we shall present a series of closed subgroups $G \subset GL_n(K)$. In order to see that they are even Lie groups we use

Remark 1. Let $G \subset GL_n(K)$ be a closed subgroup, such that $G = F^{-1}(0)$ with a map $F : GL_n(K) \longrightarrow K^m$ satisfying $F(AX) = F(X)$ for all $A \in G$. If then $DF(E) : K^{n,n} \longrightarrow K^m$ is onto, the subgroup $G \subset GL_n(K)$ carries a natural differentiable resp. complex structure, and with respect to that structure G is a K -Lie group. It suffices to check that

$$DF(A) : K^{n,n} \longrightarrow K^m$$

is onto for all $A \in G$. Denote $\lambda_A : GL_n(K) \rightarrow GL_n(K)$, $X \mapsto AX$ the left multiplication with A , a diffeomorphism. Since $F = F \circ \lambda_A$, we obtain

$$DF(E) = DF(A) \circ D(\lambda_A)(E) = DF(A) \circ \lambda_A,$$

where we have used the fact that λ_A as a linear map coincides with its own Jacobian - here we denote also λ_A the map $K^{n,n} \rightarrow K^{n,n}$, $X \mapsto AX$. But A being invertible, $\lambda_A : K^{n,n} \rightarrow K^{n,n}$ is an isomorphism of vector spaces, so with $DF(E)$ the map $DF(A)$ is surjective as well. The affine subspace

$$E + \ker DF(E) \subset K^{n,n}$$

is the best approximation of $F^{-1}(0) = G$ at E by an affine subspace, it can naturally be identified with the "tangent space" $T_E G$ of $G = F^{-1}(0)$ at E , to be defined in the next chapter.

Now let us continue with our examples:

Example 2.4. 1. *The special linear group*

$$SL_n(K) \equiv \{A \in K^{n,n}; \det(A) = 1\}$$

is a closed normal subgroup (as the kernel of the continuous q -homomorphism $\det : GL_n(K) \rightarrow K^*$). Take $F(X) = \det(X) - 1$. Since

$$DF(E) = D(\det)(E) = \text{Tr}$$

with the (surjective) trace map $\text{Tr} : K^{n,n} \rightarrow K$, $A = (\alpha_{ij}) \mapsto \sum_{i=1}^n \alpha_{ii}$, we can apply Remark 1. So $SL_n(K)$ is a K -Lie group.

2. We consider a non degenerate bilinear form $\sigma : K^n \times K^n \rightarrow K$, write $\sigma(x, y) = x^T S y$ with a matrix $S \in K^{n,n}$. Then a matrix $A \in GL_n(K)$ preserves σ , i.e. $\sigma(Ax, Ay) = \sigma(x, y)$ for all $x, y \in K^n$ iff $A^T S A = S$. Obviously the set of all such " σ -isometries" forms a closed subgroup of $GL_n(K)$. We look at the map

$$F : GL_n(K) \rightarrow K^{n,n}, X \mapsto X^T S X - S.$$

Then the Jacobian of F at E is

$$DF(E) : K^{n,n} \rightarrow K^{n,n}, X \mapsto X^T S + S X.$$

Assume now S is either symmetric: $S^T = S$ or antisymmetric: $S^T = -S$. Then $F(GL_n(K)) \subset S_n(K)$ resp. $F(GL_n(K)) \subset A_n(K)$, where $S_n(K)$ denotes the vector space of all symmetric matrices and $A_n(K)$ the vector space of all anti-symmetric matrices. Thus we may replace the target of both F and $DF(E)$ with $S_n(K)$ resp. $A_n(K)$. Since a given matrix $A \in S_n(K)$ resp. $A \in A_n(K)$ is of the form $A = DF(E)(X)$ with $X \equiv \frac{1}{2}(S^{-1}A)$, the Jacobian map of F at E is onto and hence our isometry group even a Lie group. If we take $S = E$ we obtain the K -orthogonal group

$$O_n(K) \equiv \{A \in GL_n(K); A^T A = E\},$$

while for even $n = 2m$ and $S = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ the analogous group

$$Sp_n(K) \equiv \{A \in GL_n(K); A^T S A = S\}$$

is called the K -symplectic group. We have

$$\dim O_n(K) = n^2 - \dim S_n(K) = \frac{1}{2}n(n-1)$$

and

$$\dim Sp_n(K) = n^2 - \dim A_n(K) = \frac{1}{2}n(n+1).$$

We remark that the real orthogonal group $O_n(\mathbb{R})$ is compact, and that $\det(A) = \pm 1$ for $A \in O_n(K)$ as well as for $A \in Sp_n(K)$.

3. Now let us consider the hermitian form $\sigma : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$, $\sigma(z, w) = \bar{z}^T w$. The corresponding isometry group is the unitary group

$$U(n) \equiv \{A \in GL_n(\mathbb{C}); \bar{A}^T A = E\},$$

while

$$F : GL_n(\mathbb{C}) \rightarrow H_n, X \mapsto \bar{X}^T X - E.$$

describes $U(n) = F^{-1}(0)$ as before. Here $H_n \subset \mathbb{C}^{n,n}$ denotes the real subspace of all Hermitian matrices. The Jacobian map

$$DF(E) : \mathbb{C}^{n,n} \rightarrow H_n, X \mapsto \bar{X}^T + X$$

is again onto. Note that $U(n)$ is not a complex Lie group, since F is not holomorphic!

4. For $G \subset GL_n(K)$ let $SG \equiv G \cap SL_n(K)$. Since matrices $A \in O_n(K)$ have determinant ± 1 , $SO_n(K)$ is an open subgroup of $O_n(K)$ of index 2, while $Sp(n)$, as we hopefully shall see later on, is connected, hence not only $\det(A) = \pm 1$, but even $\det(A) = 1$ for all $A \in Sp_n(K)$. The group $SU(n) \subset GL_n(\mathbb{C})$ can be realized as follows: First note that $\det(A) \in \mathbf{S}^1$ for $A \in U(n)$. Let $W \equiv GL_n(\mathbb{C}) \setminus \det^{-1}(\mathbb{R}_{<0})$ and consider the map

$$F : W \rightarrow H_n \times \mathbb{R}, X \mapsto (\bar{X}^T X - E, \arg(\det(X)))$$

where, say, $-\pi < \arg(\cdot) < \pi$, with $SU(n) = F^{-1}(0)$ and Jacobian

$$DF(E) : \mathbb{C}^{n,n} \rightarrow H_n \times \mathbb{R}, X \mapsto (\bar{X}^T + X, \text{IM}(\text{Tr}(X))),$$

which is onto, since $(A, \lambda) \in H_n \times \mathbb{R}$ has $X = \frac{1}{2}(A + in^{-1}\lambda E)$ as an inverse image.

5. Let $V_0 \equiv \{0\} \subset V_1 \subset \dots \subset V_n = K^n$ be an increasing sequence of subspaces (a "flag"). If $V_i = Ke_1 + \dots + Ke_i$ the subgroup

$$UT_n(K) \equiv \{A \in GL_n(K); A(V_i) \subset V_i\}$$

is a K -Lie group, consisting of the invertible upper triangular matrices, indeed the underlying differentiable or complex manifold is nothing but $(K^*)^n \times K^{n(n-1)/2}$ and we may argue as in the case of $GL_n(K)$.

There is a canonical q -homomorphism

$$UT_n(K) \rightarrow GL(V_n/V_{n-1}) \times \dots \times GL(V_2/V_1) \times GL(V_1) \cong (K^*)^n,$$

its kernel $UU_n(K) \subset UT_n$ is K -Lie group, it consists of all upper triangular matrices with diagonal entries equal to 1 ("unipotent" matrices).

All our above Lie groups are closed subgroups of $GL_n(K)$.

Recall the definition of q -Lie group [6].

Definition 2.5. We define the following commutative ring [6]:

$$(F, +, \cdot, 1) \equiv \mathbb{R}[\text{Sin}_q, \text{Cos}_q, \text{Sinh}_q, \text{Cosh}_q, E_q]. \quad (1)$$

Then the q -Lie group $U_q(n)$ is defined by

$$U_q(n) \equiv \{A \in F^{(n,n)} \mid A \cdot_q A = I\}, \quad (2)$$

where the q -multiplication \cdot_q is twisted by using well-known formulas for q -trigonometric functions, and the function argument for $E_q(x)$ is multiplied by q -Ward numbers.

Definition 2.6. The q -conjugation $I_g(h) : G_{n,q} \rightarrow G_{n,q}$ is given by

$$I_g(h) : h \rightarrow g \cdot h \cdot_q \tau(g^{-1}). \quad (3)$$

Theorem 2.7. The mapping E_q is surjective for $G_{n,q} = SU_q(2)$.

Proof. According to formula (61), every element in $SU_q(2)$ is q -conjugate with an element

$$\begin{pmatrix} \text{Cos}_q(\psi) & -\text{Sin}_q(\psi) \\ \text{Sin}_q(\psi) & \text{Cos}_q(\psi) \end{pmatrix}.$$

□

Definition 2.8. A q -Lie group G_q is called compact if the matrix elements of all matrices in G_q are bounded functions and the set of matrices in G_q is closed under the two operations \cdot and \cdot_q .

Definition 2.9. A q -torus in a compact q -Lie group is a q -Lie subgroup, which is a finite tensor product of matrices of the forms

$$\begin{pmatrix} \text{Cos}_q(\psi) & -\text{Sin}_q(\psi) \\ \text{Sin}_q(\psi) & \text{Cos}_q(\psi) \end{pmatrix}, \tag{4}$$

and

$$\begin{pmatrix} E_q(i\phi_1) & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & E_q(i\phi_{n-1}) & 0 \\ 0 & 0 & 0 & E_q(i\phi_n) \end{pmatrix}, \{\phi_i\}_{i=1}^n \in [0, \xi(q, 1)]. \tag{5}$$

A maximal q -torus is a q -torus H_q such that if T_q is another q -torus with $H_q \subset T_q$, then $H_q = T_q$.

The notion of maximal q -torus plays a special role in the theory of q -Lie groups, since all maximal q -tori are conjugate.

Theorem 2.10. For a compact q -Lie group, all maximal q -tori are conjugate to each other.

Proof. The q -conjugation map (3) $I_g(h)$ is an isomorphism. Then the image of H_q under $I_g(h)$ is isomorphic to H_q , and is therefore a q -torus. Assume $\tilde{h} \in \tilde{H}_q \subset G_q$. If \tilde{H}_q were a higher dimensional q -torus containing $g \cdot h \cdot_q \tau(g^{-1})$, then $I_{g^{-1}}(\tilde{h})$ would be a higher dimensional q -torus containing H_q . This is impossible, so $I_g(h)$ must be maximal. □

Theorem 2.11. For $n \geq 1$, $U_q(n)$ acts smoothly on \mathbb{C}^n by matrix multiplication. The conjugates of the maximal q -torus T_q cover $U_q(n)$.

Proof. Similar to [1, p. 215]. Let $\vec{v}_1, \dots, \vec{v}_n$ denote an orthonormal basis of eigenvectors for $A \in U_q(n)$. Assume that B sends

$$e_j \text{ to } v_j. \tag{6}$$

Then BAB^{-1} sends e_j to v_j to $\lambda_j v_j$ to $\lambda_j e_j$. This implies that we can compute the corresponding diagonal elements of T_q from the λ_j . □

In this section we will rely on the very similar results from Curtis [2].

Definition 2.12. A q -one parameter subgroup of a q -Lie group G_q over the normed field K is a q -homomorphism $M : K \mapsto G_q$ such that

$$M(x \oplus_q y) = M(x)M(y). \tag{7}$$

Theorem 2.13. A q -analogue of [2, p.93]. If γ, σ are q -one parameter subgroups of an abelian q -Lie group G_q , then $\gamma \times \sigma$ is a q -one parameter subgroup of G_q .

Proof. We use definitions 29 and 30 in [6]. For the convenience of the reader, we briefly repeat the first one.

$$(g_{11}, g_{21}) \cdot (g_{12}, g_{22}) = (g_{11} \cdot_1 g_{12}, g_{21} \cdot_2 g_{22}), \tag{8}$$

and

$$(\mathfrak{g}_{11}, \mathfrak{g}_{21}) \cdot_q (\mathfrak{g}_{12}, \mathfrak{g}_{22}) = (\mathfrak{g}_{11} \cdot_{1:q} \mathfrak{g}_{12}, \mathfrak{g}_{21} \cdot_{2:q} \mathfrak{g}_{22}). \tag{9}$$

Because of the abelian property only the first formula comes into consideration. We find that

$$\begin{aligned} \gamma \cdot \sigma(s \oplus_q t) &\stackrel{\text{by(8)}}{=} \gamma(s \oplus_q t) \cdot \sigma(s \oplus_q t) \\ &\stackrel{\text{by}^*}{=} \gamma(s) \cdot \gamma(t) \cdot \sigma(s) \cdot \sigma(t) \\ &= \gamma(s) \cdot \sigma(s) \cdot \gamma(t) \cdot \sigma(t) \\ &= \gamma \cdot \sigma(s) \cdot \gamma \cdot \sigma(t), \end{aligned} \tag{10}$$

where we used the abelian property \cdot^* in the second step. Obviously, $\gamma \times \sigma$ is a submanifold of G_q . □

Theorem 2.14. *A q -analogue of [2, p.95]. Any compact connected abelian q -Lie group is a q -torus.*

Theorem 2.15. *Compare with [3]. A q -analogue of [2, p.97].*

$$\left(\begin{array}{cccc} E_q(i\phi_1) & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & E_q(i\phi_{n-1}) & 0 \\ 0 & 0 & 0 & E_q(i\phi_n) \end{array} \right), \{\phi_i\}_{i=1}^n \in [0, \xi(q, 1)] \tag{11}$$

is a maximal q -torus in $U_q(n)$.

Proof. The q -Lie subgroup T_q of all diagonal matrices in $U_q(n)$ is clearly isomorphic to T_q^n . It is actually a maximal q -torus, for if there is a strictly larger one, then one can find some element g in $U_q(n)$ which commutes with all elements of T_q . But T_q contains diagonal matrices with n distinct eigenvalues, and any matrix which commutes with such matrices must be diagonal, so we get a contradiction. □

Theorem 2.16.

$$\left(\begin{array}{cccc} E_q(i\phi_1) & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & E_q(i\phi_{n-1}) & 0 \\ 0 & 0 & 0 & E_q(-i(\phi_1 \oplus_q \dots \oplus_q \phi_{n-1})) \end{array} \right) \tag{12}$$

is a maximal q -torus in $SU_q(n)$.

Proof. Obviously, the above matrix has q -determinant 1. This implies that it is just the intersection of $SU_q(n)$ with the maximal q -torus given for $U_q(n)$. □

Definition 2.17. *A k - q -torus is a Cartesian product of k q -tori.*

Definition 2.18. *A maximal q -torus of a q -Lie group is a k , q -torus and which is not contained in any larger q -torus.*

Definition 2.19. [3] *The q -determinant of an $n \times n$ matrix $M_n \equiv [m_{ij}]_{i,j=0}^{n-1}$ is defined by*

$$\det_q M_n \equiv \sum_{\pi \in S_n} \text{sign} \pi m_{\pi(0)0} \tau(m_{\pi(1)1}) m_{\pi(2)2} \tau(m_{\pi(3)3}) \dots \xi(m_{\pi(n-1)n-1}), \tag{13}$$

where ξ is the identity if n is odd, and $\xi = \tau$ if n is even. In particular, the q -determinant of a 2×2 matrix is given by the formula $\det_q \alpha = a_{00} \tau(a_{11}) - a_{10} \tau(a_{01})$. The q -determinant of a tensor product of q -tori is defined as the product of the corresponding q -determinants.

Definition 2.20. Compare with [3]

$$\mathrm{SL}_q(n, K) \equiv \{A \in \mathrm{GL}_q(n, K) \mid \det_q A = 1\}. \quad (14)$$

Theorem 2.21. A q -analogue of [11, p.18]. The q -determinant function $\det_q : \mathrm{GL}_q \rightarrow K^*$ for q -Lie groups is a q -Lie group q -homomorphism. The kernel of \det_q is SL_q .

Proof. Apply the two matrix multiplications to the q -tori of the corresponding q -Lie groups. \square

2.1 The q -Lie group $\mathrm{SL}_q(2, K)$

We are going to give two different examples of realizations of $\mathrm{SL}_q(2, K)$.

Example 2.22. A q -analogue of [19, p. 57]. The q -Lie algebra of $\mathrm{SL}_q(2, K)$ has a basis which consists of the well-known matrices

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (15)$$

We can form three q -one-parameter subgroups by the mapping $X_i \rightarrow E_q(tX_i)$.

$$a_1(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, a_2(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, a_3(t) = \begin{pmatrix} E_q(t) & 0 \\ 0 & E_q(-t) \end{pmatrix}. \quad (16)$$

Example 2.23. A q -analogue of [19, p. 363]. The q -Lie algebra of $\mathrm{SL}_q(2, K)$ has a basis which consists of the matrices

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (17)$$

Similarly, we can form three q -one-parameter subgroups.

$$a'_1(\psi) = \begin{pmatrix} \mathrm{Cosh}_q(\psi) & \mathrm{Sinh}_q(\psi) \\ \mathrm{Sinh}_q(\psi) & \mathrm{Cosh}_q(\psi) \end{pmatrix}, a'_2(\psi) = \begin{pmatrix} \mathrm{Cos}_q(\psi) & -\mathrm{Sin}_q(\psi) \\ \mathrm{Sin}_q(\psi) & \mathrm{Cos}_q(\psi) \end{pmatrix}, \quad (18)$$

$$a'_3(t) = \begin{pmatrix} E_q(t) & 0 \\ 0 & E_q(-t) \end{pmatrix}.$$

In the same way as for Lie groups we can show that the E_q mapping is not surjective on the whole $\mathrm{SL}_q(2)$.

We can define a q -hyperbolic plane as (compare with \mathbb{H}_q^2)

$$\frac{\mathrm{SL}_q(2)}{\mathrm{SO}_q(2)}. \quad (19)$$

According to Iwasawa a real semisimple, non-compact Lie algebra \mathfrak{g} regarded as a vector space is a q -direct sum of subalgebras:

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad (20)$$

where \mathfrak{a} is abelian and \mathfrak{n} is nilpotent. We have the following q -analogue assuming that we have a matrix q -Lie algebra:

$$\mathfrak{g} = \mathfrak{t} \oplus_{q'} \mathfrak{a} \oplus_{q'} \mathfrak{n}. \quad (21)$$

Then we can form q -Lie groups with the function $E_q(x)$. For the Lie group case this corresponds to

Theorem 2.24. For each $A \in GL(n, \mathbb{R})$ there is a $S \in O(n)$, a diagonal matrix D with real positive diagonal elements and an upper triangular matrix U , such that $A = SDU$. This decomposition is unique.

The Iwasawa decomposition plays a key role in the representation theory. There is a corresponding q -Iwasawa decomposition for $SL_q(2)$.

Theorem 2.25. The q -Iwasawa decomposition for matrices $A_{n,q} \in SL_q(2)$ is $G_{n,q} = a'_2(\psi) \cdot a'_3(t) \cdot a_1(x)$, where the a -matrices are defined by examples 2.22 and 2.23. This decomposition is unique.

Proof. Assume that $\alpha, \gamma > 0$. Otherwise use negative ψ . A multiplication of the three matrices gives

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \text{Cos}_q(\psi)E_q(t) & x\text{Cos}_q(\psi)E_q(t) - \text{Sin}_q(\psi)E_q(-t) \\ \text{Sin}_q(\psi)E_q(t) & x\text{Sin}_q(\psi)E_q(t) + \text{Cos}_q(\psi)E_q(-t) \end{pmatrix}. \quad (22)$$

This matrix has q -determinant 1. We can compute ψ, t and x explicitly from (22) as

$$\tan_q \psi = \frac{\gamma}{\alpha}, \quad (23)$$

$$E_q(t) = \frac{\alpha}{\text{Cos}_q(\psi)}, \quad (24)$$

$$x = \frac{\beta + \text{Sin}_q(\psi)E_q(-t)}{\text{Cos}_q(\psi)E_q(t)}. \quad (25)$$

Equation (23) always has a solution, since $\frac{\gamma}{\alpha} > 0$. This proves the uniqueness. \square

2.2 Centers, q -central subgroups and coverings by maximal q -tori

We first conclude that most of the centers for maximal q -tori have almost equivalent q -analogues.

Definition 2.26. The center, $Z(G_q)$, of a q -Lie group G_q is the set of elements of G_q that commute with all other elements.

The center must lie in each maximal q -torus. More specifically, $Z(G_q) = \lambda I$, where $\lambda \in \mathbb{C}$.

Proof. By Schur's lemma, $Z(G_q)$ is a multiple of the unit matrix. \square

Theorem 2.27. $Z(U_q(n)) = \{E_q(i\vartheta)I\}$.
 $Z(SU_q(n)) = \{wI \mid w^n = 1\}$.

Proof. If $B \in Z(U_q(n))$ we infer that B must be diagonal with diagonal elements $E_q(a_i)$. Therefore

$$B = \begin{pmatrix} E_q(ia_1) & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & E_q(ia_{n-1}) & 0 \\ 0 & 0 & 0 & E_q(ia_n) \end{pmatrix}. \quad (26)$$

Put

$$A \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (27)$$

Then $AB = BA$ shows that $a_1 = a_2$ etc., so all a_i are equal. Clearly any $E_q(i\vartheta)I$ is in the center, which proves the theorem. \square

Theorem 2.28. $Z(\mathrm{SO}_q(2n+1)) = \{I\}$. $Z(\mathrm{SO}_q(2n)) = \{I, -I\}$.

Definition 2.29. A q -central subgroup H_q of a q -Lie group is defined by $H_q \subset Z(G_q) \iff g \cdot h = h \cdot g \ \forall g \in G_q, h \in H_q$.

Theorem 2.30. A q -discrete subgroup H_q of a q -Lie group G_q has the property $e \in H_q$ is an isolated point is equivalent to all $h \in H_q$ are isolated points in H_q . If G_q is connected, then every q -discrete subgroup is q -central.

Theorem 2.31. A q -analogue of [2, p. 110]. The q -Lie group $U_q(n)$ is covered by the conjugates of its maximal q -tori.

Theorem 2.32. A q -analogue of [2, p. 110]. The q -Lie group $SU_q(n)$ is covered by the conjugates of its maximal q -tori.

3 A theory for q -Lie algebras

We assume that the reader is familiar with the theory of q -Lie algebras from our previous article [6].

3.1 The universal covering q -group

A q -Lie group q -homomorphism $\varphi : G_q \rightarrow H_q$ induces a q -Lie algebra q -homomorphism φ_* . In this section we ask when for given q -Lie groups G_q, H_q and a q -Lie algebra q -homomorphism $\psi : \mathfrak{g}_q \rightarrow \mathfrak{h}_q$ we can find a q -Lie group q -homomorphism $\varphi : G_q \rightarrow H_q$ inducing ψ , i.e. such that $\psi = \varphi_*$. The strategy is as follows: If $\varphi : G_q \rightarrow H_q$ is a q -Lie group q -homomorphism, then its graph

$$\Gamma_\varphi \equiv \{(g, \varphi(g)); g \in G_q\} \subset G_q \times H_q$$

is a q -Lie subgroup of $G_q \times H_q$ with q -Lie algebra

$$\mathrm{Lie}(\Gamma_\varphi) = \{(X, \varphi_*(X)); X \in \mathfrak{g}_q\}.$$

So given $\psi : \mathfrak{g}_q \rightarrow \mathfrak{h}_q$ we look at the connected q -Lie subgroup $\Gamma_q \subset G_q \times H_q$ with

$$\mathrm{Lie}(\Gamma_q) = \{(X, \psi(X)); X \in \mathfrak{g}_q\}.$$

The inclusion followed by the projection onto the first factor

$$\pi : \Gamma_q \hookrightarrow G_q \times H_q \xrightarrow{\mathrm{pr}_{G_q}} G_q$$

has obviously bijective q -differential

$$\pi_* : \mathrm{Lie}(\Gamma_q) \rightarrow \mathfrak{g}_q.$$

Since both q -Lie groups, G_q and Γ_q are connected, it is a surjective q -homomorphism with discrete kernel. And if it is even an isomorphism, we can take $\varphi \equiv \mathrm{pr}_{H_q} \circ \pi^{-1}$. Indeed, there are q -Lie groups, where π necessarily is an isomorphism.

Now the basic idea in the study of q -differentiable q -groups is to replace the commutator map

$$K : G_q \times G_q \rightarrow G_q, (x, y) \mapsto xyx^{-1}y^{-1}$$

with the "bilinear part" of its q -Taylor expansion at (e, e) - here $e \in G_q$ denotes the neutral element of the q -group G_q . Let us explain that: Denote $DK(e, e) \in \mathbb{R}^{m, 2m}$ the q -Jacobian matrix of K at (e, e) - the linear part of the q -Taylor expansion. Then we have for small $\xi, \eta \in \mathbb{R}^m$ the expansion

$$K(e + \xi, e + \eta) = K((e, e) + (\xi, \eta))$$

$$e + DK(e, e) \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \sum_{1 \leq i, j \leq m} D_{q, x_i} D_{q, y_j} K(e, e) \xi_i \eta_j + \dots$$

Now the "bilinear term"

$$[\xi, \eta] \equiv \sum_{1 \leq i, j \leq m} D_{q, x_i} D_{q, y_j} K(e, e) \xi_i \eta_j$$

defines a bilinear map

$$[\dots, \dots] : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

It turns out that that map determines the group law near $(e, e) \in G_q \times G_q$ completely, so one can replace the local study of q -differentiable q -groups with the study of certain bilinear maps $[\dots, \dots] : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$.

Let us discuss the example $G_q = GL_n(\mathbb{R}) \subset \mathbb{R}^{n,n}$ and compute the map $[\dots, \dots] : \mathbb{R}^{n,n} \times \mathbb{R}^{n,n} \longrightarrow \mathbb{R}^{n,n}$. For a matrix $A = (\alpha_{ij})$ we define its norm by

$$\|A\| \equiv n \max\{|\alpha_{ij}|; 1 \leq i, j \leq n\}$$

and note that it is even well behaved with respect to products: $\|AB\| \leq \|A\| \cdot \|B\|$. Now denote $E \in GL_n(\mathbb{R})$ the unit matrix (replacing $e \in G_q$) and take $X, Y \in \mathbb{R}^{n,n}$ of norm < 1 (replacing ξ and η). Then we have $E + X \in GL_n(\mathbb{R})$ with

$$(E + X)^{-1} = E - X + X^2 - X^3 + \dots,$$

a convergent series. Consequently

$$\begin{aligned} K(E + X, E + Y) &= (E + X)(E + Y)(E - X + X^2 - \dots)(E - Y + Y^2 - \dots) \\ &= E + (X + Y - X - Y) + (X^2 + Y^2 + XY - X^2 - XY - YX - Y^2 + XY) + \dots \end{aligned}$$

with the dots representing terms of total degree > 2 in X and Y . Thus the linear term vanishes and

$$[X, Y] = XY - YX$$

is the commutator of the matrices $X, Y \in \mathbb{R}^{n,n}$. This implies the following equivalent definition for q -Lie algebras.

Definition 3.1. The q -Lie bracket of two elements A, B in a q -Lie algebra is defined by

$$[A, B] \equiv AB - BA. \tag{28}$$

A matrix q -Lie algebra over K satisfies the Jacobi identity:

1. (antisymmetry) $[x, x] = 0$
2. (Jacobi identity) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

Definition 3.2. A q -Lie algebra q -homomorphism is a K -linear map $\varphi_q : \mathfrak{g}_q \rightarrow \mathfrak{h}_q$, that preserves the q -Lie brackets:

$$\varphi_q : ([x, y]_{\mathfrak{g}_q} = [\varphi_q(x), \varphi_q(y)]_{\mathfrak{h}_q}). \tag{29}$$

Definition 3.3. The kernel of a q -Lie algebra q -homomorphism $\varphi_q : \mathfrak{g}_q \rightarrow \mathfrak{h}_q$ is the set $\ker \varphi_q \equiv \{x \in \mathfrak{g}_q \mid \varphi_q([x, \mathfrak{h}_q]) = 0\}$.

Let E_q denote q -exponentiation of matrices. We have the following commutative diagram, almost a q -analogue of [2, p.192]:

$$\begin{array}{ccc} \mathfrak{g}_q & \xrightarrow{\text{ad}} & \mathfrak{g}_q \\ \downarrow E_q & & \downarrow E_q \\ G_q & \xrightarrow{\text{Ad}} & G_q \end{array} \tag{30}$$

Definition 3.4. Let $X \in \mathfrak{g}_q$ and define the adjoint representation $\text{ad}_X : \mathfrak{g}_q \mapsto \mathfrak{g}_q$ by

$$\text{ad}_X(Y) \equiv [X, Y]. \quad (31)$$

Definition 3.5. The adjoint representation Ad of a q -Lie group G_q is defined by

$$\text{Ad}(E_q(x)) \equiv E_q(\text{ad}_x), \quad x \in \mathfrak{g}_q. \quad (32)$$

Definition 3.6. A representation of a q -Lie group G_q is a q -Lie group q -homomorphism $G_q \rightarrow \text{GL}(V)$ from G_q into the general linear group $\text{GL}(V)$ of a finite dimensional K -vector space V .

Theorem 3.7. The set of q -Lie groups belong to the category of smooth manifolds.

Proof. Similar to the proof for Lie groups. The morphisms are the two matrix multiplications. \square

Theorem 3.8. A q -analogue of [2, p. 192]. The center of G_q is the kernel of the adjoint representation map.

Proof. Consider a maximal q -torus of G_q . We have $\text{Ad}(E_q(x)) = E_q(\text{ad}_x)$ is the identity $\Leftrightarrow \text{ad}_x = 0 \Leftrightarrow E_q(x) \in Z(G_q)$. \square

Since we are looking for q -analogues of real and complex Lie algebras in matrix form, we can rely on the following "embedding theorem":

Theorem 3.9 (Theorem of Ado). Any finite dimensional K - q -Lie algebra \mathfrak{g}_q is isomorphic to a q -Lie subalgebra $\mathfrak{h}_q \subset \mathfrak{gl}_{n,q}(K)$ for some $n \in \mathbb{N}$.

As a consequence we see that any q -Lie algebra \mathfrak{g}_q is isomorphic to the q -Lie algebra $\text{Lie}(G_q)$ of a q -Lie group G_q .

Definition 3.10. The derived q -Lie algebra, $\text{Der}(\mathfrak{g}_q)$, is the q -subalgebra of \mathfrak{g}_q , which consists of all q -Lie brackets of pairs of elements of \mathfrak{g}_q . It is also called commutator q -Lie algebra.

Theorem 3.11. The derived q -Lie algebra is a q -ideal.

Proof. We have $[\text{Der}(\mathfrak{g}_q), \mathfrak{g}_q] \subseteq \text{Der}(\mathfrak{g}_q)$. \square

Theorem 3.12. The kernel of a q -Lie q -homomorphism is always a q -ideal.

Proof. Pick out a q -Lie q -homomorphism $\Phi : \mathfrak{g}_q \mapsto \mathfrak{h}_q$. To show that $\ker\Phi$ is a q -ideal we need to show that $[h, X] \in \ker\Phi \forall X \in \mathfrak{g}_q$ and for every $H \in \ker\Phi$. Assume that $H \in \ker\Phi$. Then $\Phi([H, X]) = [\Phi(H), \Phi(X)] = [0, \Phi(X)] = 0 \in \ker\Phi$. By surjectivity, we see that $[H, X] \in \ker\Phi$. Therefore $\ker\Phi$ is a q -ideal. \square

Definition 3.13. The two matrices $x, y \in \mathfrak{g}_q$ are said to commute when $[x, y] = 0$.

Definition 3.14. The q -ideal $Z(\mathfrak{g}_q)$ of elements of \mathfrak{g}_q which commute with everything in \mathfrak{g}_q is called the q -center of \mathfrak{g}_q .

Definition 3.15. The q -Lie algebra \mathfrak{g}_q is called abelian when $Z(\mathfrak{g}_q) = \mathfrak{g}_q$.

3.2 The classical q -Lie algebras

The classical q -Lie algebras are defined by

Definition 3.16.

$$\mathfrak{sl}_q(n, \mathbb{C}) \equiv \{A \in \mathbb{C}[\mathbb{Z}_q]^{(n,n)} \mid \text{tr}_q(A) \sim \theta\}. \tag{33}$$

$$\mathfrak{su}_q(n, \mathbb{C}) \equiv \{A \in \mathbb{C}[\mathbb{Z}_q]^{(n,n)} \mid A^* = -A^T\}. \tag{34}$$

$$\mathfrak{so}_q(n, \mathbb{C}) \equiv \{A \in \mathbb{C}[\mathbb{Z}_q]^{(n,n)} \mid A + A^T = 0\}. \tag{35}$$

$$\mathfrak{sp}_q(n, \mathbb{C}) \equiv \{A \in \mathbb{C}[\mathbb{Z}_q]^{(2n,2n)} \mid JA + A^T J = 0\}, J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{36}$$

The q -Lie algebras $\mathfrak{su}_q(n, \mathbb{C})$ and $\mathfrak{so}_q(n, \mathbb{C})$ are called skew-Hermitian and skew-symmetric, respectively.

Theorem 3.17. *The q -Lie algebras $\mathfrak{sl}_q(n, \mathbb{C})$, $n \geq 2$, $\mathfrak{so}_q(n, \mathbb{C})$ $n \geq 3$ and $\mathfrak{sp}_q(n, \mathbb{C})$, $n \geq 1$ are semisimple.*

They have no q -center except in very low dimensions.

Theorem 3.18. *The real q -Lie algebras $\mathfrak{sl}_q(n, \mathbb{R})$ and $\mathfrak{sp}_q(n, \mathbb{R})$ are all semisimple.*

Proof. A real q -Lie algebra \mathfrak{g}_q is semisimple if $\mathfrak{g}_q \otimes \mathbb{C}$ is. □

Theorem 3.19. *The (finitedimensional) q -Lie algebras together with the q -Lie homomorphisms form a category.*

Theorem 3.20. *A q -analogue of [2, p. 173]. Denote the q -Lie algebra of the q -Lie group G_q by $\mathcal{L}(G_q)$. The mapping $G_q \rightarrow \mathcal{L}(G_q)$ defines a covariant functor*

$$F : C_1 \mapsto C_2 \tag{37}$$

from the q -Lie group category C_1 to the q -Lie algebra category C_2 , which sends objects to objects and homomorphisms to homomorphisms.

Proof. Assume that α and β are q -Lie group homomorphisms. This follows from the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{F(A)} & \\ \downarrow \alpha & & \downarrow F(\alpha) \\ B & \xrightarrow{F(B)} & \\ \downarrow \beta & & \downarrow F(\beta) \\ C & \xrightarrow{F(C)} & \end{array} \quad \text{with } F(\beta \circ \alpha) = F(\beta) \circ F(\alpha) \tag{38}$$

□

Theorem 3.21. *The composition of two q -homomorphisms is a q -homomorphism.*

Proof. Use composition of mappings and morphism properties. □

3.3 solvable and nilpotent q -Lie algebras

Back to q -Lie algebras! Note first that, given a q -ideal $\mathfrak{a} \subset \mathfrak{g}_q$ of a q -Lie algebra \mathfrak{g}_q we can endow the factor vector space with a natural q -Lie bracket

$$[X \oplus_q \mathfrak{a}, Y \oplus_q \mathfrak{a}] \equiv [X, Y] \oplus_q \mathfrak{a},$$

the resulting q -Lie algebra being called the q -factor algebra $\mathfrak{g}_q/\mathfrak{a}$ of $\mathfrak{g}_q \bmod(\text{ulo})$ the q -ideal \mathfrak{a} . Furthermore, we have $\mathfrak{g}_q = \mathfrak{a}_D$ if $\dim \mathfrak{g}_q = \dim \mathfrak{a} + 1$ and $D = \text{ad}(X)$ for some $X \in \mathfrak{g}_q \setminus \mathfrak{a}$.

Definition 3.22. *The derived series of a q -Lie algebra \mathfrak{g}_q is the sequence of q -ideals*

$$\mathfrak{g}_q^{(0)}, \mathfrak{g}_q^{(1)}, \mathfrak{g}_q^{(2)}, \dots \quad (39)$$

which is defined by the recursion

$$\mathfrak{g}_q^{(0)} = \mathfrak{g}_q, \mathfrak{g}_q^{(i+1)} = [\mathfrak{g}_q^{(i)}, \mathfrak{g}_q^{(i)}]. \quad (40)$$

We have an equivalent definition.

Definition 3.23. *The q -Lie algebra \mathfrak{g}_q is called solvable if there is a finite sequence of q -Lie subalgebras*

$$\mathfrak{g}_{0,q} = 0 \subset \mathfrak{g}_{1,q} \subset \dots \subset \mathfrak{g}_{r,q} = \mathfrak{g}_q,$$

such that $\mathfrak{g}_{i,q} \subset \mathfrak{g}_{i+1,q}$ is a q -ideal of $\mathfrak{g}_{i+1,q}$ for $i < r$ with abelian q -factor algebra $\mathfrak{g}_{i+1,q}/\mathfrak{g}_{i,q}$ (or equivalently $[\mathfrak{g}_{i+1,q}, \mathfrak{g}_{i+1,q}] \subset \mathfrak{g}_{i,q}$).

Example 3.24. 1. *If \mathfrak{g}_q is solvable, then any subspace $\mathfrak{h}_q \subset \mathfrak{g}_q$ with $\mathfrak{g}_{i,q} \subset \mathfrak{h}_q \subset \mathfrak{g}_{i+1,q}$ is a q -Lie subalgebra and even a q -ideal in $\mathfrak{g}_{i+1,q}$. Hence we may refine a given strictly increasing sequence as in Def. 3.23 in such a way that finally $r = \dim \mathfrak{g}_q$ and $\dim \mathfrak{g}_{i+1,q} = \dim \mathfrak{g}_{i,q} + 1$. In particular we see, that a solvable q -Lie algebra can be constructed by a repeated application of the $\mathfrak{g}_{D,q}$ -construction for a q -Lie algebra \mathfrak{g}_q together with a derivation $D \in \text{Der}(\mathfrak{g}_q)$.*

2. *Denote $C(\mathfrak{g}_q) \equiv [\mathfrak{g}_q, \mathfrak{g}_q]$ the commutator q -Lie subalgebra of \mathfrak{g}_q . A q -Lie algebra is solvable iff the decreasing sequence of successive commutator q -Lie subalgebras $C^i(\mathfrak{g}_q)$, i.e.*

$$C^0(\mathfrak{g}_q) \equiv \mathfrak{g}_q, C^{i+1}(\mathfrak{g}_q) \equiv C(C^i(\mathfrak{g}_q))$$

terminates at the trivial q -Lie subalgebra.

3. *Let $\mathfrak{a}_q \subset \mathfrak{g}_q$ be a q -ideal. If \mathfrak{a}_q is solvable as well as $\mathfrak{g}_q/\mathfrak{a}_q$, so is \mathfrak{g}_q . In particular, if $\mathfrak{a}_q, \mathfrak{b}_q \subset \mathfrak{g}_q$ are solvable q -ideals, so is $\mathfrak{a}_q \oplus_{\mathfrak{a}_q'} \mathfrak{b}_q$. Hence there is a unique maximal solvable q -ideal in a q -Lie algebra \mathfrak{g}_q :*

Definition 3.25. *The maximal solvable q -ideal $\mathfrak{r}_q \subset \mathfrak{g}_q$ is called the radical of the q -Lie algebra \mathfrak{g}_q .*

For solvable (q)-Lie algebras we have:

Theorem 3.26 (Theorem of Lie). *Any (finite dimensional) module V over a solvable complex q -Lie algebra \mathfrak{g}_q admits a one dimensional submodule $L \subset V$. In particular an irreducible \mathfrak{g}_q -module over a solvable complex q -Lie algebra \mathfrak{g}_q has dimension $\dim V = 1$.*

Corollary 3.27. 1. *Any (finite dimensional) module V over a solvable complex q -Lie algebra \mathfrak{g}_q admits an invariant flag*

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = V$$

of submodules $V_i \subset V$ of dimension i .

2. *A solvable complex q -Lie algebra is isomorphic to a subalgebra of $\text{ut}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*

Definition 3.28. *A q -Lie algebra \mathfrak{g}_q is called nilpotent if the following decreasing sequence of q -Lie subalgebras $N^i(\mathfrak{g}_q)$ terminates at $\{0\}$:*

$$N^0(\mathfrak{g}_q) \equiv \mathfrak{g}_q, N^{i+1}(\mathfrak{g}_q) = [\mathfrak{g}_q, N^i(\mathfrak{g}_q)].$$

Remark 2. *We have $N(\mathfrak{g}_q) = C(\mathfrak{g}_q)$ and $C^i(\mathfrak{g}_q) \subset N^i(\mathfrak{g}_q)$ for $i > 1$, hence a nilpotent algebra is solvable.*

- Example 3.29.** 1. The q -Lie algebra $u_{n,q}(K)$ consisting of all upper triangular matrices with zeros on the diagonal is nilpotent.
2. The q -Lie algebra $\mathfrak{g}_q = KX \oplus_{q'} KZ$ with $[X, Z] = Z$ is solvable, but not nilpotent: We have $N^i(\mathfrak{g}_q) = KZ$ for all $i \in \mathbb{N}_{>0}$.
3. Abelian q -Lie algebras are nilpotent.
4. q -Lie subalgebras and q -factor algebras of nilpotent q -Lie algebras are nilpotent.

Let $B_{n,q}(K)$ denote the upper triangular matrices with 1:s along the diagonal. Then $u_{n,q}(K)$ is the q -Lie algebra of $B_{n,q}(K)$. $B_{n,q}$ is not semisimple since the subset with 0 everywhere except for the upper left hand corner is a q -ideal. There is a common pattern in the following exposition. Many of the functions, which exist in q -(Lie) group theory reappear with the same name in q -Lie algebra theory. The reason is of course that these functions mean the same thing in either category. As the following exposition shows these terms can also be used in the case of q -Lie algebras (and q -Lie groups). In a nilpotent q -Lie algebra, \mathfrak{g}_q , the Baker-Campbell-

Hausdorff series

$$C(X, Y) = \sum_{i=1}^{\infty} C_i(X, Y) = (X \oplus_{q'} Y) + \frac{1}{2}[X, Y] + C_3(X, Y) + \dots$$

is finite and hence defines a polynomial map

$$C : \mathfrak{g}_q \times \mathfrak{g}_q \longrightarrow \mathfrak{g}_q.$$

Indeed, it provides \mathfrak{g}_q with a group law: Consider the q -exponential map $E_q : \mathfrak{g}_q \longrightarrow G_q$ for the simply connected q -Lie group with $\text{Lie}(G_q) \cong \mathfrak{g}_q$. Then, E_q being diffeomorphic near $0 \in \mathfrak{g}_q$, the conditions for a group law are satisfied near the origin and hence everywhere by the identity theorem for polynomial maps (A map $V \times V \longrightarrow K$ for a K -vector space V is called polynomial if it is a K -linear combination of products of linear forms in one of both variables. A map $V \times V \longrightarrow V$ is polynomial, if the composition with any linear functional $V \longrightarrow K$ is polynomial.). Indeed the q -exponential map $E_q : \mathfrak{g}_q \longrightarrow G_q$ turns out to be an isomorphism of q -Lie groups. So we can replace G_q with $(\mathfrak{g}_q, C(\cdot, \cdot))$, the expression for C in terms of Lie monomials being independent from the nilpotent q -Lie algebra \mathfrak{g}_q . Note that the n -th power of $X \in \mathfrak{g}_q$ for $n \in \mathbb{Z}$ with respect to $C(\cdot, \cdot)$ is nX .

Example 3.30. The q -exponential map $E_q : u_{n,q}(K) \longrightarrow UU_n(K)$ is polynomial:

$$E_q(X) = \sum_{i=0}^{n-1} \frac{X^i}{\{i\}_q!}.$$

In order to understand all q -Lie groups with nilpotent q -Lie algebra we have to consider factor groups of $(\mathfrak{g}_q, C(\cdot, \cdot)) \bmod q$ -central discrete q -Lie subgroups $D_q \subset (\mathfrak{g}_q, C(\cdot, \cdot))$. We claim that the center of the q -Lie group $(\mathfrak{g}_q, C(\cdot, \cdot))$ is the q -center $\ker(\text{ad})$ of the q -Lie algebra \mathfrak{g}_q : Since $C(X, 0) = 0 = C(0, Y)$, any of the q -Lie bracket monomials in $C(X, Y)$ contains both X and Y as factor, hence $C(Z, X) = Z \oplus_{q'} X = X \oplus_{q'} Z = C(X, Z)$ for $Z \in \ker(\text{ad})$ and any $X \in \mathfrak{g}_q$. On the other hand, for a q -central element Z of the q -Lie group $(\mathfrak{g}_q, C(\cdot, \cdot))$, all its integral powers nZ are q -central as well; hence $C(nZ, mX) = C(mX, nZ)$ for all $n, m \in \mathbb{Z}$. Both expressions being polynomials in $n, m \in \mathbb{Z}$, comparison of the bilinear term yields $[Z, X] = [X, Z]$ resp. $[Z, X] = 0$. So normal discrete q -Lie subgroups are exactly the lattices in the subspace $\ker(\text{ad}) \subset \mathfrak{g}_q$. Note furthermore that the connected q -Lie subgroups of $(\mathfrak{g}_q, C(\cdot, \cdot))$ are exactly the q -Lie subalgebras $\mathfrak{h}_q \subset \mathfrak{g}_q$ (the q -exponential map being the identity on \mathfrak{g}_q and subspaces being maximal connected submanifolds).

Theorem 3.31. 1. A q -Lie algebra \mathfrak{g}_q is solvable iff its commutator q -Lie algebra $C(\mathfrak{g}_q)$ is nilpotent.

2. A q -Lie algebra is nilpotent iff $\text{ad}(X) \in \mathfrak{g}_q^l(\mathfrak{g}_q)$ is nilpotent for every $X \in \mathfrak{g}_q$.

3. Any nilpotent q -Lie algebra is isomorphic to a q -Lie subalgebra of $u_{n,q}(K)$ for some $n \in \mathbb{N}$.

Example 3.32. *A non-linear q -Lie group: Consider $G_q = (\mathfrak{g}_q, C(\cdot, \cdot))/D$ with $D = \mathbb{Z} \cdot [X, Y]$ with a q -central element $[X, Y]$. Assume $\varphi : G_q/D \xrightarrow{\cong} H_q \subset \mathrm{GL}_q(n, K)$ is an isomorphism of q -Lie groups. Then, \mathfrak{g}_q being solvable we may assume $\mathfrak{h}_q \subset \mathrm{ut}_n(K)$, see Theorem 3.26 resp. $H_q \subset \mathrm{UT}_n(K)$, see Cor. 3.27. Consider the element $Z \equiv 2^{-1}[X, Y]$. Since $C(\mathrm{ut}_n(K)) \subset \mathfrak{u}_{n,q}(K)$ we find $\varphi_*(Z) \in \mathfrak{u}_{n,q}(K)$, hence $\varphi(Z) = E_q(\varphi_*(Z)) \in \mathrm{UU}_n(K)$ (note that G_q is here identified with its q -Lie algebra), but $\mathrm{UU}_n(K)$ contains no non-trivial elements of finite order!*

Now let us consider semisimple algebras and modules:

Corollary 3.33. *A semisimple \mathfrak{g}_q -module V is the q -direct sum of irreducible \mathfrak{g}_q -modules*

$$V = \bigoplus_{i=1, q'}^s V_i,$$

the factors V_1, \dots, V_s being unique up to isomorphy and order.

But note that for $v \in V$ the subspace $\mathfrak{g}_q v \equiv \{Xv; X \in \mathfrak{g}_q\}$ in general is not a (\mathfrak{g}_q) -submodule, so the irreducible factors are not necessarily q -factor algebras of \mathfrak{g}_q (e.g. $X(Yv)$ need not belong to $\mathfrak{g}_q v$). On the other hand $V \equiv \mathfrak{g}_q$ is a \mathfrak{g}_q -module with the q -Lie bracket as "scalar multiplication" (corresponding to the adjoint representation $\mathfrak{g}_q \rightarrow \mathfrak{gl}_q(\mathfrak{g}_q), X \mapsto \mathrm{ad}(X)$). Then the irreducible factors are q -ideals of \mathfrak{g}_q . Calling a q -Lie algebra \mathfrak{g}_q simple if it is semisimple and admits no nontrivial q -ideals we obtain:

Theorem 3.34. *A semisimple q -Lie algebra \mathfrak{g}_q is the q -direct sum of simple q -Lie algebras*

$$\mathfrak{g}_q = \bigoplus_{i=1, q'}^s \mathfrak{g}_{i,q},$$

the factors $\mathfrak{g}_{1,q}, \dots, \mathfrak{g}_{s,q}$ being unique up to isomorphy and order.

q -Ideals of semisimple algebras are direct factors and thus semisimple as well. As a consequence, no non-trivial solvable algebra is semisimple, since otherwise we would find that the one-dimensional q -Lie algebra K is semisimple. Furthermore, a semisimple algebra has trivial radical. Indeed the reverse implication holds as well:

Theorem 3.35 (Theorem of Weyl). *A q -Lie algebra \mathfrak{g}_q with trivial radical is semisimple.*

Example 3.36. *Complex semisimple q -Lie algebras: The first point applies only to real q -Lie algebras, since there are no compact simply connected q -Lie groups except the trivial group. (For $q = 1$, the only connected compact complex Lie groups are the tori $G = \mathbb{C}^m/\Lambda$ with a lattice $\Lambda \cong \mathbb{Z}^{2m}$ of maximal rank.) But we can weaken our assumption: It is sufficient that $\mathfrak{g}_q = \mathfrak{k} \oplus_{q'} i\mathfrak{k}$ with a real q -Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}_q$ belonging to a simply connected compact real q -Lie group K_q (not to be confused with the base field). Given now a \mathfrak{g}_q -submodule $U \subset V$, its orthogonal complement with respect to a K -invariant inner product on V is a K -invariant complex vector subspace, hence also \mathfrak{k} and $\mathfrak{g}_q = \mathfrak{k} \oplus_{q'} i\mathfrak{k}$ -invariant. As example take*

$$\mathfrak{g}_q = \mathfrak{sl}_{n,q}(\mathbb{C}) = \mathfrak{su}_{n,q} \oplus_{q'} i\mathfrak{su}_{n,q}.$$

or

$$\mathfrak{g}_q = \mathfrak{so}_q(\times, \mathbb{C}) = \mathfrak{so}_{n,q}(\mathbb{R}) \oplus_{q'} i\mathfrak{so}_{n,q}(\mathbb{R}).$$

Again, any complex semisimple q -Lie algebra is obtained in that way.

In the remaining part of this section we explain the classification of complex simple q -Lie algebras.

Definition 3.37. *A q -Lie subalgebra $\mathfrak{h}_q \subset \mathfrak{g}_q$ of a q -Lie algebra \mathfrak{g}_q is called a q -Cartan subalgebra (CSA), if it is nilpotent and satisfies*

$$[X, \mathfrak{h}_q] \subset \mathfrak{h}_q \implies X \in \mathfrak{h}_q, \quad \forall X \in \mathfrak{g}_q.$$

This can also be expressed in the following two definitions:

Definition 3.38. Let \mathfrak{h}_q be a q -Lie subalgebra of a q -Lie algebra \mathfrak{g}_q . Then $N_{\mathfrak{g}_q}(\mathfrak{h}_q) \equiv \{a \in \mathfrak{h}_q \mid [a; \mathfrak{h}_q] \subset \mathfrak{h}_q\}$ is a q -Lie subalgebra of \mathfrak{g}_q , called the q -normalizer of \mathfrak{h}_q .

Definition 3.39. A q -Cartan subalgebra of a q -Lie algebra \mathfrak{g}_q is a q -Lie subalgebra \mathfrak{h}_q , which satisfies the following two conditions:

1. \mathfrak{h}_q is a nilpotent q -Lie algebra
2. $N_{\mathfrak{g}_q}(\mathfrak{h}_q) = \mathfrak{h}_q$

Example 3.40. For $\mathfrak{g}_q = \mathfrak{sl}_{n,q}(\mathbb{C})$ the q -Lie subalgebra $\mathfrak{h}_q \equiv \mathfrak{sd}_q n(\mathbb{C})$ consisting of all diagonal matrices in $\mathfrak{sl}_q(\times, \mathbb{C})$ is a q -Cartan subalgebra.

Definition 3.41. Let \mathfrak{g}_q be a q -Lie algebra. The mapping $\sigma : \mathfrak{g}_q \mapsto \mathfrak{g}_q$ that preserves the algebraic operations on \mathfrak{g}_q is called a σ -automorphism of \mathfrak{g}_q .

Theorem 3.42. In a complex q -Lie algebra \mathfrak{g}_q any two CSA $\mathfrak{h}_q, \mathfrak{h}'_q \subset \mathfrak{g}_q$ are conjugate under an automorphism $f = E_q(\text{ad}(X))$ for some $X \in \mathfrak{g}_q$, i.e. $\mathfrak{h}'_q = f(\mathfrak{h}_q)$.

We first give an alternative characterization of a CSA.

Theorem 3.43. \mathfrak{h}_q is a CSA if and only if $\mathfrak{h}_q = \mathfrak{g}_{0,q}\text{ad}(Z)$, where $\mathfrak{g}_{0,q}\text{ad}(Z)$ contains no proper subalgebra of the form $\mathfrak{g}_{0,q}\text{ad}(X)$.

Proof. Suppose $\mathfrak{h}_q = \mathfrak{g}_{0,q}\text{ad}(Z)$ which is minimal in the sense of the proposition. Then we know that \mathfrak{h}_q is its own q -normalizer. Also, $\mathfrak{h}_q \subset \mathfrak{g}_{0,q}, \text{ad}(X) \forall X \in \mathfrak{h}_q$. Hence $\text{ad}(X)$ acts nilpotently on \mathfrak{h}_q for all $X \in \mathfrak{h}_q$. Hence, by Engel's theorem, \mathfrak{h}_q is nilpotent and hence is a CSA. Suppose that \mathfrak{h}_q is a CSA. Since \mathfrak{h}_q is nilpotent, we have $\mathfrak{h}_q \subset \mathfrak{g}_{0,q}\text{ad}(X), \forall X \in \mathfrak{h}_q$. Choose a minimal Z . Then,

$$\mathfrak{g}_{0,q}\text{ad}(Z) \subset \mathfrak{g}_{0,q}\text{ad}(X), \forall X \in \mathfrak{h}_q. \quad (41)$$

Thus \mathfrak{h}_q acts nilpotently on $\mathfrak{g}_{0,q}\text{ad}(Z)/\mathfrak{h}_q$. If this space were not zero, we could find a non-zero common eigenvector with eigenvalue zero by Engel's theorem. This means that there is a $Y \in \mathfrak{h}_q$ with $[y, \mathfrak{h}_q] \subset \mathfrak{h}_q$ contradicting the fact \mathfrak{h}_q is its own q -normalizer. \square

Lemma 3.44. If $\Phi : \mathfrak{g}_q \mapsto \mathfrak{g}'_q$ is a surjective q -homomorphism and \mathfrak{h}_q is a CSA of \mathfrak{g}_q then $\Phi(\mathfrak{h}_q)$ is a CSA of \mathfrak{g}'_q .

Proof. Clearly $\Phi(\mathfrak{h}_q)$ is nilpotent. Let $k = \text{Ker}\Phi$ and identify $\mathfrak{g} = \mathfrak{g}/k$ so $\Phi(\mathfrak{h}_q) = \mathfrak{h}_q \oplus_{\mathfrak{q}'_q} \mathfrak{k}_q$. If $X \oplus_{\mathfrak{q}'_q} \mathfrak{k}_q$ normalizes $\mathfrak{h}_q \oplus_{\mathfrak{q}'_q} \mathfrak{k}_q$ then X normalizes $\mathfrak{h}_q \oplus_{\mathfrak{q}'_q} \mathfrak{k}_q$. But $\mathfrak{h}_q = \mathfrak{g}_{0,q}(\text{ad}Z)$ for some minimal such Z , and as an algebra containing a $\mathfrak{g}_{0,q}(\text{ad}z)$, $\mathfrak{h}_q \oplus_{\mathfrak{q}'_q} \mathfrak{k}_q$ is self-normalizing. So $X \in \mathfrak{h}_q \oplus_{\mathfrak{q}'_q} \mathfrak{k}_q$. \square

Lemma 3.45. Let $\Phi : \mathfrak{g}_q \mapsto \mathfrak{g}'_q$ be surjective, as above, and \mathfrak{h}_q a CSA of \mathfrak{g}_q . Any CSA \mathfrak{h}_q of $\mathfrak{m}_q \equiv \Phi^{-1}(\mathfrak{h}'_q)$ is a CSA of \mathfrak{g}_q .

Proof. \mathfrak{h}_q is nilpotent by assumption. We must show it is its own q -normalizer in \mathfrak{g}_q . By the preceding lemma, $\Phi(\mathfrak{h}_q)$ is a Cartan subalgebra of \mathfrak{h}_q . But $\Phi(\mathfrak{h}_q)$ is nilpotent and hence would have a common eigenvector with eigenvalue zero in $\mathfrak{h}/\Pi(\mathfrak{h})$, contradicting the selfnormalizing property of $\Phi(\mathfrak{h}_q)$ unless $\Phi(\mathfrak{h}_q) = \mathfrak{h}_q$. So $\Phi(\mathfrak{h}_q) = \mathfrak{h}_q$. If $x \in \mathfrak{g}_q$ normalizes \mathfrak{h}_q , then $\Phi(x)$ normalizes \mathfrak{h}_q . Hence $\Phi(x) \in \mathfrak{h}'_q$ so $x \in \mathfrak{m}_q$ so $x \in \mathfrak{h}_q$. \square

Definition 3.46. The q -Lie subalgebra \mathfrak{h}_q of \mathfrak{g}_q is a real form of \mathfrak{g}_q if there exists a \mathbb{C} -linear isomorphism $\phi : \mathfrak{h}_q^{\mathbb{C}} \mapsto \mathfrak{g}_q$ such that $\phi|_{\mathfrak{h}_q} = \text{I}$, where $\mathfrak{h}_q^{\mathbb{C}}$ denotes the complexification of \mathfrak{h}_q .

Definition 3.47. Let \mathfrak{h}_q be a real q -Lie algebra and let $+_q$ denote a twisted q -addition. Its complexification $\mathfrak{h}_q^{\mathbb{C}}$ is the complexification of \mathfrak{h}_q as a vector space together with the q -Lie bracket

$$[x +_q iy, z +_q iw] \equiv [x, z] - [y, w] + i([y, z][x, w]), \quad x, y, z, w \in \mathfrak{h}_q. \quad (42)$$

Each complex semisimple q -Lie algebra has a compact real form.

For a semisimple q -algebra we have

Theorem 3.48. Let \mathfrak{g}_q be a complex semisimple q -Lie algebra. Then a q -Lie subalgebra $\mathfrak{h}_q \subset \mathfrak{g}_q$ is a CSA if and only if the following conditions are satisfied:

1. The q -Lie subalgebra $\mathfrak{h}_q \subset \mathfrak{g}_q$ is a maximal abelian q -Lie subalgebra.
2. All homomorphisms $\text{ad}(H_q) : \mathfrak{g}_q \rightarrow \mathfrak{g}_q$ are diagonalizable.

Since \mathfrak{h}_q is abelian, the endomorphisms $\text{ad}(H_q)$ for $H_q \in \mathfrak{h}_q$ commute one with the other, hence can be diagonalized simultaneously, and satisfy $\text{ad}(H_q)|_{\mathfrak{h}_q} = 0$. So defining $\mathfrak{g}_{\alpha, q} \subset \mathfrak{g}_q$ for a linear form $\alpha_q : \mathfrak{h}_q \rightarrow \mathbb{C}$ by

$$\mathfrak{g}_{\alpha, q} \equiv \{X \in \mathfrak{g}_q; \text{ad}(H_q)(X) = \alpha_q(H_q)X, \forall H_q \in \mathfrak{h}_q\}$$

we can write the q -direct sum

$$\mathfrak{g}_q = \mathfrak{h}_q \oplus_{q'} \bigoplus_{\alpha_q \in \Phi_q} \mathfrak{g}_{\alpha, q}$$

with the following finite subset $\Phi_q \subset \mathfrak{h}_q^*$:

$$\Phi_q \equiv \{\alpha_q \in \mathfrak{h}_q^* \setminus \{0\}; \mathfrak{g}_{\alpha, q} \neq \{0\}\}.$$

The case $\alpha_q = 0$ does not occur, \mathfrak{h}_q being a maximal abelian q -Lie subalgebra.

Example 3.49. For $\mathfrak{g}_q = \mathfrak{sl}_q(n, \mathbb{C})$, $\mathfrak{h}_q = \mathfrak{sd}_q(n, \mathbb{C}) \equiv$ the diagonal matrices with q -trace 0 we have

$$\Phi_q = \{\alpha_{ij, q} \equiv \beta_{i, q} - \beta_{j, q}; 1 \leq i, j \leq n, i \neq j\}$$

where

$$\beta_i : \mathfrak{sd}_q(\times, \mathbb{C}) \rightarrow \mathbb{C}, D = (z_k \delta_{k\ell}) \mapsto z_i$$

with

$$\mathfrak{g}_{ij} \equiv \mathfrak{g}_{\alpha_{ij, q}} = \mathbb{C}E_{ij}$$

with the matrix $E_{ij} \equiv (\varepsilon_{k\ell} \equiv \delta_{ki} \delta_{\ell j})$.

Since $\text{ad}(H_q) \in \text{Der}(\mathfrak{g}_q)$ we find

$$[\mathfrak{g}_{\alpha, q}, \mathfrak{g}_{\beta, q}] \subset \mathfrak{g}_{\alpha+\beta, q},$$

indeed

$$[\mathfrak{g}_{\alpha, q}, \mathfrak{g}_{\beta, q}] = \mathfrak{g}_{\alpha+\beta, q}.$$

Furthermore one can show

$$\dim \mathfrak{g}_{\alpha, q} = 1, \quad \forall \alpha_q \in \Phi_q$$

and that

$$\Phi_q \cap \mathbb{C}\alpha_q = \{\pm\alpha_q\}, \quad \forall \alpha_q \in \Phi_q.$$

The set Φ_q spans a real subspace

$$\mathfrak{h}_{\mathbb{R}, q}^* \equiv \sum_{\alpha_q \in \Phi_q} \mathbb{R}\alpha_q \subset \mathfrak{h}_q^*$$

with $\mathfrak{h}_q^* = \mathfrak{h}_{\mathbb{R}, q}^* \oplus_{q, i} \mathfrak{ih}_{\mathbb{R}, q}^*$. For a more detailed description of Φ_q we need a natural inner product on $\mathfrak{h}_{\mathbb{R}, q}^*$.

3.4 q -Cartan Killing form

First of all on a q -Lie algebra \mathfrak{g}_q one defines:

Definition 3.50. Let \mathfrak{g}_q be a q -Lie algebra. The q -Cartan Killing form is the following bilinear symmetric form

$$\langle \cdot, \cdot \rangle : \mathfrak{g}_q \times \mathfrak{g}_q \longrightarrow K, (X, Y) \mapsto \langle X, Y \rangle \equiv \text{tr}_q(\text{ad}(X)\text{ad}(Y)).$$

Note that we changed trace to q -trace in the definition.

Let us mention:

Theorem 3.51. Let \mathfrak{g}_q be a q -Lie algebra with q -Cartan Killing form $\langle \cdot, \cdot \rangle : \mathfrak{g}_q \times \mathfrak{g}_q \longrightarrow K$. Then \mathfrak{g}_q is

1. solvable iff $\langle \mathfrak{g}_q, \mathfrak{C}(\mathfrak{g}_q) \rangle = \{0\}$ and
2. semisimple iff its q -Cartan Killing form is nondegenerate.

This is Cartan's criterion. Moreover in the latter case its restriction to $\mathfrak{h}_q \times \mathfrak{h}_q$ with a CSA $\mathfrak{h}_q \subset \mathfrak{g}_q$ is nondegenerate as well, and its dual form, also denoted

$$\langle \cdot, \cdot \rangle : \mathfrak{h}_q^* \times \mathfrak{h}_q^* \longrightarrow \mathbb{C},$$

is real valued on $\mathfrak{h}_{\mathbb{R},q}^* \times \mathfrak{h}_{\mathbb{R},q}^*$ and even positive definite.

Remark 3. Define $H_{\alpha,q} \in \mathfrak{h}_q$ by $\langle H_{\alpha,q}, H \rangle = \alpha_q(H)$. Then

$$[X, Y] = \langle X, Y \rangle H_{\alpha,q} \neq 0$$

for $X \in \mathfrak{g}_{\alpha,q} \setminus \{0\}$, $Y \in \mathfrak{g}_{-\alpha,q} \setminus \{0\}$.

In particular we see that

$$\mathbb{C}H_{\alpha,q} \oplus_{q'} \mathfrak{g}_{\alpha,q} \oplus_{q'} \mathfrak{g}_{-\alpha,q} \cong \mathfrak{sl}_{2,q}(\mathbb{C}).$$

3.5 q -root systems

The classification of complex semisimple q -Lie algebras now depends on a better understanding of the set $\Phi_q \subset \mathfrak{h}_{\mathbb{R},q}^*$. Indeed it has a remarkable symmetry property: It forms a q -root system:

Definition 3.52. A finite subset $\Phi_q \subset V \setminus \{0\}$ of a finite dimensional euclidean vector space V with inner product $\langle \cdot, \cdot \rangle$ is called a q -root system if the following conditions are satisfied:

1. $V = \text{span}(\Phi_q)$.
2. $\Phi_q \cap \mathbb{R}\alpha_q = \{\pm\alpha_q\}$ for all $\alpha_q \in \Phi_q$.
3. For any q -root (i.e. element) $\alpha_q \in \Phi_q$ the reflection

$$s_\alpha : V \longrightarrow V, v \mapsto v - \frac{2\langle v, \alpha_q \rangle}{\langle \alpha_q, \alpha_q \rangle} \alpha_q \quad (43)$$

on the hyperplane $\alpha_q^\perp \subset V$ leaves Φ_q invariant:

$$s_\alpha(\Phi_q) = \Phi_q, \forall \alpha_q \in \Phi_q.$$

4. For all $\beta_q, \alpha_q \in \Phi_q$ we have

$$\chi(\beta_q, \alpha_q) \equiv \frac{2\langle \beta_q, \alpha_q \rangle}{\langle \alpha_q, \alpha_q \rangle} \in \mathbb{Z}.$$

The fourth condition is a very strong condition on the possible angles between the roots: Denote $\vartheta \in [0, \pi]$ the angle between the non proportional q -roots α_q, β_q . Then

$$\chi(\beta_q, \alpha_q)\chi(\alpha_q, \beta_q) = 4 \cos^2(\vartheta) \in \mathbb{Z},$$

hence $\cos(\vartheta) \in \{0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}\}$ resp. $\vartheta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$.

Furthermore $\chi(\beta_q, \alpha_q) = \pm 1, \pm 2, \pm 3$, if α_q, β_q are neither proportional nor orthogonal. A more detailed analysis of that situation shows that in case $\|\beta_q\| \geq \|\alpha_q\|$ we have

$$\|\beta_q\|^2 = 4 \cos^2(\vartheta) \cdot \|\alpha_q\|^2 = \chi(\beta_q, \alpha_q)\chi(\alpha_q, \beta_q)\|\alpha_q\|^2.$$

Indeed, that means nothing but $\beta_q = \pm(\alpha_q - s_\beta(\alpha_q))$, the sign depending on whether $\vartheta < \frac{\pi}{2}$ or $\vartheta > \frac{\pi}{2}$.

The proof that the above conditions hold for $\Phi_q \subset \mathfrak{h}_q^*$ relies on the knowledge of the irreducible $\mathfrak{sl}_{2,q}(\mathbb{C})$ -modules. Indeed, given nonproportional α_q, β_q , the subspace

$$\bigoplus_{k \in \mathbb{Z}, 'q'} \mathfrak{g}_{\beta_q + k\alpha_q}$$

is a module over the q -Lie subalgebra

$$\mathbb{C}H_{\alpha,q} \oplus_{'q'} \mathfrak{g}_{\alpha,q} \oplus_{'q'} \mathfrak{g}_{-\alpha,q} \cong \mathfrak{sl}_{2,q}(\mathbb{C}).$$

3.6 Weyl groups and Weyl chambers

Now let us concentrate on abstract q -root systems:

Definition 3.53. Let $\Phi_q \subset V$ be a q -root system, and denote by $O_q(V)$ the group of linear q -isometries of the euclidean space V . The Weyl group $W(\Phi_q) \subset O(V)$ is defined as the subgroup of $O(V)$ generated by the reflections $s_\alpha, \alpha_q \in \Phi_q$.

The symmetries of a q -root system Φ_q given by the action of the Weyl group W make it possible to compress the information contained in it in a basis B :

Definition 3.54. A subset $B \subset \Phi_q$ is called a basis of the q -root system $\Phi_q \subset V$ if B is a basis of the vector space V and every $\beta_q \in \Phi_q$ is an integral linear combination

$$\beta_q = \sum_{\alpha_q \in B} k_\alpha \cdot \alpha_q,$$

where the coefficients $k_\alpha \in \mathbb{Z}$ satisfy either $k_\alpha \geq 0$ for all $\alpha_q \in B$ or $k_\alpha \leq 0$ for all $\alpha_q \in B$.

A basis of a q -root system $\Phi_q \subset V$ gives rise to a decomposition

$$\Phi_q = \Phi_q^+ + \Phi_q^-, \quad (44)$$

with $\Phi_1^+ \equiv \Phi_1 \cap (\sum_{\alpha_1 \in B} \mathbb{N}_{\geq 0} \alpha_1)$. On the other hand, starting with certain decompositions we get all the bases of a q -root system Φ_q :

Theorem 3.55. Let $\Phi_q \subset V$ be a q -root system and $H_q \subset V$ a hyperplane with $H_q \cap \Phi_q = \emptyset$. Given a connected component V_0 of $V \setminus H$ the indecomposable elements in $\Phi_q \cap V_0$, i.e. those which can not be written as a sum $\beta_{1,q} + \beta_{2,q}$ with $\beta_{1,q}, \beta_{2,q} \in \Phi_q \cap V_0$, constitute a basis of the q -root system Φ_q . Indeed, any basis of Φ_q is obtained in that way. In particular any root can be realized as an element of a suitable basis $B \subset \Phi_q$. Furthermore the angle between two base vectors is obtuse, i.e. $\in [\frac{\pi}{2}, \pi)$.

Theorem 3.56. Let $\Phi_q \subset V$ be a q -root system.

1. The action of the Weyl group W on the set of bases of Φ_q is simply transitive.
2. Given a basis B , the reflections $\sigma_\alpha, \alpha_q \in B$, generate W .

As a consequence we see that $|W| < \infty$. Furthermore that given a basis B we can recover the Weyl group W as well as W_B .

A q -root system is called reduced if $\alpha_q, \lambda\alpha_q \in \Phi_q$ implies that $\lambda = \pm 1$. If Φ_q is a q -root system in V , the q -coroot α_q^\vee of a q -root $\alpha_q \in \Phi_q$ is defined by

$$\alpha_q^\vee \equiv \frac{2}{(\alpha_q, \alpha_q)} \alpha_q. \quad (45)$$

The set of q -coroots also forms a q -root-system Φ_q^\vee in V , called the dual q -root system.

An element of Φ_q^+ is called a simple q -root if it cannot be written as the sum of two elements of Φ_q^+ , and the corresponding s_α defined by (43) is called a simple reflection.

Let us explain a little bit more in detail the different bases a root system Φ_q admits: Writing a hyperplane as $H = P_\gamma \equiv \gamma^\perp$ for $\gamma \in V$ we see that it is a separating hyperplane for Φ_q , i.e., $P_\gamma \cap \Phi_q = \emptyset$, iff $\gamma \notin V \setminus \bigcup_{\alpha_q \in \Phi_q} P_{\alpha_q}$.

Definition 3.57. Let $\Phi_q \subset V$ be a q -root system. The connected components of $V \setminus \bigcup_{\alpha_q \in \Phi_q} P_{\alpha_q}$ are called Weyl chambers. An element $\gamma \in V$ is called regular if $\gamma \notin V \setminus \bigcup_{\alpha_q \in \Phi_q} P_{\alpha_q}$. For such an element γ denote $Ch(\gamma)$ the Weyl chamber containing γ .

Theorem 3.58. For a regular element γ denote $B_\gamma \subset \Phi_q$ the unique basis of Φ_q with $\langle \gamma, B_\gamma \rangle > 0$. Then $B_\delta = B_\gamma$ for all $\delta \in Ch(\gamma)$ and $Ch(\gamma) \mapsto B_\gamma$ is a bijection between the set of Weyl chambers of Φ_q and the set of bases of Φ_q .

3.7 Dynkin diagram, table of all simply connected complex simple q -Lie groups

Now the information contained in a basis $B \subset \Phi_q$ can be encoded in a so called *Dynkin diagram*, a graph whose vertices are the base roots $\alpha_q \in B$. Two vertices α_q, β_q are connected by $\chi(\beta_q, \alpha_q)\chi(\alpha_q, \beta_q) = 4 \cos^2(\vartheta)$ edges, i.e. by one edge, if the angle $\vartheta \in [0, \pi)$ equals $\frac{2\pi}{3}$, by two edges, if $\vartheta = \frac{3\pi}{4}$ and by three edges if $\vartheta = \frac{5\pi}{6}$. In the last two cases the two or three edges are even oriented, the arrow pointing from the longer root to the smaller one. Note that the diagram does not depend on the choice of the basis B .

Let us come back to q -Lie algebras: A q -Lie algebra can be reconstructed - up to isomorphy - from its q -root system, and a q -root system from one of its bases resp. - again up to isomorphy - from its Dynkin diagram. First of all, it is connected if and only if the corresponding algebra is simple, and there is a complete classification of the connected Dynkin diagrams. Here it is, the index counting the number of vertices:

1. $A_\ell, \ell \geq 1$: A linear string with ℓ vertices and only simple edges.
2. $B_\ell, \ell \geq 2$: A linear string with ℓ vertices with $\ell - 2$ simple edges and one double edge at one of its ends, the arrow pointing to the end point (for $\ell \geq 3$).
3. $C_\ell, \ell \geq 3$: A linear string with ℓ vertices with $\ell - 2$ simple edges and one double edge at one of its ends, the arrow pointing to the inner point. Note that $C_2 = B_2$.
4. $D_\ell, \ell \geq 4$: A string with $\ell - 2$ vertices, with the two remaining vertices connected to the same end point of the string. All edges are simple. Note that $D_3 = A_3$.
5. $V_\ell, \ell = 6, 7, 8$: A string with $\ell - 1$ vertices, the ℓ -th vertex being connected to the third vertex of the string. All edges are simple.
6. F_4 : A linear string with 4 vertices, the outer edges being simple, the inner one being a double edge (the orientation is not important, since both choices give isomorphic Dynkin diagrams).
7. G_2 : Two vertices connected by three edges.

Indeed, for every of the above Dynkin diagrams, there is a complex simple q -Lie algebra realizing it.

Now let us look at complex semisimple q -Lie groups, i.e. complex connected q -Lie groups G_q with semisimple q -Lie algebra $\mathfrak{g}_q = \text{Lie}(G_q)$.

First of all the adjoint representation $\text{Ad} : G_q \longrightarrow \text{GL}_n(K)$ is a covering of the q -Lie subgroup $\text{Ad}(G_q) \subset \text{Aut}(\mathfrak{g}_q) \subset \text{GL}_n(K)$, since its differential $\text{ad} : \mathfrak{g}_q \longrightarrow \mathfrak{gl}(\mathfrak{g}_q)$ is injective for a semisimple algebra \mathfrak{g}_q , its q -center

being trivial. So there is not only a maximal q -Lie group - the simply connected one - but also a minimal q -Lie group with q -Lie algebra \mathfrak{g}_q , since $\text{Ad}(G_q)$ only depends on \mathfrak{g}_q : It is the connected q -Lie subgroup of $\text{Aut}(\mathfrak{g}_q)$ with q -Lie algebra $\text{ad}(\mathfrak{g}_q)$. Indeed, for a semisimple algebra we have $\text{ad}(\mathfrak{g}_q) = \text{Der}(\mathfrak{g}_q)$ and hence $\text{Ad}(G_q) = \text{Aut}^0(\mathfrak{g}_q)$, the component of the identity of $\text{Aut}(\mathfrak{g}_q)$.

If G_q is simply connected, any other connected q -Lie group with q -Lie algebra \mathfrak{g}_q is of the form G_q/D_q with a discrete q -Lie subgroup $D_q \subset Z(G_q)$. Indeed the center $Z(G_q) \subset G_q$ is finite, it can even be read off from the q -root system Φ_q associated to a CSA $\mathfrak{h}_q \subset \mathfrak{g}_q = \text{Lie}(G_q)$: If we denote $\Gamma_0 \subset V \equiv \mathfrak{h}_{\mathbb{R},q}^*$ the lattice generated by Φ_q (in fact $\Gamma_0 = \bigoplus_{\alpha_q \in B} \mathbb{Z}\alpha_q$ with any basis $B \subset \Phi_q$) and $\Gamma \equiv \{\gamma \in V; \chi(\gamma, \Phi_q) \subset \mathbb{Z}\}$, then, obviously $\Gamma_0 \subset \Gamma$ and (less obviously)

$$Z(G_q) \cong \Gamma/\Gamma_0.$$

The connected q -Lie subgroup $H_q \subset G_q$ is a maximal (complexified) q -torus $(\mathbb{C}^*)^\ell$, a closed q -Lie subgroup of G_q containing the center $Z(G_q)$.

Furthermore semisimple q -Lie groups can be realized as algebraic (in particular closed) q -Lie subgroups of $\text{GL}(V)$ for some complex vector space V ; and even better, any q -homomorphism $G_q \rightarrow \text{GL}(W)$ for an arbitrary vector space W is algebraic. Indeed this follows from the knowledge of all irreducible \mathfrak{g}_q -modules W resp. all irreducible representations $\mathfrak{g}_q \rightarrow \mathfrak{gl}_q(W)$.

Here is the table of all simply connected complex simple q -Lie groups:

Dynkin diagram	simply connected q -Lie group	Dimension	Center
$A_\ell, \ell \geq 1$	$\text{SL}_{\ell+1,q}(\mathbb{C})$	$\ell(\ell + 2)$	$\mathbb{Z}_{\ell+1}$
$B_\ell, \ell \geq 2$	$\text{Spin}_{2\ell+1,q}(\mathbb{C})$	$\ell(2\ell + 1)$	\mathbb{Z}_2
$C_\ell, \ell \geq 3$	$\text{Sp}_{2\ell,q}(\mathbb{C})$	$\ell(2\ell + 1)$	\mathbb{Z}_2
$D_\ell, \ell \geq 4, \text{ even}$	$\text{Spin}_{2\ell,q}(\mathbb{C})$	$\ell(2\ell - 1)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$D_\ell, \ell \geq 5, \text{ odd}$	$\text{Spin}_{2\ell,q}(\mathbb{C})$	$\ell(2\ell - 1)$	\mathbb{Z}_4
E_6	—	78	\mathbb{Z}_3
E_7	—	133	\mathbb{Z}_2
E_8	—	248	0
F_4	—	52	0
G_2	$\text{Aut}(\mathbb{O}_q(\mathbb{C}))$	14	0

Remark 4. 1. Here $\text{Spin}_{n,q}(\mathbb{C})$ denotes the universal covering group of $\text{SO}_{n,q}(\mathbb{C})$, and $\mathbb{O}_q(\mathbb{C})$ is the complexified algebra of q -Cayley numbers (q -octonions) [9]. The remaining exceptional q -groups do not have an immediate geometric realization.

- Note that an arbitrary semisimple complex q -Lie group is of the form G_q/D_q , where G_q is a finite product of copies of the above q -Lie groups and $D_q \subset Z(G_q)$, with $Z(G_q)$ being the direct product of the centers of the simple factors.
- The q -Lie algebra G_2 has a base of 14 8×8 matrices; it is also given by $\text{Der}(\mathbb{O}_q(\mathbb{C}))$. There are two types of (q -)octonions: The standard octonions and the split-octonions, an 8-dimensional nonassociative algebra over the real numbers.

3.8 Complex semisimple and real compact q -Lie groups

Finally, let us comment on the relation between complex semisimple q -Lie groups and real compact q -Lie groups. Let us start with a semisimple q -Lie algebra \mathfrak{g}_q and look for a "real compact form" $\mathfrak{k} \subset \mathfrak{g}_q$, i.e., a real q -Lie subalgebra $\mathfrak{k} \cong \text{Lie}_q(G_q)$ for some real compact q -Lie group G_q satisfying $\mathfrak{g}_q = \mathfrak{k} \oplus_{i\mathbb{R}'} i\mathfrak{k}$. We start with a

CSA $\mathfrak{h}_q \subset \mathfrak{g}_q$ and write

$$\mathfrak{g}_q = \mathfrak{h}_q \oplus_{q'} \bigoplus_{\alpha \in \Phi_q} \mathfrak{g}_{\alpha,q},$$

noting that, with respect to the q -Cartan Killing form $\langle \cdot, \cdot \rangle$ we have $\mathfrak{h}_q \perp \mathfrak{g}_{\alpha,q}$ for all $\alpha_q \in \Phi_q$ as well as $\mathfrak{g}_{\alpha,q} \perp \mathfrak{g}_{\beta,q}$ for $\alpha_q + \beta_q \neq 0$. We shall represent $\mathfrak{k} = \text{Fix}(\tau)$ as fix algebra of a conjugation

$$\tau : \mathfrak{g}_q \longrightarrow \mathfrak{g}_q,$$

i.e. an involutive automorphism of \mathfrak{g}_q as real q -Lie algebra ($\tau^2 = I_{\mathfrak{g}_q}$) satisfying $\tau(iX) = -iX$. Setting

$$\mathfrak{g}_{0,q} = \mathfrak{h}_{\mathbb{R},q} \oplus_{q'} \bigoplus_{\alpha \in \Phi_q} \mathbb{R}Z_{\alpha,q} \subset \mathfrak{g}_q$$

with $\mathfrak{h}_{\mathbb{R},q} \equiv \text{span}(H_{\alpha,q}; \alpha_q \in \Phi_q)$ (see Remark 3) we have $\mathfrak{g}_q = \mathfrak{g}_{0,q} \oplus_{q'} i\mathfrak{g}_{0,q}$ and a conjugation $\sigma \equiv I_{\mathfrak{g}_{0,q}} \oplus_{q'} -I_{i\mathfrak{g}_{0,q}}$.

We want to take $\tau = \varphi \circ \sigma$ with a complex automorphism $\varphi : \mathfrak{g}_q \longrightarrow \mathfrak{g}_q$: In order to define φ , we choose elements $Z_{\alpha,q} \in \mathfrak{g}_{\alpha,q}, Z_{-\alpha,q} \in \mathfrak{g}_{-\alpha,q}$ with $\langle Z_{\alpha,q}, Z_{-\alpha,q} \rangle = -1$. Then the linear map $\varphi : \mathfrak{g}_q \longrightarrow \mathfrak{g}_q$ with $\varphi|_{\mathfrak{h}_q} = -I_{\mathfrak{h}_q}$ and $\varphi(Z_{\alpha,q}) = Z_{-\alpha}$ is the desired q -Lie algebra automorphism.

Example 3.59. For $\mathfrak{g}_q = \mathfrak{sl}_q(n, \mathbb{C}), \mathfrak{h}_q = \mathfrak{sd}_q(\times, \mathbb{C})$ we have $\sigma(A) = \bar{A}$ and $\varphi(A) = -A^T$.

4 Brief introduction to q -homogeneous spaces

4.1 On q -differentials

Definition 4.1. Let $\mathbb{R}_{\oplus q}$ denote the set generated by NWA of at most two letters in \mathbb{R} .

Assume that $x \oplus_q y \in \mathbb{R}_{\oplus q}$. Since $\mathbb{R} \times \mathbb{R} \cong \mathbb{C}$, this implies that the topological reasoning for \mathbb{C} can be repeated. A connected open set is a domain. A domain, together with some of its boundary opints, is called a region. We can express the Nalli–Ward q -Taylor formula as follows:

$$F(x \oplus_q y) = \sum_{k=0}^{n-1} \frac{y^k}{\{k\}_q!} D_q^k F(x) + \int_{t=0}^y D_{q,t}^n [F(x \oplus_q t)] \frac{(-t)^{n-1}}{\{n-1\}_q!} q^{\binom{n}{2}} d_q(t). \tag{46}$$

For $n = 2$ this becomes

$$F(x \oplus_q y) = F(x) + yD_q F(x) + \int_{t=0}^y D_{q,t}^2 [F(x \oplus_q t)] (-t)_q d_q(t). \tag{47}$$

Formula (47) (for formal power series) can obviously be expressed in the form

$$F(x \oplus_q y) - F(x) = yD_q F(x) + r(y), \tag{48}$$

with remainder term $r(y)$ such that $\lim_{y \rightarrow 0} \frac{r(y)}{|y|} = 0$. This means that the difference between two function values

$$\Delta_q F \equiv F(x \oplus_q y) - F(x)$$

can be approximated with a first degree polynomial in y :

$$d_q F = yD_{q,y} F(x).$$

$d_q F$ is called the q -differential of F in x . Furthermore, the q -differential of a multivariable function $F(x_1, \dots, x_n)$ is given by

$$d_q F = \sum_{k=1}^n x_k D_{q, x_k}. \quad (49)$$

The following expositions are q -analogues of [16].

Lemma 4.2. *Let G_q be a q -Lie group and let H_q be a closed q -Lie subgroup of G_q . Denote the q -Lie algebra of G_q by \mathfrak{g}_q and the q -Lie algebra of H_q by \mathfrak{h}_q . If \mathfrak{m}_q is a linear complement to \mathfrak{h}_q , i.e. $\mathfrak{g}_q = \mathfrak{h}_q \oplus_{q'} \mathfrak{m}_q$ and we endow G_q/H_q with the quotient topology, let $\pi : G_q \rightarrow G_q/H_q$, $\mathfrak{g}_q \mapsto \mathfrak{g}_q H_q$, be the canonical projection onto the quotient, then π becomes an open map, since $\pi^{-1}(\pi(V)) = \bigcup_{h_q \in H_q} V h_q$.*

Theorem 4.3. *The map*

$$\pi \circ E_q : \mathfrak{m}_q \rightarrow G_q/H_q \quad (50)$$

is a local homomorphism at 0.

Proof. Let φ be the map

$$\varphi : \mathfrak{g}_q = \mathfrak{h}_q \oplus_{q'} \mathfrak{m}_q \rightarrow G_q, \quad \varphi(X, Y) = E_q(X)E_q(Y). \quad (51)$$

Then $d_q \varphi(0, 0) \stackrel{\text{by(49)}}{=} X + Y$. By the equivalent to the inverse function theorem, φ is a diffeomorphism from an open neighborhood $U_0 \times V_0$ of $(0, 0)$. In order to show that $\pi \circ \varphi$ is one-to-one, we define some sets. The set $E_q(V_0)$ is an open neighborhood of I_h in H_q with the subspace topology, so $E_q(V_0) = K_q \cap H_q$ for some open set K_q in G_q . So there exists an open set

$$U_1 \times V_1 \subseteq U_0 \times V_0 \quad (52)$$

such that

$$\varphi(U_1 \times V_1) \cap H_q \subseteq K_q \cap H_q = E_q(V_0). \quad (53)$$

Suppose that $X \in U_1$, $Y \in V_1$ and $\varphi(X, Y) \in H_q$ then

$$E_q(X)E_q(Y) = E_q(Y'), \quad \text{where } Y' \in V_0. \quad (54)$$

Since φ is a diffeomorphism on $U_1 \times V_1$ so $X = 0$ and $Y = Y'$ and we conclude that

$$\varphi(U_1 \times V_1) \cap H_q = E_q(V_1). \quad (55)$$

In the next step we show the injectivity. Let $U_2 \subseteq U_1$ be a neighbourhood of 0 in \mathfrak{m}_q such that

$$E_q(-U_2)E_{\frac{1}{q}}(U_2) \subseteq \varphi(U_1 \times V_1). \quad (56)$$

Then $\pi \circ E_q(U_2)$ is injective, since if $X', X'' \in U_2$ and

$$\pi(E_q(X')) = \pi(E_q(X'')) \quad (57)$$

then

$$E_q(-X')E_{\frac{1}{q}}(X'') \in \varphi(U_1 \times V_1) \cap H_q, \quad \text{so } X' = X''. \quad (58)$$

The surjectivity on a neighborhood follows since φ is surjective from $U_2 \times \{0\}$ onto $\varphi(U_2, 0)$. By definition $\pi \circ E_q(\mathfrak{m}_q)$ is continuous. If N is an open subset of U_2 then $\pi(E_q(N)) = \pi \circ \varphi(N, V_1)$, which is open since φ is a diffeomorphism here and we have the quotient topology. So the inverse is continuous. Thus $\pi \circ E_q : \mathfrak{m}_q \rightarrow G_q/H_q$ is a local homomorphism at 0 in the quotient topology of G_q/H_q . \square

Theorem 4.4. *The q -homogeneous space G_q/H_q is a Hausdorff space.*

Proof. Choose the quotient topology on G_q/H_q . Consider the map

$$\varphi : G_q \times G_q \longrightarrow G_q, \quad \varphi(g_1, g_2) = g_2 \cdot_q g_1^{-1}, \quad (59)$$

which is continuous, and since H_q is closed the inverse image $\varphi^{-1}(H_q)$ is closed. Now if $H_q g_1 \neq H_q g_2$ then $g_2 \cdot_q g_1^{-1} \notin H_q$ so there are open sets W_1, W_2 such that $(g_1, g_2) \in W_1 \times W_2$ and $W_1 \times W_2 \cap \varphi^{-1}(H_q) = \emptyset$. Assuming $H_q g \in H_q W_i$ then there is a $H_i \in H_q$ such that $H_i g \in W_i$. So if $H_q g \in H_q W_1 \cap H_q W_2$ then $H_1 g \in W_1$ and $H_2 g \in W_2$. Therefore $(H_1 g, H_2 g) \in W_1 \times W_2$ is mapped to $H_2 \cdot_q H_1^{-1} \in H_q$, which contradicts that $W_1 \times W_2 \cap \varphi^{-1}(H_q) = \emptyset$. And so G_q/H_q is Hausdorff. \square

4.2 The q -Lie groups $SU_q(2)$ and $SU_q(1, 1)$

In Audrey Terras' book [18] the first symmetric space to be treated was the sphere (chapter 2), followed by the upper half plane H (chapter 3). We somehow follow the same plan and give formal definitions of the most important q -analogues of homogeneous spaces in each section. In [3, (19) p.157] we defined the following general form of a matrix in $SU_q(2)$ as:

$$U_{\Psi, \phi, \gamma} = \begin{pmatrix} \text{Cos}_q(\Psi)E_q(i\phi) & -\text{Sin}_q(\Psi)E_q(i\gamma) \\ \text{Sin}_q(\Psi)E_q(-i\gamma) & \text{Cos}_q(\Psi)E_q(-i\phi) \end{pmatrix}. \quad (60)$$

Every element of $SU_q(2)$ has an inverse for the multiplication $\cdot_{\psi_1, q}$, namely $U_{\psi_1, \phi_1, \alpha_1}^{-1} = U_{\psi_2, \phi_2, \alpha_2}$, with the following three conditions on the umbrae: $\alpha_1 \sim \alpha_2$, $\phi_1 \sim -\phi_2$, $\psi_1 \sim -\psi_2$. A straightforward calculation shows that formula (60) is equal to

$$\begin{pmatrix} E_q(i\alpha) & 0 \\ 0 & E_q(-i\alpha) \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}_q(\psi) & -\text{Sin}_q(\psi) \\ \text{Sin}_q(\psi) & \text{Cos}_q(\psi) \end{pmatrix} \cdot \begin{pmatrix} E_q(i\beta) & 0 \\ 0 & E_q(-i\beta) \end{pmatrix}, \quad (61)$$

where

$$\alpha \sim \frac{\phi \oplus_q \gamma}{2_q} \text{ and } \beta \sim \frac{\phi \ominus_q \gamma}{2_q}. \quad (62)$$

This last expression can be written in the form $E_q(T)$, where

$$T \sim \begin{pmatrix} i \frac{\phi \oplus_q \gamma}{2_q} & 0 \\ 0 & -i \frac{\phi \oplus_q \gamma}{2_q} \end{pmatrix} \oplus_{q'} \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix} \oplus_{q'} \begin{pmatrix} i \frac{\phi \ominus_q \gamma}{2_q} & 0 \\ 0 & -i \frac{\phi \ominus_q \gamma}{2_q} \end{pmatrix}, \quad (63)$$

where the matrices don't commute.

By the Weyl unitary trick, we find the following expression for the noncompact q -Lie group $SU_q(1, 1)$:

$$\begin{pmatrix} E_q(i\alpha) & 0 \\ 0 & E_q(-i\alpha) \end{pmatrix} \cdot \begin{pmatrix} \text{Cosh}_q(\psi) & \text{Sinh}_q(\psi) \\ \text{Sinh}_q(\psi) & \text{Cosh}_q(\psi) \end{pmatrix} \cdot \begin{pmatrix} E_q(i\beta) & 0 \\ 0 & E_q(-i\beta) \end{pmatrix} = \\ E_q \left(\begin{pmatrix} i \frac{\phi \oplus_q \gamma}{2_q} & 0 \\ 0 & -i \frac{\phi \oplus_q \gamma}{2_q} \end{pmatrix} \oplus_{q'} \begin{pmatrix} 0 & \psi \\ \psi & 0 \end{pmatrix} \oplus_{q'} \begin{pmatrix} i \frac{\phi \ominus_q \gamma}{2_q} & 0 \\ 0 & -i \frac{\phi \ominus_q \gamma}{2_q} \end{pmatrix} \right), \quad (64)$$

where the matrices don't commute.

By the Weyl unitary trick, we find the following expression for the noncompact q -Lie group $SU_q(1, 1)$, a q -analogue of [10, p. 202]:

$$\begin{pmatrix} E_q(i\beta) & 0 \\ 0 & E_q(-i\beta) \end{pmatrix} \cdot \begin{pmatrix} \text{Cosh}_q(\frac{\alpha}{2}) & (\widehat{\alpha}_1 - i\widehat{\alpha}_2)\text{Sinh}_q(\frac{\alpha}{2}) \\ (\widehat{\alpha}_1 + i\widehat{\alpha}_2)\text{Sinh}_q(\frac{\alpha}{2}) & \text{Cosh}_q(\frac{\alpha}{2}) \end{pmatrix}, \quad (65)$$

$$\text{where } \widehat{\alpha}_1 = \frac{\alpha_1}{\alpha}, \quad \widehat{\alpha}_2 = \frac{\alpha_2}{\alpha}.$$

This corresponds to the q -Lie algebra decomposition

$$su_q(1, 1) = \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix} \oplus_{q'} \begin{pmatrix} 0 & \frac{1}{2}(\alpha_1 - i\alpha_2) \\ \frac{1}{2}(\alpha_1 + i\alpha_2) & 0 \end{pmatrix}. \quad (66)$$

Example 4.5. Put $G_{n,q}^* = \text{SU}_q(1, 1)$ and $K_q^* = \text{SO}_q(2)$, Then the quotient space is the q -deformed hyperboloid (compare [10, p. 202])

$$e_1 \cdot_q e_1 = 1, e_1 = (x, y, z), \quad (67)$$

where we use the following indefinite q -scalar product:

$$(\alpha \cdot_{q;\text{indef}} \beta)_{ij} \equiv a_{i0} \tau(\beta_{0j}) - \sum_{m=1}^3 a_{im} \tau(\beta_{mj}). \quad (68)$$

We obtain the q -homogeneous space

$$\frac{\text{SU}_q(1, 1)}{\text{SO}_q(2)} \cong \mathbb{H}_q^2. \quad (69)$$

For this q -hyperbolic space we take

$$p_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{H}_q^2.$$

For the q -Lie algebra

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}_q(1, 1),$$

the q -geodesic starting at p_0 is

$$\begin{pmatrix} \text{Cosh}_q(t) & \text{Sinh}_q(t) & 0 \\ \text{Sinh}_q(t) & \text{Cosh}_q(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \text{Cosh}_q(t) \\ \text{Sinh}_q(t) \\ 0 \end{pmatrix}.$$

This q -geodesic starting at p_0 has direction

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

A general q -geodesic is obtained by multiplication by an arbitrary element of $\text{SO}_q(3)$:

$$\begin{pmatrix} \text{Cos}_q(\gamma) & \text{Sin}_q(\gamma) & 0 \\ -\text{Sin}_q(\gamma) & \text{Cos}_q(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{Cosh}_q(t) \\ \text{Sinh}_q(t) \\ 0 \end{pmatrix}.$$

4.3 The q -Lie groups $\text{SO}_q(3)$ and $\text{SO}_q(2)$

A general element of $\text{SO}_q(3)$ is denoted O_ψ , where $\psi \in \mathbb{R}_q$. We can write O_ψ as follows:

$$O_\psi = \begin{pmatrix} \text{Cos}_q(\psi) & -\text{Sin}_q(\psi) & 0 \\ \text{Sin}_q(\psi) & \text{Cos}_q(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (70)$$

The q -Lie algebra $\mathfrak{so}_q(3)$ of $\text{SO}_q(3)$ has the following basis in the space of skew-symmetric real (3×3) matrices:

$$Z(1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, Z(2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, Z(3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (71)$$

whose commutation relations are

$$[Z(2), Z(3)] = Z(1), [Z(3), Z(1)] = Z(2), \text{ and } [Z(1), Z(2)] = Z(3). \quad (72)$$

Definition 4.6. The q -deformed sphere, S_q^2 , a manifold, is defined by q -spherical coordinates with unit radius:

$$\begin{cases} x_1 = \text{Sin}_q(\gamma)\text{Cos}_q(\varphi) \\ x_2 = \text{Sin}_q(\gamma)\text{Sin}_q(\varphi) \\ x_3 = \text{Cos}_q(\gamma), \end{cases} \quad (73)$$

where $0 \leq \gamma < \xi(q)$; $0 < \varphi < \xi(q, 2)$.

Then we have

$$\frac{\text{SO}_q(3)}{\text{SO}_q(2)} \cong S_q^2. \quad (74)$$

For the q -sphere we take

$$p_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in S_q^2.$$

For the q -Lie algebra

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

the q -geodesic starting at p_0 is

$$\begin{pmatrix} \text{Cos}_q(t) & \text{Sin}_q(t) & 0 \\ -\text{Sin}_q(t) & \text{Cos}_q(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \text{Cos}_q(t) \\ -\text{Sin}_q(t) \\ 0 \end{pmatrix}.$$

This q -geodesic has direction

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

Again, a general q -geodesic is obtained by multiplication by an arbitrary element of $\text{SO}_q(3)$:

$$\begin{pmatrix} \text{Cos}_q(\gamma) & \text{Sin}_q(\gamma) & 0 \\ -\text{Sin}_q(\gamma) & \text{Cos}_q(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}_q(t) \\ -\text{Sin}_q(t) \\ 0 \end{pmatrix}.$$

We can already guess the structure of the q -Lie algebra for $\text{SO}_q(n)$. It is given by the algebra of $n \times n$ antisymmetric matrices $A^{(n)}$ according to the following mapping, a q -analogue of [10, (3.19) p.19]:

$$A^{(n)} \xrightarrow{E_q} \text{SO}_q(n). \quad (75)$$

Theorem 4.7. The mapping E_q is surjective for $G_{n,q} = \text{SU}_q(2)$.

Proof. According to formula (61), every element in $\text{SU}_q(2)$ is q -conjugate with an element

$$\begin{pmatrix} \text{Cos}_q(\psi) & -\text{Sin}_q(\psi) \\ \text{Sin}_q(\psi) & \text{Cos}_q(\psi) \end{pmatrix}.$$

□

4.4 Sample q -roots and q -Cartan subalgebras

We will now study some special q -Lie groups. The results will be very similar to the classical Lie groups [2] and we only list formulas which differ from this classical case. In particular, we state q -roots $\in \mathbb{R}_q$ and q -Cartan subalgebras. For the computations of these q -roots see [2, p.197]. For $SO_q(4)$ we have the four q -roots

$$\alpha_1 \oplus_q \alpha_2, -\alpha_1 \boxplus_q \alpha_2, \alpha_1 \boxminus_q \alpha_2, -\alpha_1 \ominus_q \alpha_2. \quad (76)$$

For $SU_q(3)$ with maximal q -torus (12) we have three q -roots

$$\alpha_1 \boxminus_q \alpha_2, \alpha_2 \boxminus_q \alpha_3, \alpha_1 \boxminus_q \alpha_3. \quad (77)$$

For $SU_q(2)$ we have the q -root $\alpha_1 \boxminus_q \alpha_2$. For $Sp_q(1)$ we have the q -root α . And for $SO_q(3)$ we have one q -root, which we denote $-\alpha$. For $SU_q(n)$ the q -Cartan subalgebras are diagonal matrices

$$D \equiv \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \quad (78)$$

with $\{d_j\}_{j=1}^n \in \mathbb{R}_q$ and

$$\oplus_{q,j=1}^n d_j \sim \theta.$$

4.5 A q -deformed semidirect product

The following considerations from Simon [15] are very important in (q -Lie) group theory.

Definition 4.8. Let N_q be a q -Lie group. A mapping $f : N_q \mapsto N_q$ is called a q -automorphism if f is bijective, and both f and f^{-1} are q -Lie group homomorphisms.

Definition 4.9. A q -analogue of [15, p.6]. Let $N_{n,q}$ and $H_{n,q}$ be two q -Lie groups. Let α_h be a q -automorphism $N_q \rightarrow N_q$ indexed by an element $h \in H_{n,q}$.

The operations in $N_{n,q}$ are \odot and \odot_q . The operations in $H_{n,q}$ are \cdot and \cdot_q . The q -semidirect product $N_q \otimes_{\alpha,q} H_{n,q}$ of $N_{n,q}$ and $H_{n,q}$ is the q -Lie group of all ordered pairs (n, h) with the multiplication \cdot defined by

$$(n_1, h_1) \cdot (n_2, h_2) \equiv (n_1 \odot (n_2), h_1 \cdot h_2), \quad (79)$$

and the multiplication \cdot_q defined by

$$(n_1, h_1) \cdot_q (n_2, h_2) \equiv (n_1 \odot_q \alpha_{h_1}(n_2), h_1 \cdot_q h_2). \quad (80)$$

The unit element of $N_q \otimes_{\alpha,q} H_{n,q}$ is $e = (I_N, I_H)$ and the inverse element of a pair (n, h) is the pair

$$(n, h)^{-1} = (\alpha_{h^{-1}}(n^{-1}), h^{-1}). \quad (81)$$

Theorem 4.10. N_q is a normal q -Lie subgroup of $N_q \otimes_{\alpha,q} H_{n,q}$.

Proof. Assume that (n_1, h) and (n_2, I_H) are two elements of $N_q \otimes_{\alpha,q} H_{n,q}$. Then a straightforward calculation shows

$$\begin{aligned} & (\alpha_{h^{-1}}(n_1^{-1}), h^{-1}) \cdot_q (n_2, I_H) \cdot_q (n_1, h) \\ & = ((\alpha_{h^{-1}}(n_1^{-1})) \odot \alpha_{h^{-1}}(n_2), h^{-1} \cdot_q I_H) \cdot_q (n_1, h) \in N_q. \end{aligned} \quad (82)$$

□

Definition 4.11. A sequence of q -Lie groups is defined by

$$1 \longrightarrow G_{1,q} \xrightarrow{f_1} \cdots \longrightarrow G_{i-1,q} \xrightarrow{f_{i-1}} G_{i,q} \xrightarrow{f_i} G_{i+1,q} \longrightarrow \cdots \xrightarrow{f_{k-1}} G_{k,q}, \quad (83)$$

where f_i are q -Lie group homomorphisms. The sequence (83) is said to be exact at $G_{i,q}$ if

$$\text{Im}(f_{i-1}) = \ker(f_i).$$

If the sequence is exact at every $G_{i,q}$, it is called an exact sequence.

Theorem 4.12. Compare with [17, p. 578]. If the sequence (83) is exact, then

$$0 = \dim G_{1,q} - \dim G_{2,q} + \dim G_{3,q} - \cdots + (-1)^{k-1} \dim G_{k,q}. \quad (84)$$

Contemplate a short exact sequence

$$1 \longrightarrow N_q \xrightarrow{i} A_q \xrightarrow{j} G_{n,q} \longrightarrow 1$$

of q -Lie groups. A splitting for $j : A_q \mapsto G_{n,q}$ is a q -Lie group q -homomorphism $s : G_{n,q} \mapsto A_q$ such that $j \circ s$ is the identity mapping on $G_{n,q}$.

Theorem 4.13. Let

$$1 \longrightarrow N_q \xrightarrow{i} A_q \xrightarrow{j} G_{n,q} \longrightarrow 1 \quad (85)$$

be an exact sequence of q -Lie groups. Then there exists a splitting $s : G_{n,q} \mapsto A_q$ for $j : A_q \mapsto G_{n,q}$ if and only if there exist a q -Lie group q -homomorphism $K : G_{n,q} \mapsto \text{Aut}(N_q)$ and a q -Lie group isomorphism $f : A_q \mapsto N_q \otimes_q G_{n,q}$ such that

$$f \circ i(n) = (n, I_G) \text{ and } j \circ f^{-1}(n, g) = g.$$

Proof. The proof goes along the same lines as for groups and is omitted. \square

Corollary 4.14. If $A_q = N_q \otimes_q G_{n,q}$, the quotient A_q/N_q is a q -Lie group and $A_q/N_q \cong G_{n,q}$.

Example 4.15. A q -analogue of [12, p.23]:

$$\begin{aligned} 1 &\longrightarrow \text{SO}_q(n) \xrightarrow{i} O_q(n) \xrightarrow{j} \mathbb{Z}_2 \longrightarrow 1, \\ 1 &\longrightarrow \text{SU}_q(n) \xrightarrow{i} U_q(n) \xrightarrow{j} U_q(1) \longrightarrow 1, \\ 1 &\longrightarrow \text{SL}_q(n, \mathbb{R}) \xrightarrow{i} \text{GL}_q(n, \mathbb{R}) \xrightarrow{j} \mathbb{R}^* \longrightarrow 1. \end{aligned} \quad (86)$$

The mappings i and j are the inclusion map and \det_q . We have the following q -Lie group isomorphisms:

$$\begin{aligned} O_q(n) &\cong \text{SO}_q(n) \otimes_q \mathbb{Z}_2, \\ U_q(n) &\cong \text{SU}_q(n) \otimes_q U_q(1), \\ \text{GL}_q(n, \mathbb{R}) &\cong \text{SL}_q(n, \mathbb{R}) \otimes_q \mathbb{R}^*. \end{aligned} \quad (87)$$

We have the following q -homogeneous spaces:

$$\begin{aligned} O_q(n)/\text{SO}_q(n) &\cong \mathbb{Z}_2, \\ U_q(n)/\text{SU}_q(n) &\cong U_q(1), \\ \text{GL}_q(n, \mathbb{R})/\text{SL}_q(n, \mathbb{R}) &\cong \mathbb{R}^*. \end{aligned} \quad (88)$$

5 Conclusion

Various forms of q -symmetric spaces in quantum group form have appeared in the literature. Our approach to q -homogeneous spaces is completely different and much more promising, since it gives concrete formulas

to be applied in physics. The Wigner functions $D_{m,n}^j$ characterize the states in the Coulomb problem, because the three indices, j, m, n are, respectively, associated with the eigenstates of the Hamiltonian, the z -component of angular momentum, and the z -component of the Runge-Lenz vector. Similarly, the q -Wigner functions from [3] provide applicable q -analogues of these objects. The q -semidirect product will be used later to define the q -Euclidean group and the q -Poincaré group, which are both matrix q -Lie groups. Furthermore, we mention that quantum groups and complex motion groups are treated in the papers [13] and [14].

Acknowledgements: I thank Karl-Heinz Fieseler for his assistance. Large parts of the paper are based on his lectures. Arne Meurman has kindly checked several parts of the manuscript, in particular the parts on Weyl groups.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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