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Enumeration of weighted paths on a digraph and block hook determinant

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Abstract: In this article, we evaluate determinants of "block hook" matrices, which are block matrices consist of hook matrices. In particular, we deduce that the determinant of a block hook matrix factorizes nicely. In addition we give a combinatorial interpretation of the aforesaid factorization property by counting weighted paths in a suitable weighted digraph.

Keywords: block hook matrix; determinants; weighted path; combinatorial proof

MSC: 05A10; 05A05; 11C20; 05C30; 05C38

1 Introduction

The evaluation of determinants is a nice topic, and fascinating for many people [4, 6, 7, 12, 16]. A huge amount of such evaluation has been collected in [6, 7]. Specially, the problem of calculating the determinant of a 2×2 block matrix has been long studied [12, 16]. Block matrices are applied all over in mathematics and physics. They appear naturally in the description of systems with multiple discrete variables [13, 14]. Moreover, block matrices are utilized in many computational methods familiar to researchers of fluid dynamics [10]. Also the determinants of these matrices are found over a large number of area for both analytical and numerical applications [8, 11]. The purpose of this paper is to evaluate the determinantal formulas of some special classes of block matrices, known as *block hook matrix* (defined later). In particular, we will show that the determinants of these block hook matrices admit nice product formulas. Now, let us define hook matrix in a precise way. First we need to define the following.

Definition 1.1. A square matrix is called an *hook* matrix if the pattern of the entries satisfy one of the following four conditions;
(a) all the entries right and below of the entry at the $(i, i)^{\text{th}}$ $(i = 1, \cdots, m)$ position are same
(b) all the entries left and above of the entry at the $(m+1-i, m+1-i)^{\text{th}}$ $(i = 1, \cdots, m)$ position are same
(c) all the entries right and above of the entry at the $(m+1-i, i)^{\text{th}}$ $(i = 1, \cdots m)$ position are same
(d) all the entries left and below of the entry at the $(i, m+1-i)^{\text{th}}$ $(i = 1, \cdots m)$ position are same.

Now consider the matrices in (1), (2), (3) and (4).

$$A_m(x_1, \cdots, x_m) = \begin{pmatrix} x_m & x_m & \cdots & x_m & x_m \\ x_m & x_{m-1} & \cdots & x_{m-1} & x_{m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_m & x_{m-1} & \cdots & x_2 & x_2 \\ x_m & x_{m-1} & \cdots & x_2 & x_1 \end{pmatrix} \tag{1}$$

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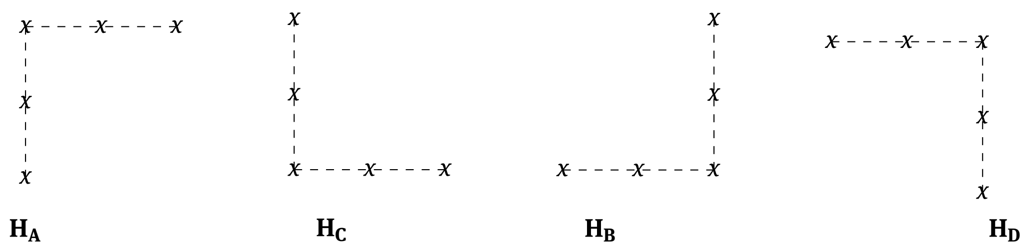

Figure 1: H_A, H_C, H_B, H_D are four different shapes hook.

$$B_m(x_1,\cdots,x_m)=\begin{pmatrix} x_1 & x_2 & \cdots & x_{m-1} & x_m \\ x_2 & x_2 & \cdots & x_{m-1} & x_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1} & x_{m-1} & \cdots & x_{m-1} & x_m \\ x_m & x_m & \cdots & x_m & x_m \end{pmatrix} \tag{2}$$

$$C_m(x_1,\cdots,x_m)=\begin{pmatrix} x_m & x_{m-1} & \cdots & x_2 & x_1 \\ x_m & x_{m-1} & \cdots & x_2 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_m & x_{m-1} & \cdots & x_{m-1} & x_{m-1} \\ x_m & x_m & \cdots & x_m & x_m \end{pmatrix}. \tag{3}$$

$$D_m(x_1,\cdots,x_m)=\begin{pmatrix} x_m & x_m & \cdots & x_m & x_m \\ x_{m-1} & x_{m-1} & \cdots & x_{m-1} & x_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2 & x_2 & \cdots & x_{m-1} & x_m \\ x_1 & x_2 & \cdots & x_{m-1} & x_m \end{pmatrix}. \tag{4}$$

For each of the above matrix the pattern should be clear: In case of the matrix in (1), all the entries right and below of the entry at the $(i,i)^{\text{th}}$ $(i=1,\cdots,m)$ position are same and clearly these entries form a hook of shape like H_A in Figure 1. Similarly, for the matrix in (4), all the entries left and below of the entry at the $(i,m+1-i)^{\text{th}}$ $(i=1,\cdots m)$ position are same and these entries form a hook of shape like H_D in Figure 1. Similarly the entries in the matrices in (2) and (3) form a hook of shape like H_B and H_C (in Figure 1) respectively. Moreover we have the following nice product formulas regarding the determinant of these hook matrices.

Proposition 1.1. *Let $A_m(x_1,\cdots,x_m)$ be a matrix defined as* (1). *Then*

$$\det(A_m(x_1,\cdots,x_m))=\prod_{i=1}^{m}(x_i-x_{i+1}),\ \textit{where } x_{m+1}=0.$$

Proof. We apply the following row operations on the matrix $A_m(x_1,\cdots,x_m)$;

$$R_i'=R_i-R_{i-1},\ \text{for } i=2,\cdots,m.$$

Clearly, the resulting matrix (matrix obtained after row operations) is a diagonal matrix with diagonal entries (x_i-x_{i+1}), for $i=m,m-1,\cdots,1$, where $x_{m+1}=0$. Hence the result. □

Proposition 1.2. *Let $B_m(x_1,\cdots,x_m)$ be a matrix defined as* (2). *Then*

$$\det B_m(x_1,\cdots,x_m)=\prod_{i=1}^{m}(x_i-x_{i+1}),\ \textit{where } x_{m+1}=0.$$

Proof. We apply the following row and column interchange on the matrix $A_m(x_1, \cdots, x_m)$;

$$R_i \leftrightarrow R_{m-i+1} \text{ and } C_i \leftrightarrow C_{m-i+1}, \text{ for } i = \begin{cases} 1, 2, \cdots, \frac{m}{2}, & \text{if } m \text{ is even} \\ 1, 2, \cdots, \frac{m+1}{2} - 1, & \text{if } m \text{ is odd}, \end{cases} \tag{5}$$

Evidently the resulting matrix is $B_m(x_1, \cdots, x_m)$. This completes the proof. □

Proposition 1.3. *Let $C_m(x_1, \cdots, x_m)$ be a matrix defined as* (3). *Then*

$$\det C_m(x_1, \cdots, x_m) = \begin{cases} (-1)^{\frac{m}{2}} \times \prod_{i=1}^{m} (x_i - x_{i+1}), & \text{if } m \text{ is even} \\ (-1)^{\frac{m-1}{2}} \times \prod_{i=1}^{m} (x_i - x_{i+1}), & \text{if } m \text{ is odd}, \end{cases}$$

where $x_{m+1} = 0$.

Proof. Here we apply the same row operations depicted as (5) on the matrix $A_m(x_1, \cdots, x_m)$, and as a result we get the matrix $C_m(x_1, \cdots, x_m)$. So the proposition. □

Proposition 1.4. *Let $D_m(x_1, \cdots, x_m)$ be a matrix defined as* (4). *Then*

$$\det D_m(x_1, \cdots, x_m) = \begin{cases} (-1)^{\frac{m}{2}} \times \prod_{i=1}^{m} (x_i - x_{i+1}), & \text{if } m \text{ is even} \\ (-1)^{\frac{m-1}{2}} \times \prod_{i=1}^{m} (x_i - x_{i+1}), & \text{if } m \text{ is odd}, \end{cases}$$

where $x_{m+1} = 0$.

Proof. Applying the same column operations described as (5) on the matrix $A_m(x_1, \cdots, x_m)$, we get the matrix $D_m(x_1, \cdots, x_m)$. Hence the proposition. □

Definition 1.2. An $N \times N (N \geq 2)$ block matrix M is called a *block hook matrix* if each block of M is a hook matrix.

Remark 1. *If $N = 1$, then the block hook matrix reduces to a hook matrix.*

Example 1.5.

$$\left(\begin{array}{cc|cc} x_2 & x_2 & y_2 & y_2 \\ x_2 & x_1 & y_2 & y_1 \\ \hline z_2 & z_2 & w_2 & w_2 \\ z_2 & z_1 & w_2 & w_1 \end{array}\right), \left(\begin{array}{cc|cc} x_1 & x_2 & y_2 & y_1 \\ x_2 & x_2 & y_2 & y_2 \\ \hline z_1 & z_2 & w_2 & w_1 \\ z_2 & z_2 & w_2 & w_2 \end{array}\right), \left(\begin{array}{cc|cc} x_1 & x_2 & y_2 & y_1 \\ x_2 & x_2 & y_2 & y_2 \\ \hline z_2 & z_2 & w_2 & w_2 \\ z_1 & z_2 & w_2 & w_1 \end{array}\right), \left(\begin{array}{cc|cc} x_2 & x_2 & y_2 & y_2 \\ x_2 & x_1 & y_1 & y_2 \\ \hline z_2 & z_2 & w_2 & w_2 \\ z_2 & z_1 & w_1 & w_2 \end{array}\right).$$

Example 1.5 contains four 2×2 block matrices, in which each block is a hook. In fact, the 1^{st} matrix contains four hooks of shape H_A as depicted in Figure 1, whereas the hooks in the last matrix are of different shapes. Also it can be shown that, the determinant of any one of the above matrices gives a nice product formula. So, one natural question occurs: Is it true that the determinant of any block hook matrix admits such a product formula? In this paper, we formulate different classes of block hook matrices of order Nm ($N \times N$ block matrix and order of each block is m) and prove that the determinants of all these block hook matrices can be written as $\prod_{i=1}^{m} \det(X_{(i,i+1)})$ (upto sign), where

$$X_{(i,i+1)} = \begin{pmatrix} x_i^{(1,1)} - x_{i+1}^{(1,1)} & x_i^{(1,2)} - x_{i+1}^{(1,2)} & \cdots & x_i^{(1,N)} - x_{i+1}^{(1,N)} \\ x_i^{(2,1)} - x_{i+1}^{(2,1)} & x_i^{(2,2)} - x_{i+1}^{(2,2)} & \cdots & x_i^{(2,N)} - x_{i+1}^{(2,N)} \\ \vdots & \vdots & \ddots & \vdots \\ x_i^{(N,1)} - x_{i+1}^{(N,1)} & x_i^{(N,2)} - x_{i+1}^{(N,2)} & \cdots & x_i^{(N,N)} - x_{i+1}^{(N,N)} \end{pmatrix}. \tag{6}$$

Moreover, we give combinatorial explanation of these product formulas. Throughout this paper we denote $i \times j$ zero matrix by $O_{i,j}$, $R_i \leftrightarrow R_j (C_i \leftrightarrow C_j)$ denotes the row (column) interchange between i^{th} and j^{th} row (column) of a matrix, we denote the set $\{1, 2, \cdots, m\}$ by $[m]$.

Now, let us briefly summarize the content. In Section 2, we consider block hook matrices containing same shape hooks and derive the product formulas for the determinants of these matrices. In Section 3, we focus on the block hook matrices formed by hooks of two different shapes and we show that the determinant of these matrices admit nice product formulas. In Section 4, we deal with block hook matrices containing hooks of four different shapes and find similar product formulas. In Section 5, we give combinatorial interpretations of factorization property of block hook determinants.

2 Block matrices containing hooks of same shape

In this section, we evaluate the determinants of block hook matrices, which are block matrices, in which each block is a hook and shape of all hooks are same. Now, we introduce an $N \times N$ block matrix $A(N, m)$ in the following way;

$$A(N, m) = (A_{ij}), \text{ where } 1 \le i, j \le N \text{ and each block } A_{ij} = A_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}). \tag{7}$$

So $A(N, m)$ is a hook matrix of order Nm. For example,

$$A(3,2) = \left(\begin{array}{cc|cc|cc} x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)} \\ x_2^{(1,1)} & x_1^{(1,1)} & x_2^{(1,2)} & x_1^{(1,2)} & x_2^{(1,3)} & x_1^{(1,3)} \\ \hline x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)} \\ x_2^{(2,1)} & x_1^{(2,1)} & x_2^{(2,2)} & x_1^{(2,2)} & x_2^{(2,3)} & x_1^{(2,3)} \\ \hline x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)} \\ x_2^{(3,1)} & x_1^{(3,1)} & x_2^{(3,2)} & x_1^{(3,2)} & x_2^{(3,3)} & x_1^{(3,3)} \end{array}\right) \tag{8}$$

is a block hook matrix, whose each block is a hook like (1).

Theorem 2.1. *Let $A(N, m)$ be a block hook matrix of order Nm defined as (7). Then*

$$\det(A(N, m)) = \prod_{i=1}^{m} \det(X_{(i,i+1)}),$$

where $X_{(i,i+1)}$ is defined as (6) and $x_{m+1}^{(i,j)} = 0$, for all $1 \le i, j \le N$.

Proof. We prove the theorem by applying induction on m. For $m = 1$, clearly the matrix

$$A(N, 1) = \left(\begin{array}{c|c|c|c} x_1^{(1,1)} & x_1^{(1,2)} & \cdots & x_1^{(1,N)} \\ \hline x_1^{(2,1)} & x_1^{(2,2)} & \cdots & x_1^{(2,N)} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline x_1^{(N,1)} & x_1^{(N,2)} & \cdots & x_1^{(N,N)} \end{array}\right) = X_{(1,2)}, \text{ with } x_2^{(i,j)} = 0, \text{ for all } 1 \le i, j \le N.$$

So, the base case is true. Suppose the result is true for all such matrices $A(N, (m-1))$. Now to complete the proof we want to go through the following row operations on the matrix $A(N, m)$;

$$R_i' = R_i - R_{i-1}, \text{ for } i = m, 2m, \cdots, Nm.$$

Then the matrix, we obtain thereby is as follows:

All the rows of each block A_{ij} remain unchanged except the last row. The last row of each block A_{ij} looks like $0, 0, \cdots, (x_1^{(i,j)} - x_2^{(i,j)})$. So for $\ell = 1, 2, \cdots, N$, the $(\ell m)^{\text{th}}$ row of the matrix $A(N, m)$ is

$$0, 0, \cdots, (x_1^{(\ell,1)} - x_2^{(\ell,1)}), 0, 0, \cdots, (x_1^{(\ell,2)} - x_2^{(\ell,2)}), \cdots, 0, 0, \cdots, (x_1^{(\ell,N)} - x_2^{(\ell,N)}).$$

Now we apply the successive row interchange on the resulting matrix (after performing row operations on the matrix $A(N, m)$) to arrange the rows in the following order;

$$R_m, R_{2m}, \cdots, R_{Nm}, R_1, \cdots, R_{m-1}, R_{m+1}, \cdots, R_{2m-1}, \cdots, R_{(N-1)m+1}, \cdots, R_{Nm-1}.$$

Then by successive column interchange we arrange the columns in the following order;

$$C_m, C_{2m}, \cdots, C_{Nm}, C_1, \cdots, C_{m-1}, C_{m+1}, \cdots, C_{2m-1}, \cdots, C_{(N-1)m+1}, \cdots, C_{Nm-1}.$$

And finally we obtain the matrix of the form

$$\left(\begin{array}{c|c} X_{(1,2)} & O_{N,(Nm-N)} \\ \hline \star & A(N,(m-1)) \end{array}\right),$$

where $A(N, (m-1))$ is block matrix and each block is $A_{m-1}(x_2^{(i,j)}, \cdots, x_m^{(i,j)})$. Therefore, $A(N, (m-1))$ is a block hook matrix of order $N(m-1)$. Now, without doubt we can write

$$\det(A(N,m)) = \left(\begin{array}{c|c} X_{(1,2)} & O_{N,(Nm-N)} \\ \hline \star & A(N,(m-1)) \end{array}\right). \tag{9}$$

Again by the Laplace expansion

$$\det\left(\begin{array}{c|c} X_{(1,2)} & O_{N,(Nm-N)} \\ \hline \star & A(N,(m-1)) \end{array}\right) = \det(X_{(1,2)}) \times \det(A(N,(m-1))). \tag{10}$$

Now we set $x_k^{(i,j)} = y_{k-1}^{(i,j)}$, for $k = 2, 3, \cdots, m$ and $1 \le i, j \le N$. Then $A_{m-1}(x_2^{(i,j)}, \cdots, x_m^{(i,j)}) = A_{m-1}(y_1^{(i,j)}, \cdots, y_{m-1}^{(i,j)})$. Again by inductive hypothesis we can write

$$\det(A(N,(m-1))) = \prod_{i=1}^{m-1} \det(Y_{(i,i+1)}), \tag{11}$$

where $Y_{(i,i+1)}$ is the matrix obtained by putting $x_{i+1}^{(r,s)} - x_{i+2}^{(r,s)} = y_i^{(r,s)} - y_{i+1}^{(r,s)}$ in $X_{(i+1,i+2)}$ for all $1 \le r, s \le N$, and $y_m^{(i,j)} = x_{m+1}^{(i,j)} = 0$. Clearly

$$\prod_{i=1}^{m-1} \det(Y_{(i,i+1)}) = \prod_{i=2}^{m} \det(X_{(i,i+1)}). \tag{12}$$

Therefore, by (9), (10), (11) and (12) we get

$$\det(A(N,m)) = \prod_{i=1}^{m} \det(X_{(i,i+1)}).$$

□

Here we think about the determinantal formulas for the block hook matrix formed by the hook matrices like $B_m(x_1, \cdots, x_m)$. Now we define an $N \times N$ block matrix $B(N, m)$ in the following way;

$$B(N,m) = (B_{ij}), \text{ where } 1 \le i, j \le N \text{ and } B_{ij} = B_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}). \tag{13}$$

So $B(N,m)$ is a block hook matrix of order Nm. For example,

$$B(3,2)=\left(\begin{array}{cc|cc|cc} x_1^{(1,1)} & x_2^{(1,1)} & x_1^{(1,2)} & x_2^{(1,2)} & x_1^{(1,3)} & x_2^{(1,3)} \\ x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)} \\ \hline x_1^{(2,1)} & x_2^{(2,1)} & x_1^{(2,2)} & x_2^{(2,2)} & x_1^{(2,3)} & x_2^{(2,3)} \\ x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)} \\ \hline x_1^{(3,1)} & x_2^{(3,1)} & x_1^{(3,2)} & x_2^{(3,2)} & x_1^{(3,3)} & x_2^{(3,3)} \\ x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)} \end{array}\right). \tag{14}$$

Theorem 2.2. *Let $B(N,m)$ be a block hook matrix of order Nm defined as* (13). *Then*

$$\det(B(N,m))=\prod_{i=1}^{m}\det(X_{(i,i+1)}),$$

where $X_{(i,i+1)}$ is defined by (6) *and $x_{m+1}^{(i,j)}=0$, for all $1\le i,j\le N$.*

Proof. We prove this theorem by performing some row and column operations on $A(N,m)$, so that the resulting matrix is $B(N,m)$. In fact, we perform the following row and column operations;

$$R_{km+i}\leftrightarrow R_{km+(m-i+1)} \text{ and } C_{km+i}\leftrightarrow C_{km+(m-i+1)}, \text{ where } k=0,1,\cdots,(N-1) \text{ and} \tag{15}$$

$$i=\begin{cases}1,2,\cdots,\frac{m}{2}, & \text{if } m \text{ is even}\\ 1,2,\cdots,\frac{m+1}{2}-1, & \text{if } m \text{ is odd}.\end{cases} \tag{16}$$

Obviously, the resulting matrix is $B(N,m)$. Hence the result. □

Now we evaluate the determinant of the block hook matrix $C(N,m)$ formed by the hook matrices of the form $C_m(x_1,\cdots,x_m)$. Let us define the matrix $C(N,m)$ as follows:

$$C(N,m)=(C_{ij}), \text{ where } C_{ij}=C_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}) \text{ and } 1\le i,j\le N. \tag{17}$$

For example,

$$C(3,2)=\left(\begin{array}{cc|cc|cc} x_2^{(1,1)} & x_1^{(1,1)} & x_2^{(1,2)} & x_1^{(1,2)} & x_2^{(1,3)} & x_1^{(1,3)} \\ x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)} \\ \hline x_2^{(2,1)} & x_1^{(2,1)} & x_2^{(2,2)} & x_1^{(2,2)} & x_2^{(2,3)} & x_1^{(2,3)} \\ x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)} \\ \hline x_2^{(3,1)} & x_1^{(3,1)} & x_2^{(3,2)} & x_1^{(3,2)} & x_2^{(3,3)} & x_1^{(3,3)} \\ x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)} \end{array}\right). \tag{18}$$

Theorem 2.3. *Let $C(N,m)$ be a block hook matrix of order Nm defined as* (17). *Then*

$$\det(C(N,m))=\begin{cases}(-1)^{\frac{Nm}{2}}\times\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \textit{if } m \textit{ is even}\\ (-1)^{\frac{N(m-1)}{2}}\times\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \textit{if } m \textit{ is odd,}\end{cases}$$

where $X_{(i,i+1)}$ is defined by (6) *and $x_{m+1}^{(i,j)}=0$ for all $1\le i,j\le N$.*

Proof. If we apply the same row operations on the matrix $A(N,m)$ as in the proof of Theorem 2.2, we get the matrix $C(N,m)$. Hence the theorem. □

In this place we want to establish the determinantal formula for the block hook matrix $D(N,m)$, where $D(N,m)$ is an Nm ordered matrix defined as follows:

$$\text{Now, } D(N,m)=(D_{ij}),\ \text{where } D_{ij}=D_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}) \text{ and } 1\le i,j\le N. \tag{19}$$

For example,

$$D(3,2)=\left(\begin{array}{cc|cc|cc}
x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)}\\
x_1^{(1,1)} & x_2^{(1,1)} & x_1^{(1,2)} & x_2^{(1,2)} & x_1^{(1,3)} & x_2^{(1,3)}\\ \hline
x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)}\\
x_1^{(2,1)} & x_2^{(2,1)} & x_1^{(2,2)} & x_2^{(2,2)} & x_1^{(2,3)} & x_2^{(2,3)}\\ \hline
x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)}\\
x_1^{(3,1)} & x_2^{(3,1)} & x_1^{(3,2)} & x_2^{(3,2)} & x_1^{(3,3)} & x_2^{(3,3)}
\end{array}\right). \tag{20}$$

Theorem 2.4. *Let $D(N,m)$ be a block hook matrix of order Nm defined as* (19). *Then*

$$\det(D(N,m))=\begin{cases}(-1)^{\frac{Nm}{2}}\times\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \text{if } m \text{ is even}\\ (-1)^{\frac{N(m-1)}{2}}\times\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \text{if } m \text{ is odd,}\end{cases}$$

where $X_{(i,i+1)}$ is defined by (6) *and $x_{m+1}^{(i,j)}=0$ for all $1\le i,j\le N$.*

Proof. Applying the same column operations as in the proof of Theorem 2.2 on the matrix $A(N,m)$, we get the matrix $D(N,m)$. This proves the theorem. □

3 Block matrices containing hooks of two different shapes

In this section, we present determinantal formulas for the block hook matrices containing hooks of two different shapes. In fact, we construct four different such block matrices and evaluate their determinants. Let $E(N,m)$ be a block hook matrix defined in the following way;

$$E(N,m)=(E_{ij})_{N\times N},\ \text{where } E_{ij}=\begin{cases}C_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}), & \text{if } i \text{ is even}\\ A_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}), & \text{if } i \text{ is odd .}\end{cases} \tag{21}$$

For example,

$$E(3,2)=\left(\begin{array}{cc|cc|cc}
x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)}\\
x_2^{(1,1)} & x_1^{(1,1)} & x_2^{(1,2)} & x_1^{(1,2)} & x_2^{(1,3)} & x_1^{(1,3)}\\ \hline
x_2^{(2,1)} & x_1^{(2,1)} & x_2^{(2,2)} & x_1^{(2,2)} & x_2^{(2,3)} & x_1^{(2,3)}\\
x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)}\\ \hline
x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)}\\
x_2^{(3,1)} & x_1^{(3,1)} & x_2^{(3,2)} & x_1^{(3,2)} & x_2^{(3,3)} & x_1^{(3,3)}
\end{array}\right). \tag{22}$$

Theorem 3.1. *Let $E(N,m)$ be a block hook matrix of order Nm defined as (21). Then*

$$\det(E(N,m)) = \begin{cases} (-1)^{\frac{Nm}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is even, } m \textit{ is even} \\ (-1)^{\frac{N(m-1)}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is even, } m \textit{ is odd} \\ (-1)^{\frac{(N-1)m}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is odd, } m \textit{ is even} \\ (-1)^{\frac{(N-1)(m-1)}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is odd, } m \textit{ is odd}, \end{cases}$$

where $X_{(i,i+1)}$ is defined as (6) and $x_{m+1}^{(i,j)} = 0$ for all $1 \le i, j \le N$.

Proof. Here we apply the following elementary row operations on the matrix $A(N,m)$;

$$R_{km+i} \leftrightarrow R_{km+(m-i+1)}, \text{ where} \tag{23}$$

$$i = \begin{cases} 1, 2, \cdots, \frac{m}{2}, & \text{if } m \text{ is even} \\ 1, 2, \cdots, \frac{m+1}{2} - 1, & \text{if } m \text{ is odd} \end{cases} \text{ and } k = \begin{cases} 1, 3, \cdots, N-1, & \text{if } N \text{ is even} \\ 1, 3, \cdots, N-2, & \text{if } N \text{ is odd}. \end{cases} \tag{24}$$

Evidently the resulting matrix is $E(N,m)$. Hence the theorem. □

Let us introduce a block hook matrix $E'(N,m)$ in the following way;

$$E'(N,m) = (E'_{ij})_{N\times N}, \text{ where } E'_{ij} = \begin{cases} C_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i \text{ is odd} \\ A_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i \text{ is even}. \end{cases} \tag{25}$$

For example,

$$E'(3,2) = \left(\begin{array}{cc|cc|cc} x_2^{(1,1)} & x_1^{(1,1)} & x_2^{(1,2)} & x_1^{(1,2)} & x_2^{(1,3)} & x_1^{(1,3)} \\ x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)} \\ \hline x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)} \\ x_2^{(2,1)} & x_1^{(2,1)} & x_2^{(2,2)} & x_1^{(2,2)} & x_2^{(2,3)} & x_1^{(2,3)} \\ \hline x_2^{(3,1)} & x_1^{(3,1)} & x_2^{(3,2)} & x_1^{(3,2)} & x_2^{(3,3)} & x_1^{(3,3)} \\ x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)} \end{array}\right). \tag{26}$$

Theorem 3.2. *Let $E'(N,m)$ be a block hook matrix of order Nm defined as (25). Then*

$$\det(E'(N,m)) = \begin{cases} (-1)^{\frac{Nm}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is even, } m \textit{ is even} \\ (-1)^{\frac{N(m-1)}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is even, } m \textit{ is odd} \\ (-1)^{\frac{(N-1)m}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is odd, } m \textit{ is even} \\ (-1)^{\frac{(N-1)(m-1)}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \textit{if } N \textit{ is odd, } m \textit{ is odd}, \end{cases}$$

where $X_{(i,i+1)}$ is defined as (6) and $x_{m+1}^{(i,j)} = 0$ for all $1 \le i, j \le N$.

Proof. Applying the same row operations described by (23) and (24) on the matrix $C(N,m)$, we get the matrix $E'(N,m)$. This completes the proof. □

Here, we define another block hook matrix $F(N,m)$ as follows:

$$F(N,m)=(F_{ij})_{N\times N},\ \text{where } F_{ij}=\begin{cases} B_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}), & \text{if } i \text{ is odd}\\ D_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}), & \text{if } i \text{ is even}\ .\end{cases} \tag{27}$$

For example:

$$F(3,2)=\left(\begin{array}{cc|cc|cc} x_1^{(1,1)} & x_2^{(1,1)} & x_1^{(1,2)} & x_2^{(1,2)} & x_1^{(1,3)} & x_2^{(1,3)}\\ x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)}\\ \hline x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)}\\ x_1^{(2,1)} & x_2^{(2,1)} & x_1^{(2,2)} & x_2^{(2,2)} & x_1^{(2,3)} & x_2^{(2,3)}\\ \hline x_1^{(3,1)} & x_2^{(3,1)} & x_1^{(3,2)} & x_2^{(3,2)} & x_1^{(3,3)} & x_2^{(3,3)}\\ x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)} \end{array}\right). \tag{28}$$

Theorem 3.3. *Let $F(N,m)$ be a block hook matrix of order Nm defined as* (27). *Then*

$$\det(F(N,m))=\begin{cases} (-1)^{\frac{Nm}{4}}\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \textit{if } N \textit{ is even, } m \textit{ is even}\\ (-1)^{\frac{N(m-1)}{4}}\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \textit{if } N \textit{ is even, } m \textit{ is odd}\\ (-1)^{\frac{(N-1)m}{4}}\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \textit{if } N \textit{ is odd, } m \textit{ is even}\\ (-1)^{\frac{(N-1)(m-1)}{4}}\prod_{i=1}^{m}\det(X_{(i,i+1)}), & \textit{if } N \textit{ is odd, } m \textit{ is odd}\ , \end{cases}$$

where $X_{(i,i+1)}$ is defined as (6) *and $x_{m+1}^{(i,j)}=0$ for all $1\le i,j\le N$.*

Proof. Here we apply the same row operations described by (23) and (24) on the matrix $B(N,m)$. Evidently the resulting matrix is $F(N,m)$. Hence the theorem. □

Let us define another block hook matrix $F'(N,m)$ as follows:

$$F'(N,m)=(F'_{ij})_{N\times N},\ \text{where } F'_{ij}=\begin{cases} D_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}), & \text{if } i \text{ is odd}\\ B_m(x_1^{(i,j)},\cdots,x_m^{(i,j)}), & \text{if } i \text{ is even}\ .\end{cases} \tag{29}$$

For example,

$$F'(3,2)=\left(\begin{array}{cc|cc|cc} x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)}\\ x_1^{(1,1)} & x_2^{(1,1)} & x_1^{(1,2)} & x_2^{(1,2)} & x_1^{(1,3)} & x_2^{(1,3)}\\ \hline x_1^{(2,1)} & x_2^{(2,1)} & x_1^{(2,2)} & x_2^{(2,2)} & x_1^{(2,3)} & x_2^{(2,3)}\\ x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)}\\ \hline x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)}\\ x_1^{(3,1)} & x_2^{(3,1)} & x_1^{(3,2)} & x_2^{(3,2)} & x_1^{(3,3)} & x_2^{(3,3)} \end{array}\right). \tag{30}$$

Theorem 3.4. *Let $F'(N,m)$ be a block hook matrix of order Nm defined as* (29). *Then*

$$\det(F'(N,m)) = \begin{cases} (-1)^{\frac{Nm}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \text{if } N \text{ is even, } m \text{ is even} \\ (-1)^{\frac{N(m-1)}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \text{if } N \text{ is even, } m \text{ is odd} \\ (-1)^{\frac{(N-1)m}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \text{if } N \text{ is odd, } m \text{ is even} \\ (-1)^{\frac{(N-1)(m-1)}{4}} \prod_{i=1}^{m} \det(X_{(i,i+1)}), & \text{if } N \text{ is odd, } m \text{ is odd}, \end{cases}$$

where $X_{(i,i+1)}$ is defined as (6) *and $x_{m+1}^{(i,j)} = 0$ for all $1 \le i, j \le N$.*

Proof. In this case also we apply the same row operations described by (23) and (24) on the matrix $D(N,m)$, and we get the matrix $F'(N,m)$. This completes the proof. □

4 Block matrices containing hooks of four different shapes

In this section, we are going to derive the determinantal formulas of block hook matrices, in which the blocks are suitable combination of four different shapes of hooks. In particular, here we construct two such block matrices and show that the determinants of these matrices are also product of determinants of matrices $X_{(i,i+1)}$, $(i = 1, 2, \cdots, m)$ (defined as (6)). So let us define a block hook matrix $G(N,m)$ as follows:

$$G(N,m) = (G_{ij})_{N\times N}, \text{ where } G_{ij} = \begin{cases} B_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i,j \text{ are odd} \\ C_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i \text{ is odd, } j \text{ is even} \\ D_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i \text{ is even, } j \text{ is odd} \\ A_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i,j \text{ are even}. \end{cases} \tag{31}$$

For example,

$$G(3,2) = \left(\begin{array}{cc|cc|cc} x_1^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_1^{(1,2)} & x_1^{(1,3)} & x_2^{(1,3)} \\ x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)} \\ \hline x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)} \\ x_1^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_1^{(2,2)} & x_1^{(2,3)} & x_2^{(2,3)} \\ \hline x_1^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_1^{(3,2)} & x_1^{(3,3)} & x_2^{(3,3)} \\ x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)} \end{array}\right). \tag{32}$$

Theorem 4.1. *Let $G(N,m)$ be a block hook matrix of order Nm defined as* (31). *Then*

$$\det(G(N,m)) = \prod_{i=1}^{m} \det(X_{(i,i+1)}),$$

where $X_{(i,i+1)}$ is defined by (6) *and $x_{m+1}^{(i,j)} = 0$ for all $1 \le i, j \le N$.*

Proof. Here we apply the same row and column operations described by (15) and (16) on $B(N,m)$. Clearly after the effect of these row and column operations we get the matrix $G(N,m)$. Consequently we get the result. □

Let us introduce one more block hook matrix $G'(N, m)$ containing four different shape of hooks as follows:

$$G'(N,m) = (G'_{ij})_{N\times N}, \text{ where } G'_{ij} = \begin{cases} A_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i,j \text{ are odd} \\ D_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i \text{ is odd, } j \text{ is even} \\ C_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i \text{ is even , } j \text{ is odd} \\ B_m(x_1^{(i,j)}, \cdots, x_m^{(i,j)}), & \text{if } i,j \text{ are even .} \end{cases} \tag{33}$$

For example,

$$G'(3,2) = \left(\begin{array}{cc|cc|cc} x_2^{(1,1)} & x_2^{(1,1)} & x_2^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_2^{(1,3)} \\ x_2^{(1,1)} & x_1^{(1,1)} & x_1^{(1,2)} & x_2^{(1,2)} & x_2^{(1,3)} & x_1^{(1,3)} \\ \hline x_2^{(2,1)} & x_1^{(2,1)} & x_1^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_1^{(2,3)} \\ x_2^{(2,1)} & x_2^{(2,1)} & x_2^{(2,2)} & x_2^{(2,2)} & x_2^{(2,3)} & x_2^{(2,3)} \\ \hline x_2^{(3,1)} & x_2^{(3,1)} & x_2^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_2^{(3,3)} \\ x_2^{(3,1)} & x_1^{(3,1)} & x_1^{(3,2)} & x_2^{(3,2)} & x_2^{(3,3)} & x_1^{(3,3)} \end{array}\right). \tag{34}$$

Theorem 4.2. *Let $G'(N, m)$ be a block hook matrix of order Nm defined as* (33). *Then*

$$\det(G'(N,m)) = \prod_{i=1}^{m} \det(X_{(i,i+1)}),$$

where $X_{(i,i+1)}$ is defined by (6) *and $x_{m+1}^{(i,j)} = 0$ for all $1 \le i, j \le N$.*

Proof. Applying the same row and column operations depicted by (15) and (16) on the matrix $A(N, m)$, we get the matrix $G'(N, m)$. Hence the theorem. □

5 Combinatorial interpretations of hook determinants

Combinatorial interpretations of determinants can bring deeper understanding to their evaluations; this is especially true when the entries of a matrix have natural graph theoretic descriptions [2, 3, 5, 9, 15]. In this section, we give combinatorial interpretations of our main results (that are stated in previous sections) regarding hook determinants. Before plunging into the proof, let us recall the celebrated "Gessel-Lindström-Viennot" lemma. See [1, 5], for details. For the sake of completeness, let us reproduce the lemma from [1]. Let Γ be a weighted, acyclic digraph. The vertex set and the edge set of the graph Γ, denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively. A path in Γ is a sequence of distinct vertices $v_1, v_2, \cdots, v_r$ such that $v_i, v_{i+1} (i = 1, \cdots, r-1)$ is an edge directed from v_i to v_{i+1}. For simplicity we denote a path $v_1, v_2, \cdots, v_r$ by $v_1 v_2 \cdots v_r$ and hence an edge v_i, v_{i+1} by $v_i v_{i+1}$. The *weight* of a path P, denoted by $w(P)$ is the product of weights of all edges involved in the path and the *length* of a path P, denoted by $\ell(P)$, is the number of edges involved in the path P. Suppose that $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_n\}$ are two n-sets of vertices of Γ (not necessarily disjoint). To U and V, associate the *path matrix* $M = (m_{ij})_{n\times n}$, where $m_{ij} = \sum_{P:u_i\to v_j} w(P)$, $P : u_i \to v_j$ denotes a path from u_i to v_j. A *path system* from U to V is an ordered pair $(\mathcal{P}, \sigma)$, where σ is a permutation of n element set and $\mathcal{P}$ is a set of n paths $P_i : u_i \to v_{\sigma(i)}$. The sign of a path system $(\mathcal{P}, \sigma)$ is $\operatorname{sgn}(\sigma)$. The *weight* of $(\mathcal{P}, \sigma)$, denoted by $w(\mathcal{P}, \sigma)$ is $\prod_{i=1}^{n} w(P_i)$. We call the path system *vertex-disjoint* if no two paths have a common vertex. Let VD_Γ be the family of vertex-disjoint path systems in the graph Γ. Then the Gessel-Lindström-Viennot lemma is the following;

Lemma 5.1 (Gessel-Lindström-Viennot lemma)**.** *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a weighted acyclic digraph. Suppose $U = \{u_1, u_2, \cdots, u_n\}$, $V = \{v_1, v_2, \cdots, v_n\}$ are two n-sets of vertices of Γ (not necessarily disjoint) and M be a path matrix from U to V. Then*

$$\det(M) = \sum_{(\mathcal{P},\sigma)\in VD_\Gamma} sgn(\mathcal{P}, \sigma)w(\mathcal{P}, \sigma).$$

Now we give combinatorial proofs of theorems stated in previous sections using Lemma 5.1.

Combinatorial proof of Theorem 1.1. Consider the acyclic weighted digraph in Figure 2. We choose the sets $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_m\}$ as the initial and terminal sets of vertices respectively. Clearly the associate path matrix is the matrix $A_m(x_1, \cdots, x_m)$ defined as (1). Evidently, there is only one vertex disjoint path system $(\mathcal{P}, \mathrm{Id})$, where Id is the identity permutation of m element set and $\mathcal{P}$ is a set of m paths $P_i : u_i \to v_i (i = 1, \cdots, m)$. Now applying Lemma 5.1 we get the theorem. □

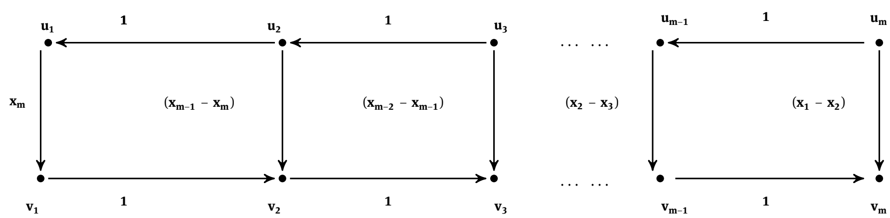

Figure 2: A weighted acyclic digraph Γ_m, and the weight of each edge is described on figure.

Remark 2. *In exactly the same way we can give combinatorial explanation of theorems 1.2, 1.3 and 1.4. In those cases also we use Γ_m as combinatorial object possibly permuting some of its vertices.*

Combinatorial proof of Theorem 2.1. To give a combinatorial interpretation of the theorem we use Gessel-Lindström-Viennot lemma. So first we have to construct a weighted acyclic digraph $\Gamma_{N,m}$ whose path matrix is the matrix $A(N, m)$ (defined as (7)). For each $i \in [m]$, let us define $\tilde{U}_i = \{u_i, u_{m+i}, \cdots, u_{(N-2)m+i}, u_{(N-1)m+i}\}$ and $\tilde{V}_i = \{v_i, v_{m+i}, \cdots, v_{(N-2)m+i}, v_{(N-1)m+i}\}$. We declare the vertex set $V(\Gamma_{N,m}) = \bigcup_{i\in[m]} (\tilde{U}_i \cup \tilde{V}_i)$. Now in $\Gamma_{N,m}$, we want the vertices $\tilde{U}_i \cup \tilde{V}_i$ to form a weighted complete bipartite digraph $\Gamma_i \cong K_{N,N}$ with bi-partition sets $\tilde{U}_i$ and $\tilde{V}_i$ for each $i \in [m]$. The direction of each edge $u_k v_\ell (k = tm + i, \ell = rm + i$, where $t, \ell \in \{0, \cdots, N-1\})$ is always taken from u_k towards v_ℓ and $w(u_k v_\ell) = x^{(t+1,r+1)}_{m-i+1} - x^{(t+1,r+1)}_{m-i+2}$. Clearly each Γ_i is acyclic. Now we finish the construction of $\Gamma_{N,m}$ by adjoining additional weighted directed edges to $\cup_{i=1}^{m} E(\Gamma_i)$, as depicted in Figure 3 Notice that $\Gamma_{N,m}$ is acyclic. Now we call $U = \{u_1, u_2, \cdots, u_{Nm}\}$ to be the initial set of vertices and $V = \{v_1, v_2, \cdots, v_{Nm}\}$ to be the terminal set of vertices of the graph $\Gamma_{N,m}$. Then the path matrix of the graph $\Gamma_{N,m}$ is the matrix $A(N, m)$.

So, by Gessel-Lindström-Viennot lemma, we can write

$$\det(A(N, m)) = \sum_{\mathcal{P}_{\Gamma_{N,m}} \in VD_{\Gamma_{N,m}}} \mathrm{sgn}(\mathcal{P}_{\Gamma_{N,m}})w(\mathcal{P}_{\Gamma_{N,m}}). \quad (35)$$

Now we have to characterize all the vertex disjoint path systems in the graph $\Gamma_{N,m}$. The following lemmas give the complete characterization of all vertex disjoint path systems in $\Gamma_{N,m}$.

Lemma 5.2. *Let $\mathcal{P}_{\Gamma_{N,m}} \in VD_{\Gamma_{N,m}}$ and P be a path in $\mathcal{P}_{\Gamma_{N,m}}$. Then $\ell(P) = 1$.*

Proof. Let P be a path in $\mathcal{P}_{\Gamma_{N,m}}$ from the initial vertex u_ℓ to the terminal vertex v_k, for some $u_\ell \in U$ and $v_k \in V$. Then we show that the length of the path P is 1. In fact, length of $P > 1$ implies, P contains at least three distinct vertices. Then P is of the form either $u_\ell u_i \cdots v_k$ or $u_\ell v_j \cdots v_k$, for some $u_i \in U$ and $v_j \in V$,

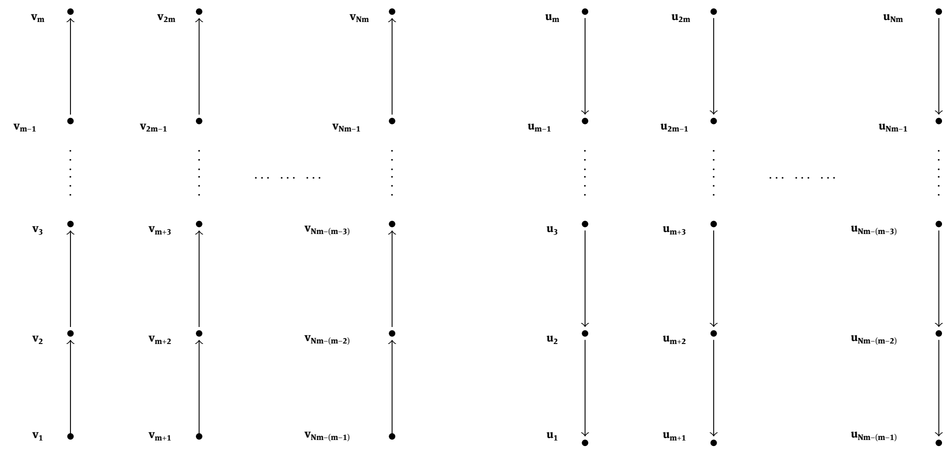

Figure 3: A weighted acyclic digraph with $w(u_{j+1}u_j) = 1 = w(v_jv_{j+1})$, for $i = 1, \cdots, Nm - 1$.

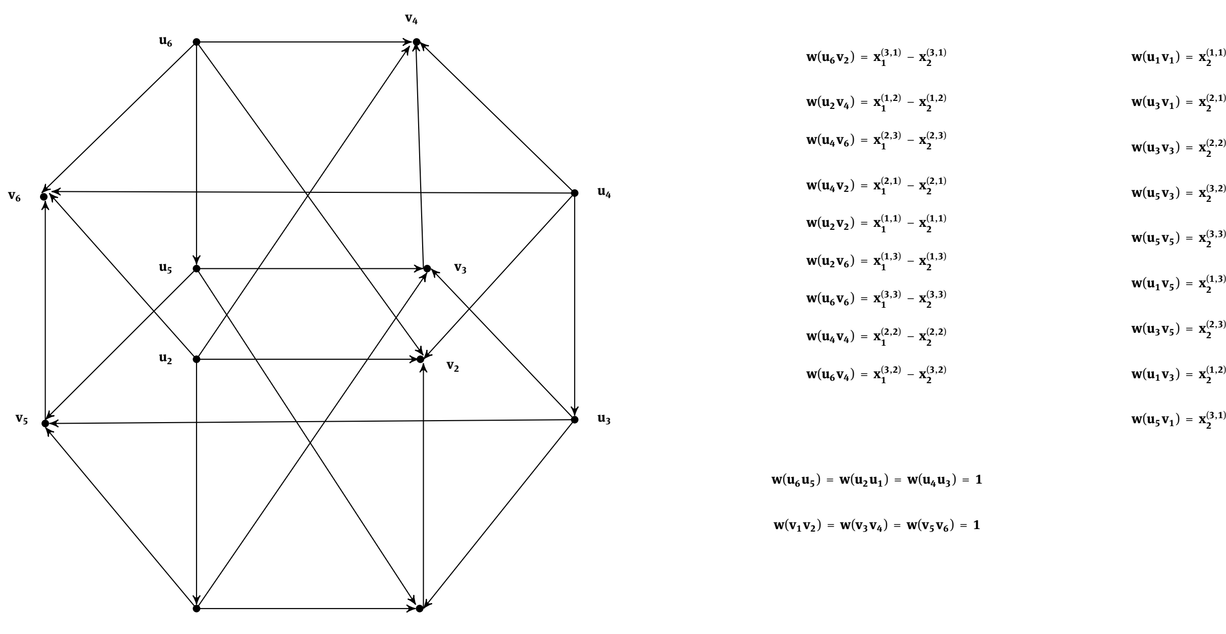

Figure 4: A weighted acyclic digraph $\Gamma_{3,2}$. The weight of the edges appear on right side of this figure.

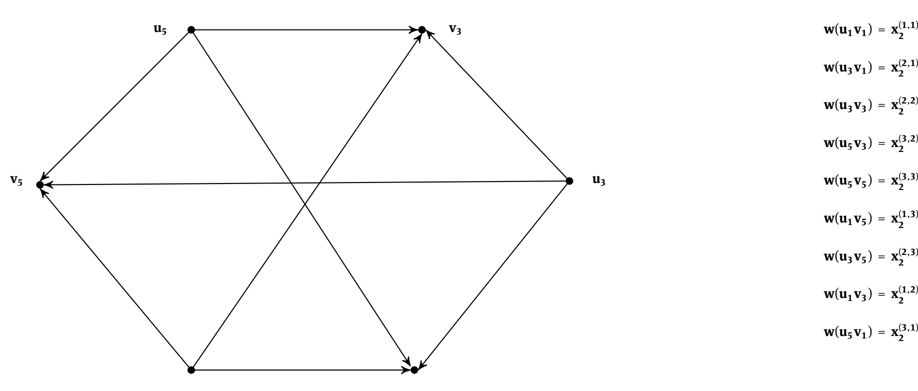

Figure 5: A weighted acyclic digraph Γ_1. The weight of the corresponding edges appear on figure.

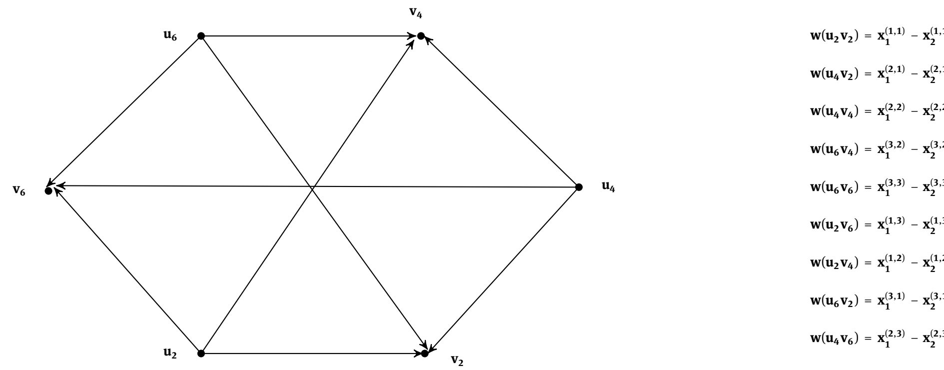

Figure 6: A weighted acyclic digraph Γ_2. The weight of the corresponding edges appear on figure.

i.e, the second vertex of P (a vertex next to the initial vertex in P) is either u_i or v_j. Suppose P is $u_\ell u_i \cdots v_k$. Since $\mathcal{P}_{\Gamma_{N,m}}$ is a path system from U to V, we must have another path $\acute{P}$ in $\mathcal{P}_{\Gamma_{N,m}}$, whose initial vertex is u_i. This contradicts that the path system $\mathcal{P}_{\Gamma_{N,m}}$ is vertex disjoint. Again if P is $u_\ell v_j \cdots v_k$, then in $\mathcal{P}_{\Gamma_{N,m}}$ we get another path $\tilde{P}$ whose terminal vertex is v_j, which also contradicts the fact $\mathcal{P}_{\Gamma_{N,m}}$ is vertex disjoint. Therefore, the length of each path in any vertex disjoint path system must be 1. □

In this portion we will think about the vertex disjoint path systems in subgraphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_m$ of the graph $\Gamma_{N,m}$. For each $\Gamma_i (i = 1, \cdots, m)$ we choose $\tilde{U}_i$, $\tilde{V}_i$ as the initial and terminal set of vertices respectively. It can be shown that the path matrix of the graph Γ_i is the matrix $X_{(m-i+1,m-i+2)}$, defined as (6). From our construction of the graph $\Gamma_{N.m}$, it is evident that

$$VD_{\Gamma_{N,m}} = \{(\mathcal{P}, \sigma) = (\mathcal{P}^1 \cup \mathcal{P}^2 \cup \cdots \cup \mathcal{P}^m, \sigma_1\sigma_2 \cdots \sigma_m),\ \text{where } (\mathcal{P}^i, \sigma_i) \text{ is a path system of } \Gamma_i\}.$$

Therefore, we can write

$$\sum_{(\mathcal{P},\sigma)\in VD_{\Gamma_{N,m}}} \operatorname{sgn}(\mathcal{P}, \sigma) w(\mathcal{P}, \sigma) =$$
$$\left(\sum_{(\mathcal{P}^1,\sigma_1)\in VD_{\Gamma_1}} \operatorname{sgn}(\mathcal{P}^1, \sigma_1) w(\mathcal{P}^1, \sigma_1) \right) \cdots \left(\sum_{(\mathcal{P}^m,\sigma_m)\in VD_{\Gamma_m}} \operatorname{sgn}(\mathcal{P}^m, \sigma_m) w(\mathcal{P}^m, \sigma_m) \right).$$

Now, $\left(\sum_{(\mathcal{P}^i,\sigma_i)\in VD_{\Gamma_i}} \operatorname{sgn}(\mathcal{P}^i, \sigma_i) w(\mathcal{P}^i, \sigma_i) \right) = \det(X_{(m-i+1,m-i+2)})$, where $i \in [m]$ and $X_{(m-i+1,m-i+2)}$ is the path matrix associated to the graph Γ_i. Hence the theorem. □

Example 5.3. *If we take $N = 3, m = 2$, then the weighted acyclic digraph $\Gamma_{3,2}$ is Figure 4 and Γ_1, Γ_2 are Figures 5, 6 respectively. Moreover the path matrix associated to $\Gamma_{3,2}$ is $A(3, 2)$ defined as (8). Consider a vertex disjoint path system $u_1v_1, u_2v_2, \cdots, u_6v_6$ in Γ, where each path $u_iv_i (i = 1, \cdots, 6)$ is an edge. Notice that, three paths u_1v_1, u_3v_3, u_5v_5 from the above path system is a vertex disjoint path system in Γ_1 and three another paths u_2v_2, u_4v_4, u_6v_6 is a vertex disjoint path system in Γ_2. Again $u_1v_5, u_2v_2, u_3v_1, u_4v_6, u_5v_3, u_6v_4$ is vertex disjoint path system in Γ. For this vertex disjoint path system, u_1v_5, u_5v_3, u_3v_1 is a vertex disjoint path system in Γ_1 and u_2v_2, u_4v_6, u_6v_4 is a vertex disjoint path system in Γ_2.*

Remark 3. *In exactly the same way we can give combinatorial explanation of all other theorems. In those cases also we use $\Gamma_{N,m}$ as combinatorial object possibly permuting some of its vertices.*

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