Research Article

Special Issue: Combinatorial Matrices

Open Access

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On the smallest singular value in the class of unit lower triangular matrices with entries in $[-a, a]$

<https://doi.org/10.1515/spma-2020-0139>

Received October 31, 2020; accepted May 24, 2021

Abstract: Given a real number $a \geq 1$, let $K_n(a)$ be the set of all $n \times n$ unit lower triangular matrices with each element in the interval $[-a, a]$. Denoting by $\lambda_n(\cdot)$ the smallest eigenvalue of a given matrix, let $c_n(a) = \min \{ \lambda_n(YY^T) : Y \in K_n(a) \}$. Then $\sqrt{c_n(a)}$ is the smallest singular value in $K_n(a)$. We find all minimizing matrices. Moreover, we study the asymptotic behavior of $c_n(a)$ as $n \rightarrow \infty$. Finally, replacing $[-a, a]$ with $[a, b]$, $a \leq 0 < b$, we present an open question: Can our results be generalized in this extension?

Keywords: real symmetric matrix, unit lower triangular matrix, GCD and LCM matrix, smallest eigenvalue, smallest singular value

MSC: 11C20, 15A18, 15A42, 15B36

1 Introduction

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers and let (x_i, x_j) and $[x_i, x_j]$ denote the greatest common divisor and the least common multiply of x_i and x_j , respectively. The $n \times n$ matrices $(S) = ((x_i, x_j))$ and $[S] = ([x_i, x_j])$ are called the GCD matrix and the LCM matrix on S , respectively. Many results on these matrices, their various generalizations and relatives have been published in the literature. For general accounts, see [2–4, 6, 8, 12] and the references therein.

One of the richest topics in the study of GCD and LCM matrices is their spectral properties. In this frame, Hong and Loewy [6] studied the asymptotic behavior of the eigenvalues of power GCD matrices and introduced a new parameter to present a lower bound for the smallest eigenvalue of the power GCD matrix. They defined the numbers c_n depending only on n as follows:

$$c_n = \min \{ \lambda_n(YY^T) : Y \in K_n \},$$

where $\lambda_n(\cdot)$ is the smallest eigenvalue and K_n is the set of all $n \times n$ nonsingular lower triangular $(0, 1)$ -matrices. Then, in the light of their MATLAB calculations for $n = 2, 3, \dots, 7$, Ilmonen, Haukkanen and Merikoski [8] conjectured that $c_n(a) = \lambda_n(Y_0 Y_0^T)$, where $Y_0 = (y_{ij}^0) \in K_n$ is defined by $y_{ij}^0 = \frac{1 - (-1)^{i+j}}{2}$ if $i > j$. Verifying the truth of the conjecture for $n = 8$ and 9 in [1], the author of the present paper, Keskin, Yıldız and Demirbüken [4] proved the conjecture and realized that there is only one matrix $Y \in K_n$ for which c_n is attained. Therefore, it was conjectured that if $c_n = \lambda_n(YY^T)$ for $Y \in K_n$, then $Y = Y_0$ [4, Conjecture 3.1]. Recently, Loewy [11] has proved the conjecture.

Due to the importance of the study of spectral properties of GCD and related matrices, some authors tried to find bounds for c_n . In this frame, assuming the truth of the conjecture, Mattila [12] obtained the following

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lower bound

$$c_n \geq \left(\frac{48}{n^4 + 56n^2 + 48n} \right)^{\frac{n-1}{2}} \text{ for even } n,$$

$$c_n \geq \left(\frac{48}{n^4 + 50n^2 + 48n - 51} \right)^{\frac{n-1}{2}} \text{ for odd } n.$$

Beside this, Merikoski, who contributed to the paper [4], improved Mattila's lower bound as follows:

$$c_n \geq \frac{2}{2F_n F_{n+1} + 1 + (-1)^n},$$

where F_n is the n th Fibonacci number. In addition to these bounds, in [2] the inequality

$$c_n \geq \left(n \sum_{k=1}^n \mu^2(k) \right)^{-1},$$

where μ is the Möbius function, was proposed for c_n by using a different method. Kaarnioja [9] improved the lower bound further and showed that

$$c_n \geq \left(\frac{1}{25} \varphi^{-4n} + \frac{3 + (-1)^n}{25} \varphi^{-2n} - \frac{2}{5\sqrt{5}} n \varphi^{-2n} + \frac{13(-1)^n - 33}{50} + n + \frac{3 + (-1)^n}{25} \varphi^{2n} + \frac{2}{5\sqrt{5}} n \varphi^{2n} + \frac{1}{25} \varphi^{4n} \right)^{-\frac{1}{2}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. He also conjectured asymptotically that

$$c_n \sim \frac{5}{\varphi^{2n}},$$

which was proved by Loewy [11].

This paper has two main goals. The first is to expand the existing results on the process of finding all matrices for which $\sqrt{c_n}$ is attained. The second is to determine the asymptotic behaviour of c_n in a larger class of matrices. Given a real number $a \geq 1$, let $K_n(a)$ denote the set of all $n \times n$ unit lower triangular matrices whose each element under diagonal is in the interval $[-a, a]$. We note that all diagonal entries of a unit lower triangular matrix are 1. Let

$$c_n(a) = \min \left\{ \lambda_n(Y Y^T) : Y \in K_n(a) \right\}, \quad (1)$$

where $\lambda_n(\cdot)$ is the smallest eigenvalue. It is clear that $\sqrt{c_n(a)}$ is the smallest singular value in $K_n(a)$. Let $Y_0 = (y_{ij}^0) \in K_n(a)$ be defined by

$$y_{ij}^0 = (-1)^{i+j+1} a \quad (2)$$

for $i > j$. In Section 2, we present a sharp upper bound for the absolute values of entries of Y^{-1} and $(Y Y^T)^{-1}$, where $Y \in K_n(a)$, and obtain all the entries of Y_0^{-1} and $(Y_0 Y_0^T)^{-1}$ in terms of a . In Section 3, we prove that $c_n(a) = \lambda_n(Y_0 Y_0^T)$. We also show that if $c_n(a) = \lambda_n(Y Y^T)$ for $Y \in K_n(a)$ then $Y = E Y_0 E$, where $E = \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ for $i = 1, 2, \dots, n$. Moreover, we determine the asymptotic behavior of $c_n(a)$. In other words, we show that $c_n(a) \sim \frac{(a+2)^2}{(a+1)^{2n-1}}$ as $n \rightarrow \infty$. Finally, we present concluding remarks in Section 4, including some open problems related to a certain generalization of the results on c_n and $c_n(a)$.

2 Preliminaries

In order to find the bound mentioned in the introduction, we present five lemmas here. In the first lemma, we obtain a recurrence relation for the entries of inverses of matrices in $K_n(a)$.

Lemma 1. Let $Y \in K_n(a)$, $W = (w_{ij}) = Y - I$ and $Y^{-1} = (b_{ij})$. Then,

$$b_{ij} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i = j, \\ -\sum_{k=j}^{i-1} w_{ik} b_{kj} & \text{if } i > j. \end{cases}$$

Proof. See [4, Lemma 2.2]. The set of matrices is different, but the proof works also here. $\square$

Lemma 2. [5] Let Y and Y^{-1} be as in Lemma 1. Then,

$$|b_{ij}| \leq a(a+1)^{i-j-1}$$

for $1 \leq j < i \leq n$.

Proof. Let $W = (w_{ij})$ be as in Lemma 1. Let $j = 1, 2, \dots, n-1$. We prove by induction on $t = 1, 2, \dots, n-j$ that $|b_{j+t,j}| \leq a(a+1)^{t-1}$. By Lemma 1, we have $|b_{j+1,j}| = |w_{j+1,j}|$, where $w_{j+1,j} \in [-a, a]$. Thus, $|b_{j+1,j}| \leq a$. Now assume that the inequality $|b_{j+t,j}| \leq a(a+1)^{t-1}$ holds for each $t = 1, 2, \dots, k-1$. We show that $|b_{j+k,j}| \leq a(a+1)^{k-1}$. By Lemma 1 and the induction hypothesis, we have

$$\begin{aligned} |b_{j+k,j}| &= \left| \sum_{i=j}^{j+k-1} w_{j+k,i} b_{ij} \right| \\ &\leq \sum_{i=j}^{j+k-1} |w_{j+k,i}| |b_{ij}| \\ &\leq a \sum_{i=j}^{j+k-1} |b_{ij}| \\ &\leq a \left(1 + \sum_{i=j+1}^{j+k-1} a(a+1)^{i-j-1} \right) \\ &= a(a+1)^{k-1}. \end{aligned}$$

The induction principle completes the proof. $\square$

Lemma 3. [5] Let Y_0 be as in (2) and $Y_0^{-1} = (c_{ij})$. Then

$$c_{ij} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i = j, \\ (-1)^{i-j} a(a+1)^{i-j-1} & \text{if } i > j. \end{cases}$$

Proof. Let $W_0 = (w_{ij}^0) = Y_0 - I$. Then

$$w_{ij}^0 = \begin{cases} 0 & \text{if } i \leq j, \\ (-1)^{i+j+1} a & \text{otherwise.} \end{cases}$$

Since $Y_0 \in K_n(a)$, we have by Lemma 1

$$c_{ij} = - \sum_{k=j}^{i-1} w_{ik}^0 c_{kj}$$

for $i > j$.

Next, we show by induction on $t = i - j$ that

$$c_{ij} = (-1)^{i-j} a(a+1)^{i-j-1}, \quad i > j. \quad (3)$$

If $t = 1$, then

$$c_{j+1,j} = -w_{j+1,j}^0 = -a.$$

Now assume that (3) holds for $t = 1, 2, \dots, k-1$. Then

$$c_{j+k,j} = - \sum_{s=j}^{j+k-1} w_{j+k,s}^0 c_{sj} = (-1)^k a + \sum_{s=j+1}^{j+k-1} (-1)^{j+k+s} a(-1)^{s-j} a(a+1)^{s-j-1} = (-1)^k a(a+1)^{k-1}.$$

The induction principle completes the proof. $\square$

In the following, we denote by $(\cdot)_{ij}$ the ij -th entry of a given matrix.

Lemma 4. [5] Let Y_0 be as in (2) and $Z_0 = Y_0 Y_0^T$. Then,

$$(Z_0^{-1})_{ij} = \begin{cases} 1 + \frac{a^2}{(a+1)^2-1} ((a+1)^{2(n-i)} - 1) & \text{if } i = j, \\ (-1)^{|i-j|} \left(a(a+1)^{|i-j|-1} + \frac{a^2}{(a+1)^2-1} (a+1)^{|i-j|} ((a+1)^{2(n-\max(i,j))} - 1) \right) & \text{if } i \neq j. \end{cases}$$

Proof. By Lemma 3, we have

$$(Z_0^{-1})_{ii} = 1 + \sum_{k=i+1}^n a^2 (a+1)^{2(k-i-1)} = 1 + \frac{a^2}{(a+1)^2-1} ((a+1)^{2(n-i)} - 1) \quad (4)$$

for all $i = 1, 2, \dots, n$ and

$$\begin{aligned} (Z_0^{-1})_{ij} &= c_{ji} + \sum_{t=j+1}^n c_{ti} c_{tj} \\ &= (-1)^{j-i} \left(a(a+1)^{j-i-1} + \sum_{t=j+1}^n a^2 (a+1)^{2t-i-j-2} \right) \\ &= (-1)^{j-i} \left(a(a+1)^{j-i-1} + a^2 (a+1)^{j-i} \frac{(a+1)^{2(n-j)} - 1}{(a+1)^2 - 1} \right) \end{aligned} \quad (5)$$

for $1 \leq i < j \leq n$. Since Z_0 is symmetric, the claim follows. $\square$

Lemma 5. [5] Let $Z = YY^T$, $Y \in K_n(a)$, and let Z_0 be as in Lemma 4. Then $|Z^{-1}| \leq |Z_0^{-1}|$, that is $|(Z^{-1})_{ij}| \leq |(Z_0^{-1})_{ij}|$ for all $1 \leq i, j \leq n$.

Proof. Let Y^{-1} be as in Lemma 1. Then, by Lemmas 2 and 4,

$$\begin{aligned} |(Z^{-1})_{ii}| &= \left| \sum_{k=1}^n b_{ki}^2 \right| \\ &= \sum_{k=1}^n |b_{ki}|^2 \\ &= \sum_{k=i}^n |b_{ki}|^2 \\ &\leq 1 + \sum_{k=i+1}^n a^2 (a+1)^{2(k-i-1)} \\ &= |(Z_0^{-1})_{ii}| \end{aligned} \quad (6)$$

for all $i = 1, 2, \dots, n$. Let $i < j$. Again, by Lemmas 2 and 4,

$$\begin{aligned} |(Z^{-1})_{ij}| &= \left| \sum_{t=1}^n b_{ti} b_{tj} \right| \\ &\leq \sum_{t=1}^n |b_{ti}| |b_{tj}| \\ &= |b_{ji}| + \sum_{t=j+1}^n |b_{ti}| |b_{tj}| \\ &\leq a(a+1)^{j-i-1} + \sum_{t=j+1}^n a^2 (a+1)^{2t-i-j-2} \end{aligned} \quad (7)$$

$$= |(Z_0^{-1})_{ij}|.$$

Since Z^{-1} and Z_0^{-1} are symmetric, we obtain $|Z^{-1}| \leq |Z_0^{-1}|$.

□

3 Main Results

Theorem 1. Let Y_0 be as in (2). Then $c_n(a) = \lambda_n(Y_0 Y_0^T)$. Conversely, if $Y \in K_n(a)$ and $\lambda_n(Y Y^T) = c_n(a)$, then $Y = E Y_0 E$, where $E = \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ for $i = 1, 2, \dots, n$.

Proof. For the first claim, we proceed as in the proof [4] of the Ilmonen-Haukkanen-Merikoski conjecture. Let $Z_0 = Y_0 Y_0^T$. Then, by (4) and (5), $\text{tr} Z_0^{-k} = \text{tr} |Z_0^{-1}|^k$ for all positive integers k . By Newton's identities [10], one can easily see that Z_0^{-1} and $|Z_0^{-1}|$ have the same characteristic polynomial. Since they are real symmetric matrices, their spectral radii are equal, that is, $\rho(|Z_0^{-1}|) = \rho(Z_0^{-1})$. Let $Y \in K_n(a)$ and $Z = Y Y^T$. By Lemma 5 and [7, Theorem 8.1.18], we obtain the following inequalities

$$\rho(Z^{-1}) \leq \rho(|Z^{-1}|) \leq \rho(|Z_0^{-1}|) = \rho(Z_0^{-1}). \quad (8)$$

Since Z^{-1} and Z_0^{-1} are positive definite, we have

$$\rho(Z^{-1}) = \lambda_1(Z^{-1}) \quad \text{and} \quad \rho(Z_0^{-1}) = \lambda_1(Z_0^{-1}),$$

where $\lambda_1(\cdot)$ is the largest eigenvalue. Hence, by (8),

$$\lambda_n(Z) = \frac{1}{\lambda_1(Z^{-1})} \geq \frac{1}{\lambda_1(Z_0^{-1})} = \lambda_n(Z_0),$$

and the claim follows.

To show that the second claim, let $Y \in K_n(a)$ satisfy $c_n(a) = \lambda_n(Z)$, $Z = Y Y^T$. Then $\lambda_n(Z) = \lambda_n(Z_0)$ and further $\rho(Z^{-1}) = \rho(Z_0^{-1})$, which implies by (8) that $\rho(|Z^{-1}|) = \rho(|Z_0^{-1}|)$. Now, by the same reasoning as in the proof of [11, Theorem 3.1], we have

$$|Z^{-1}| = |Z_0^{-1}|. \quad (9)$$

Let $Y^{-1} = (b_{ij})$ and $Y_0^{-1} = (c_{ij})$. Since Y^{-1} is a lower triangular matrix, inequalities (6) and (7), and equality (9) together imply that

$$\left| \sum_{k=j}^n b_{ki} b_{kj} \right| = \sum_{k=j}^n |b_{ki}| |b_{kj}|,$$

for $1 \leq i < j \leq n$. Here, all the summands are nonzero. Note that all the summands on the left-hand side of the last equality have the same argument. Furthermore, we have by Lemmas 2 and 3 that

$$b_{ij} = \varepsilon_{i,j} c_{ij},$$

where $\varepsilon_{i,j} = \pm 1$ and $\varepsilon_{i,i} = 1$ for all $1 \leq i, j \leq n$. Then,

$$\begin{aligned} \left| \sum_{k=j}^n \varepsilon_{k,i} \varepsilon_{k,j} c_{ki} c_{kj} \right| &= \sum_{k=j}^n |\varepsilon_{k,i} c_{ki}| |\varepsilon_{k,j} c_{kj}| \iff \left| \sum_{k=j}^n \varepsilon_{k,i} \varepsilon_{k,j} c_{ki} c_{kj} \right| = \sum_{k=j}^n |c_{ki}| |c_{kj}| \\ &\iff \varepsilon_{j,i} = \varepsilon_{j+1,i} \varepsilon_{j+1,j} = \varepsilon_{j+2,i} \varepsilon_{j+2,j} = \dots = \varepsilon_{n,i} \varepsilon_{n,j} \end{aligned}$$

for $1 \leq i < j \leq n$. This means that every $\varepsilon_{j,i}$ depends only $\varepsilon_{n,i}$ and $\varepsilon_{n,j}$. Therefore, setting $\varepsilon_i = \varepsilon_{n,i}$ for all $i = 1, 2, \dots, n$, for the sake of brevity, we have

$$b_{ij} = \varepsilon_i \varepsilon_j c_{ij}$$

or equivalently

$$Y^{-1} = EY_0^{-1}E^{-1},$$

where $E = \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$. It is clear that $E = E_1 E_2 \dots E_n$, where each E_i ($i = 1, 2, \dots, n$) is an elementary matrix obtained from the $n \times n$ identity matrix by multiplying its i th row by ε_i . So, $E = E^{-1}$. and therefore $Y = EY_0E$. This concludes the proof. $\square$

Theorem 2. We have

$$\frac{1}{n - \frac{an}{a+2} + \frac{(a+1)^{2n-1}}{(a+2)^2}} \leq c_n(a) \leq \frac{1}{1 - \frac{an}{a+2} + \frac{(a+1)^{2n-1}}{(a+2)^2}}, \quad (10)$$

and consequently

$$\lim_{n \rightarrow \infty} \frac{c_n(a)}{\frac{(a+2)^2}{(a+1)^{2n-1}}} = 1. \quad (11)$$

Proof. To prove the theorem, we use a similar but not the same method to those of Theorem 2.1 in [11].

First, let Z_0 be as in Lemma 4 and $H_n = I - Z_0^{-1}$. Now, add $a + 1$ multiple of the second column to the first column, and then add $a + 1$ multiple of the second row to the first row. Let $M_n = (m_{ij})$ be the matrix obtained by these two elementary operations from H_n . It is clear that $\det(H_n) = \det(M_n)$, $m_{11} = a(a+2)$, $m_{12} = m_{21} = a$ and $m_{1j} = m_{j1} = 0$ for $j = 3, 4, \dots, n$. By expanding $\det(M_n)$, one can easily obtain the following recurrence relation for $n \geq 3$,

$$\det(H_n) = a(a+2) \det(H_{n-1}) - a^2 \det(H_{n-2})$$

with the initial conditions $\det(H_1) = 0$ and $\det(H_2) = -a^2$.

Second, we show by induction on n that $\det(H_{n+1}) < \det(H_n)$ for $n > 1$. It is clear that $\det(H_2) < \det(H_1) = -a^2$. Suppose $\det(H_n) < \det(H_{n-1})$. Then,

$$\begin{aligned} \det(H_{n+1}) &= a(a+2) \det(H_n) - a^2 \det(H_{n-1}) \\ &< a(a+2) \det(H_n) - a^2 \det(H_n) \\ &= 2a \det(H_n) \\ &< \det(H_n) \end{aligned}$$

by the induction hypothesis and the assumption $a \geq 1$. Thus, we have $\det(H_{n+1}) < \det(H_n) < 0$ for all $n > 1$.

Next, we show that every eigenvalue of Z_0^{-1} , except its spectral radius $\rho(Z_0^{-1})$, lies in the interval $[0, 1]$. Let $\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \dots \geq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ matrix Z_0^{-1} . It is clear that $\lambda_n^{(n)} > 0$ as Z_0^{-1} is positive definite. We show again by induction on n that $\lambda_1^{(n)} > 1$ and $\lambda_2^{(n)} < 1$. For $n = 2$,

$$\lambda_1^{(n)} = \frac{a^2 + 2 + \sqrt{(a^2 + 2)^2 - 4}}{2} > 1 \quad \text{and} \quad \lambda_2^{(n)} = \frac{a^2 + 2 - \sqrt{(a^2 + 2)^2 - 4}}{2} < 1.$$

Assume $\lambda_1^{(n-1)} > 1$ and $\lambda_2^{(n-1)} < 1$. Then it follows from the Cauchy interlacing inequalities, see [7], that $\lambda_3^{(n)} \leq \lambda_2^{(n-1)} \leq 1$. Note that $\det(I - Z_0^{-1}) = \det(H_n) < 0$ from paragraph above. Moreover, the characteristic polynomial of Z_0^{-1} , evaluated at $x = 1$ is negative. So, we obtain $\lambda_2^{(n)} < 1$. Thus, it follows that

$$\rho(Z_0^{-1}) \leq \text{tr}(Z_0^{-1}) \leq \rho(Z_0^{-1}) + n - 1. \quad (12)$$

In addition to this, by (4), one can easily calculate the trace of Z_0^{-1} as

$$\begin{aligned} \text{tr}(Z_0^{-1}) &= \sum_{i=1}^n \left(1 + \frac{a^2}{(a+1)^2 - 1} ((a+1)^{2(n-i)} - 1) \right) \\ &= n - \frac{an}{a+2} + \frac{(a+1)^{2n} - 1}{(a+2)^2}. \end{aligned}$$

So, it is obvious that $\text{tr}(Z_0^{-1}) - n + 1 > 0$. The first result given in (10) follows from the inequalities in (12) and the fact $c_n(a) = \rho(Z_0^{-1})^{-1}$. For the second result, we restate the inequalities in (10) as

$$\frac{1}{2(a+2) \frac{n}{(a+1)^{2n-1}} + 1} \leq \frac{c_n(a)}{\frac{(a+2)^2}{(a+1)^{2n-1}}} \leq \frac{1}{\frac{a+2}{a} \cdot \frac{-a^2 n + (a+1)^2 - 1}{(a+1)^{2n-1}} + 1}.$$

Thus, the claim in (11) follows. $\square$

4 Concluding Remarks

In this paper, introducing a new constant $c_n(a)$, we have expanded the results on the finding all minimizing matrices of the constant c_n and its asymptotic behaviour to a larger class of matrices. We do not reckon that our constant $c_n(a)$ could be used in eigenvalue estimation of GCD and related matrices as c_n was used in the literature, see [3, 6, 8, 9, 11, 12]. However, it seems possible that the techniques of this paper could be applied to a larger class of matrices than those considered in this present paper.

Throughout the paper, we assumed that $a \geq 1$. Almost all of our results are valid for $a > 0$. However, in this case, it seems not easy to prove that $\det(H_n) < 0$ for $n > 1$, and the lower and upper bounds for $c_n(a)$ given in (10) for small values of n do not provide good estimates. For example, when we compute the three terms in the inequalities in (10) for $a = 0.1$ and $n = 5$, we find the numbers

$$0.195186787128691, \quad 0.657178123193002, \quad 0.890236025695808,$$

respectively. Also

$$\frac{(a+2)^2}{(a+1)^{2n}-1} \approx 2.76707191431876.$$

On the other hand, when $a \geq 1$, we have better lower and upper bounds for $c_n(a)$ even for small a and n . Indeed, computing the three terms in (10) for $a = 3$ and $n = 5$, we find the numbers

$$0.0000238407438312075, \quad 0.00002384095983146484, \quad 0.0000238430175723040,$$

respectively. Moreover,

$$\frac{(a+2)^2}{(a+1)^{2n}-1} \approx 0.0000238418806475455.$$

In addition to this, we believe that the results concerning $c_n(a)$ could be generalized to the interval $[a, b]$, where $a \leq 0$ and $b > 0$. Let $K_n[a, b]$ denote the set of all $n \times n$ unit lower triangular matrices with each element in the interval $[a, b]$. Let $c_n(a, b) = \min \left\{ \lambda_n(Y Y^T) : Y \in K_n[a, b] \right\}$, i.e., $\sqrt{c_n(a, b)}$ is the smallest singular value in $K_n[a, b]$. We end the paper by raising the following open problems: Can one determine such a matrix Y in $K_n[a, b]$? Is it possible to determine the asymptotic behavior of $c_n(a, b)$?

Acknowledgement: The author would like to thank the anonymous referees for their valuable remarks and corrections which helped to improve the manuscript.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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