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Perturbation analysis for the Takagi vector matrix

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Abstract: In this article, we present some perturbation bounds for the Takagi vector matrix when the original matrix undergoes the additive or multiplicative perturbation. Two numerical examples are given to illuminate these bounds.

Keywords: Takagi vector matrix; perturbation bounds; additive perturbation; multiplicative perturbation

MSC: 15A23, 15A45

1 Introduction

Suppose that A is an $n \times n$ complex symmetric matrix. Then there exists a unitary matrix V of order n such that

$$V^H A \bar{V} = \Sigma = \text{diag}(\sigma_1, \dots, \sigma_n) \quad \text{or} \quad A = V \Sigma V^T. \quad (1.1)$$

This factorization is referred to as the Takagi factorization of A , V is called the Takagi vector matrix, and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$ are called the Takagi values.

Takagi factorization was initially demonstrated by Autonne [1] and Takagi [34] and rediscovered with various proofs by a few different scholars [4, 13, 16, 26, 33]. It is also named the symmetric SVD (SSVD) [13, 36] since it can be regarded as a symmetric form of SVD. However, a pair of left-right singular vectors are unique up to complex scaling factors with unit modulus, while Takagi vectors are unique up to a sign change. Therefore, there exists a difference between the singular vector matrix and the Takagi vector matrix. For example, if $\mathbf{u}_j$ is a Takagi vector, then $(\mathbf{u}_j, \bar{\mathbf{u}}_j)$ is a pair of left-right singular vectors, but a left singular vector is not necessarily a Takagi vector; see example in [36] for detail explanation. Also, not every complex symmetric matrix is diagnoseable by a similarity transformation. For example, consider the complex symmetric matrix

$$A = \begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix}, \quad i = \sqrt{-1},$$

whose Jordan normal is as follows

$$Z^{-1} A Z = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \quad \text{where} \quad Z = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}.$$

Thus, the complex symmetric matrix can only be diagnoseable by the Takagi factorization. Therefore, the Takagi factorization of complex symmetric matrices have many important applications; see e.g. [2, 3, 7, 9, 14,

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15, 17, 28, 35]. Some authors consider its algorithms and error analysis [8, 29, 36, 37]. To our best knowledge, there is no work on perturbation analysis of the Takagi vector matrix so far.

As we know, the perturbation analysis for matrix factorizations plays an important role in the algorithms of the factorizations themselves and the problems involved with the factorizations. Many work of this topic on some matrix factorizations have been done; see e.g., [5, 6, 10–12, 18–21, 27, 31, 32].

In this article, we consider the perturbation analysis for Takagi factorization under additive or multiplicative perturbation. Some perturbation bounds for Takagi vector matrix are derived. Among them, to get the additive perturbation bounds, we use the splitting operators and Lyapunov majorants technique, which has been applied to some matrix factorizations by some authors; see e.g., [5, 6, 18, 19, 31, 32].

We mainly utilize the following notations from [6]. For example, the Frobenius norm and spectral norm of a matrix A are denoted by $\|A\|_F$ and $\|A\|_2$, respectively, $\mathbb{C}^{n \times n}$, $\mathbb{C}_D^{n \times n}$ and $\mathbb{C}_N^{n \times n}$ denote the collection of complex $n \times n$ matrices, complex $n \times n$ diagonal matrices and complex matrices $n \times n$ whose diagonal entries are zeros, respectively, and for $Y = (y_{ij}) \in \mathbb{C}^{n \times n}$, we define Y_N and Y_D by

$$(Y_N)_{ij} = \begin{cases} x_{ij} & i \neq j, \\ 0 & i = j, \end{cases} \quad (Y_D)_{ij} = \begin{cases} 0 & i \neq j, \\ x_{ij} & i = j. \end{cases}$$

2 Additive Perturbation

In this section, we will examine the perturbation bound of Takagi vector matrix V of A in the factorization (1.1) when A undergoes an additive perturbation. Firstly, we define a linear operator $\mathbf{L} : \mathbb{C}_N^{n \times n} \rightarrow \mathbb{C}_N^{n \times n}$ by

$$\mathbf{L}\left(\frac{1}{\omega}X\right) = \frac{1}{\nu}(\Sigma\bar{X} - X\Sigma), \quad (2.1)$$

where Σ is a diagonal matrix as defined in (1.1) and ω, ν are positive parameters. In addition, for $G = (g_{ij}) \in \mathbb{C}_N^{n \times n}$, we define the row vectors k_i ($i = 1, 2, \dots, n$) with respect to G by

$$k_i = (\bar{g}_{i1}, g_{i1}, \dots, \bar{g}_{i,i-1}, g_{i,i-1}, \bar{g}_{i,i+1}, g_{i,i+1}, \dots, \bar{g}_{in}, g_{in}), \quad i = 1, 2, \dots, n. \quad (2.2)$$

The following lemma will decide when the operator $\mathbf{L}$ is nonsingular, whose proof is similar to the one in [6].

Lemma 2.1. *Let $\mathbf{L}$ be the linear operator defined by (2.1). Then $\mathbf{L}$ is nonsingular if and only if all the Takagi values of A are simple i.e. $\sigma_i \neq \sigma_j$, for all $i \neq j$, $i, j = 1, \dots, n$.*

Proof: Define

$$d_{ij} = \frac{\omega}{\nu} \begin{bmatrix} \sigma_i & -\sigma_j \\ -\sigma_j & \sigma_i \end{bmatrix}, \quad i \neq j, \quad i, j = 1, \dots, n \quad (2.3)$$

and

$$L = \text{diag}(d_{12}, \dots, d_{1n}, \dots, d_{i1}, \dots, d_{i,i-1}, d_{i,i+1}, \dots, d_{in}, \dots, d_{n1}, \dots, d_{n,n-1}), \quad (2.4)$$

and define a column vector $\mathbf{cvec}(T)$ of $T = (t_{ij}) \in \mathbb{C}^{n \times n}$ by

$$\mathbf{cvec}(T) = (k_1, k_2, \dots, k_n)^T,$$

where k_i are defined by (2.2). Thus, we can write (2.1) as

$$\mathbf{cvec}\left(\frac{1}{\nu}(\Sigma\bar{X} - X\Sigma)\right) = L\mathbf{cvec}\left(\frac{1}{\omega}X\right). \quad (2.5)$$

Hence, the operator $\mathbf{L}$ is nonsingular if and only if L is a nonsingular matrix. Clearly, L is nonsingular if and only if $\sigma_i \neq \sigma_j$, $i \neq j$, $i, j = 1, 2, \dots, n$. $\square$

Now, we will define the following function

$$\Theta(\Sigma) = \min \left\{ \left\| \frac{1}{v}(\Sigma\bar{X} - X\Sigma) \right\|_F : X \in \mathbb{C}_N^{n \times n}, \left\| \frac{1}{\omega}X \right\|_F = 1 \right\}.$$

When the operator $\mathbf{L}$ is nonsingular, from (2.3)-(2.5), we have

$$\Theta(\Sigma) = \|\mathbf{L}^{-1}\|^{-1} = \frac{\omega}{v} \min_{i \neq j} |\sigma_i - \sigma_j|, \quad (2.6)$$

where $\mathbf{L}^{-1}$ is defined as

$$\|\mathbf{L}^{-1}\| = \max\{\|\mathbf{L}^{-1}(X)\| : X \in \mathbb{C}_N^{n \times n}, \|X\|_F = 1\}.$$

Based on the above preparation, we can give the additive perturbation bound for Takagi vector matrix.

Theorem 2.2. Let $A \in \mathbb{C}^{n \times n}$ be a complex symmetric matrix with distinct Takagi values and have the Takagi factorization (1.1). Let $\tilde{A} = A + \Delta A$ be a complex symmetric matrix, and let

$$\epsilon = \frac{1}{\Theta(\Sigma)} \frac{\|\Delta A\|_F}{v}, \quad \beta = \frac{\omega^2}{v\Theta(\Sigma)} \left(1 + \sqrt{\frac{1}{2} \left(1 - \frac{1}{n} \right)} \right) \|A\|_2, \quad (2.7)$$

where $\Theta(\Sigma)$ is defined by (2.6) and v is a positive constant. If ϵ satisfies the following condition

$$\epsilon \leq \frac{1}{2\beta + \sqrt{\omega^2 + 4\beta^2}}, \quad (2.8)$$

then $\tilde{A}$ has a Takagi decomposition $\tilde{A} = \tilde{V}\tilde{\Sigma}\tilde{V}^T$. Moreover,

$$\frac{\|\tilde{V} - V\|_F}{\omega} \leq \frac{\sqrt{2}\epsilon}{\sqrt{1 - 2\beta\epsilon + \sqrt{1 - 4\beta\epsilon - \omega^2\epsilon^2}}} \equiv h_v(\epsilon), \quad (2.9)$$

where ω is a positive constant.

Proof: Observe that the Frobenius norm is unitary invariant norm. So, without loss of generality, we may accept that $V = I_n$ in (1.1). Now, write $\Delta\Sigma = \tilde{\Sigma} - \Sigma$ and $\Delta V = \tilde{V} - V = \tilde{V} - I_n$. Then by $\tilde{A}\tilde{V} - AV = \tilde{V}\tilde{\Sigma} - V\Sigma$, we can obtain

$$\Delta A\tilde{V} - A\Delta\bar{V} = \tilde{V}\Delta\Sigma - \Delta V\Sigma. \quad (2.10)$$

Setting

$$E = \tilde{V}^H \Delta A \tilde{V}$$

and noting that $\tilde{V}^H = I_n + \Delta V^H$, then from (2.10), we get

$$\Sigma\Delta\bar{V} - \Delta V\Sigma = \Delta V^H \Delta V\Sigma - \Delta V^H \Sigma\Delta\bar{V} + \Delta\Sigma - E. \quad (2.11)$$

By the definition (2.1) of the linear operator $\mathbf{L}$ and (2.11), we can get

$$\mathbf{L} \left(\frac{1}{\omega} \Delta V_N \right) = \frac{1}{v} (\Delta V^H \Delta V\Sigma - \Delta V^H \Sigma\Delta\bar{V})_N - \frac{1}{v} E_N. \quad (2.12)$$

Select the Takagi factorization of $\tilde{A}$, so that the diagonal elements of $\tilde{V}$ are real. Then by $\Delta V + \Delta V^H + \Delta V^H \Delta V = 0$, we get

$$\Delta V_D = -\frac{1}{2} \text{diag}(\Delta V^H \Delta V). \quad (2.13)$$

From Lemma 2.1, we know that $\mathbf{L}$ is nonsingular. Hence, (2.12) and (2.13) can be rewritten as a continuous mapping $\Psi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ expressed by

$$\frac{1}{\omega} \Delta V_N = \frac{1}{v} \mathbf{L}^{-1} ((\Delta V^H \Delta V\Sigma - \Delta V^H \Sigma\Delta\bar{V})_N) - \frac{1}{v} \mathbf{L}^{-1}(E_N), \quad \Delta V_D = -\frac{1}{2} \text{diag}(\Delta V^H \Delta V). \quad (2.14)$$

Noting that $(\Delta V^H \Delta V \Sigma - \Delta V^H \Sigma \overline{\Delta V})_N = (\Delta V^H \Delta V \Sigma)_N - (\Delta V^H \Sigma \overline{\Delta V})_N$ and a result from [18], i.e., $\|(X^H X)_N\|_F \leq \sqrt{1/2(1 - 1/n)} \|X\|_F^2$, we have

$$\begin{aligned} \|(\Delta V^H \Delta V \Sigma)_N - (\Delta V^H \Sigma \overline{\Delta V})_N\|_F &\leq \|(\Delta V^H \Delta V \Sigma)_N\|_F + \|(\Delta V^H \Sigma \overline{\Delta V})_N\|_F \\ &\leq \omega^2 \left(1 + \sqrt{\frac{1}{2} \left(1 - \frac{1}{n}\right)}\right) \|A\|_2 \left\| \frac{1}{\omega} \Delta V \right\|_F^2. \end{aligned}$$

Also, we have

$$\|\text{diag}(\Delta V^H \Delta V)\|_F \leq \|\Delta V\|_F^2, \quad \|E_N\|_F \leq \|\Delta A\|_F.$$

Hence the mapping Ψ expressed by (2.14) fulfills

$$\left\| \frac{1}{\omega} \Delta V_N \right\|_F \leq \epsilon + \beta \left\| \frac{1}{\omega} \Delta V \right\|_F^2, \quad \left\| \frac{1}{\omega} \Delta V_D \right\|_F \leq \frac{\omega}{2} \left\| \frac{1}{\omega} \Delta V \right\|_F^2,$$

where ϵ, β are defined by (2.7). Setting $z = (b_1, b_2) \in \mathbb{C}^2$ and considering the system

$$b_1 = \epsilon + \beta(b_1^2 + b_2^2), \quad b_2 = \frac{\beta}{2}(b_1^2 + b_2^2), \quad (2.15)$$

we get the equation

$$\left(\frac{4\beta^2}{\omega} + \omega\right)b_2^2 + 2(2\beta\epsilon - 1)b_2 + \omega\epsilon^2 = 0. \quad (2.16)$$

If ϵ satisfies (2.8), then

$$b_2^* = \frac{\omega\epsilon^2}{1 - 2\beta\epsilon + \sqrt{1 - 4\beta\epsilon - \omega^2\epsilon^2}} \quad (2.17)$$

is a solution to (2.16). By $b_1 = \epsilon + \frac{2\beta}{\omega}b_2$ and (2.17), we get a solution $b^* = (b_1^*, b_2^*)^T$ to the system (2.15). Let $\Omega_{b^*} = \{X : X \in \mathbb{C}^{n \times n}, \|X_N\|_F \leq \omega b_1^*, \|X_D\|_F \leq \omega b_2^*\}$. Then, Ω_{b^*} is a bounded closed convex set of $\mathbb{C}^{n \times n}$, and the relation shows that the continuous mapping Ψ maps Ω_{b^*} to Ω_{b^*} . By the Brouwer fixed point theorem, the mapping Ψ has a fixed point $\Delta V^* \in \Omega_{b^*}$ such that

$$\left\| \frac{1}{\omega} \Delta V \right\|_F \leq \|b^*\|_2 = \sqrt{\frac{2}{\omega} b_2^*},$$

which gives the result (2.9). $\square$

Remark 2.3. Plugging $\omega = \nu = 1$ or $\nu = \|A\|_F$ and $\omega = \|V\|_F = \sqrt{n}$, the bound in (2.9) is known as the absolute or relative perturbation bound.

Remark 2.4. For small ϵ , the upper bound $h_\nu(\epsilon)$ defined by (2.9) has the Taylor expansion

$$h_\nu(\epsilon) = \epsilon + \left(\frac{\omega}{8} + 2\beta^2\right)\epsilon^3 + O(\epsilon^4), \quad \epsilon \rightarrow 0. \quad (2.18)$$

By (2.18) and (2.7), we see that the quantities $\frac{1}{\theta(\Sigma)} = \frac{1}{\min_{i \neq j} |\sigma_i - \sigma_j|}$ for $\nu = \omega = 1$ and $\frac{1}{\theta(\Sigma)} = \frac{\|A\|_F}{\sqrt{n} \min_{i \neq j} |\sigma_i - \sigma_j|}$ for $\nu = \|A\|_F, \omega = \|V\|_F$ can be regarded as the absolute and relative condition numbers of the Takagi vector matrix V .

Remark 2.5. The method and perturbation bounds in this section are similar to the ones for spectral decomposition appearing in [6]. However, the linear operator (2.1) is different from the one in [6], and the latter can't be applied to Takagi factorization for deriving the bound (2.9).

3 Multiplicative perturbation

In this section, we present the perturbation bounds for Takagi vector matrix when A undergoes a multiplicative perturbation. Let A and $\tilde{A} = D^T A D$ be two $n \times n$ complex symmetric matrices with the following Takagi factorizations

$$A = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} \tilde{V}_1 & \tilde{V}_2 \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix}, \quad (3.1)$$

where $V = \begin{pmatrix} V_1 & V_2 \end{pmatrix}$ and $\tilde{V} = \begin{pmatrix} \tilde{V}_1 & \tilde{V}_2 \end{pmatrix}$ are unitary, $V_1, \tilde{V}_1 \in \mathbb{C}^{n \times r}$ and

$$\begin{aligned} \Sigma_1 &= \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r), & \Sigma_2 &= \text{diag}(\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n), \\ \tilde{\Sigma}_1 &= \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_r), & \tilde{\Sigma}_2 &= \text{diag}(\tilde{\sigma}_{r+1}, \tilde{\sigma}_{r+2}, \dots, \tilde{\sigma}_n). \end{aligned}$$

The relative gaps were given by [24, 25]

$$Y_{ij} = \min_{\sigma \in \sigma(\Sigma_i), \tilde{\sigma} \in \sigma(\tilde{\Sigma}_j)} \frac{|\sigma - \tilde{\sigma}|}{\sqrt{|\sigma|^2 + |\tilde{\sigma}|^2}}, \quad i, j = 1, 2.$$

The canonical angle $\theta(V_1, \tilde{V}_1)$ between the column spaces $\mathfrak{R}(V_1)$ of V_1 and $\mathfrak{R}(\tilde{V}_1)$ of $\tilde{V}_1$ is defined by $\theta(V_1, \tilde{V}_1) = \arccos(V_1^H \tilde{V}_1 \tilde{V}_1^H V_1)^{1/2}$ [24, 25]. Before going to main results, we will give two lemmas from [24, 25] and [22, 23], respectively.

Lemma 3.1. Let $\Omega \in \mathbb{C}^{s \times s}$ and $\Gamma \in \mathbb{C}^{t \times t}$ be two Hermitian matrices and let $E, F \in \mathbb{C}^{s \times t}$. If $\Sigma(\Omega) \cap \Sigma(\Gamma) = \emptyset$, then $\Omega X - X\Gamma = \Omega E + F\Gamma$ has unique solution $X \in \mathbb{C}^{s \times t}$, and moreover,

$$\|X\|_F \leq \frac{\sqrt{(\|E\|_F^2 + \|F\|_F^2)}}{Y},$$

where $Y = \min_{\sigma \in \Sigma(\Omega), \tilde{\sigma} \in \Sigma(\Gamma)} \frac{|\sigma - \tilde{\sigma}|}{\sqrt{|\sigma|^2 + |\tilde{\sigma}|^2}}.$

Lemma 3.2. Let $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$ and $\tilde{\Sigma} = \text{diag}(\tilde{\sigma}_1, \dots, \tilde{\sigma}_r)$ satisfy $\Sigma X - X\tilde{\Sigma} = \Sigma E + F\tilde{\Sigma}$. Then there is a permutation ς of $\{1, 2, \dots, r\}$ such that

$$\sigma_{\min}(X) \sum_{i=1}^r \frac{|\sigma_i - \tilde{\sigma}_{\varsigma_i}|}{|\tilde{\sigma}_{\varsigma_i}|^2 + |\sigma_i|^2} \leq \|E\|_F^2 + \|F\|_F^2.$$

Theorem 3.3. Let A and $\tilde{A}$ be two $n \times n$ complex symmetric matrices with Takagi factorizations (3.1), Let D be a nonsingular matrix of order n . Then

$$Y_{12}^2 \|\sin \theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2 + Y_{11}^2 \|\cos \theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2 \leq \|(I_n - D^{-1})V_1\|_F^2 + \|(I_n - D^T)V_1\|_F^2 \quad (3.2)$$

and there is a permutation ς of $\{1, 2, \dots, r\}$ such that

$$\begin{aligned} & Y_{12}^2 \|\sin \theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2 + (1 - Y_{12}^2 \|\sin \theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2) \sum_{i=1}^r \frac{|\sigma_i - \tilde{\sigma}_{\varsigma_i}|}{|\tilde{\sigma}_{\varsigma_i}|^2 + |\sigma_i|^2} \\ & \leq \|(I_n - D^{-1})V_1\|_F^2 + \|(I_n - D^T)V_1\|_F^2. \end{aligned} \quad (3.3)$$

Proof: Simply, we have

$$\Delta A = \tilde{A} - A = \tilde{A}(I_n - D^{-1}) + (D^T - I_n)A. \quad (3.4)$$

Pre-multiplying and post-multiplying (3.4) by $\tilde{V}^H$ and $\bar{V}_1$, respectively, gives

$$\tilde{\Sigma} \tilde{V}^T \bar{V}_1 - \tilde{V}^T \bar{V}_1 \Sigma_1 = \tilde{\Sigma} \tilde{V}^T (I_n - D^{-1}) \bar{V}_1 + \tilde{V}^H (D^T - I_n) V_1 \Sigma_1.$$

The above equation can be written in the 2×2 block form

$$\begin{bmatrix} \tilde{\Sigma}_1 \tilde{V}_1^T \bar{V}_1 - \tilde{V}_1^T \bar{V}_1 \Sigma_1 \\ \tilde{\Sigma}_2 \tilde{V}_2^T \bar{V}_1 - \tilde{V}_2^T \bar{V}_1 \Sigma_1 \end{bmatrix} = \begin{bmatrix} \tilde{\Sigma}_1 \tilde{V}_1^T (I_n - D^{-1}) \bar{V}_1 + \tilde{V}_1^H (D^T - I_n) V_1 \Sigma_1 \\ \tilde{\Sigma}_2 \tilde{V}_2^T (I_n - D^{-1}) \bar{V}_1 + \tilde{V}_2^H (D^T - I_n) V_1 \Sigma_1 \end{bmatrix},$$

so,

$$\tilde{\Sigma}_1 \tilde{V}_1^T \bar{V}_1 - \tilde{V}_1^T \bar{V}_1 \Sigma_1 = \tilde{\Sigma}_1 \tilde{V}_1^T (I_n - D^{-1}) \bar{V}_1 + \tilde{V}_1^H (D^T - I_n) V_1 \Sigma_1 \quad (3.5)$$

and

$$\tilde{\Sigma}_2 \tilde{V}_2^T \bar{V}_1 - \tilde{V}_2^T \bar{V}_1 \Sigma_1 = \tilde{\Sigma}_2 \tilde{V}_2^T (I_n - D^{-1}) \bar{V}_1 + \tilde{V}_2^H (D^T - I_n) V_1 \Sigma_1. \quad (3.6)$$

To make use of Lemma 3.1 to (3.5) and (3.6) give

$$Y_{11} \|\tilde{V}_1^T \bar{V}_1\|_F^2 \leq \|\tilde{V}_1^T (I_n - D^{-1}) \bar{V}_1\|_F^2 + \|\tilde{V}_1^H (I_n - D^T) V_1\|_F^2 \quad (3.7)$$

and

$$Y_{12} \|\tilde{V}_2^T \bar{V}_1\|_F^2 \leq \|\tilde{V}_2^T (I_n - D^{-1}) \bar{V}_1\|_F^2 + \|\tilde{V}_2^H (I_n - D^T) V_1\|_F^2. \quad (3.8)$$

Observe that

$$\|\tilde{V}_1^T (I_n - D^{-1}) \bar{V}_1\|_F^2 + \|\tilde{V}_2^T (I_n - D^{-1}) \bar{V}_1\|_F^2 = \|(I_n - D^{-1}) V_1\|_F^2, \quad (3.9)$$

and

$$\|\tilde{V}_1^H (I_n - D^T) V_1\|_F^2 + \|\tilde{V}_2^H (I_n - D^T) V_1\|_F^2 = \|(I_n - D^T) V_1\|_F^2. \quad (3.10)$$

By uniting from (3.5) to (3.10), we have

$$Y_{11} \|\tilde{V}_1^T \bar{V}_1\|_F^2 + Y_{12} \|\tilde{V}_2^T \bar{V}_1\|_F^2 \leq \|(I_n - D^{-1}) V_1\|_F^2 + \|(I_n - D^T) V_1\|_F^2. \quad (3.11)$$

From [30], we have

$$\|\sin \Theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F = \|V_1^T \tilde{\bar{V}}_2\|_F = \|V_2^T \tilde{\bar{V}}_1\|_F, \quad (3.12)$$

and by definition of $\cos \Theta$, we have

$$\|\cos \Theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F = \|V_1^T \tilde{\bar{V}}_1\|_F. \quad (3.13)$$

Then the required bound (3.2) follows from (3.11)-(3.13). In order to prove (3.3), to make use of Lemma 3.2 to (3.6) gives

$$\sigma_{\min}^2(\tilde{V}_1^T \bar{V}_1) \sum_{i=1}^r \frac{|\sigma_i - \tilde{\sigma}_{\zeta_i}|}{|\tilde{\sigma}_{\zeta_i}|^2 + |\sigma_i|^2} \leq \|\tilde{V}_1^T (I_n - D^{-1}) \bar{V}_1\|_F^2 + \|\tilde{V}_1^H (I_n - D^T) V_1\|_F^2. \quad (3.14)$$

By C-S decomposition theorem [30], we have

$$\sigma_{\min}^2(\tilde{V}_1^T \bar{V}_1) = 1 - \|\sin \Theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_2^2. \quad (3.15)$$

Then the required bound (3.3) is obtained from (3.7), (3.14) and (3.15). $\square$

By C-S decomposition theorem [30] we have

$$\|\sin \Theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2 = r - \|\cos \Theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2. \quad (3.16)$$

To make use of (3.16) to the bound (3.2) gives the following corollary.

Corollary 3.4. *Under the same assumption as in Theorem 3.3, we have*

$$(Y_{12}^2 - Y_{11}^2) \|\sin \Theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2 + r Y_{11}^2 \leq \|(I_n - D^{-1}) V_1\|_F^2 + \|(I_n - D^T) V_1\|_F^2. \quad (3.17)$$

We can also give a variation of (3.17).

Theorem 3.5. *Under the same assumption as in Theorem 3.3, we have*

$$(Y_{12}^2 + Y_{21}^2 - Y_{11}^2 - Y_{22}^2) \|\sin \Theta(\bar{V}_1, \tilde{\bar{V}}_1)\|_F^2 + r Y_{11}^2 + (n - r) Y_{22}^2 \leq \|(I_n - D)^{-1}\|_F^2 + \|(I_n - D^T)\|_F^2. \quad (3.18)$$

Proof: Pre-multiplying and post-multiplying (3.4) by $\tilde{V}^H$ and $\bar{V}_2$, respectively, gives

$$\tilde{\Sigma} \tilde{V}^T \bar{V}_2 - \tilde{V}^T \bar{V}_2 \Sigma_2 = \tilde{\Sigma} \tilde{V}^T (I_n - D^{-1}) \bar{V}_2 + \tilde{V}^H (D^T - I_n) V_2 \Sigma_2.$$

By a similar discussion to the proof of Theorem 3.3 we have

$$\tilde{\Sigma}_1 \tilde{V}_1^T \bar{V}_2 - \tilde{V}_1^T \bar{V}_2 \Sigma_2 = \tilde{\Sigma}_1 \tilde{V}_1^T (I_n - D^{-1}) \bar{V}_2 + \tilde{V}_2^H (D^T - I_n) V_2 \Sigma_2 \quad (3.19)$$

and

$$\tilde{\Sigma}_2 \tilde{V}_2^T \bar{V}_2 - \tilde{V}_2^T \bar{V}_2 \Sigma_2 = \tilde{\Sigma}_2 \tilde{V}_2^T (I_n - D^{-1}) \bar{V}_2 + \tilde{V}_2^H (D^T - I_n) V_2 \Sigma_2. \quad (3.20)$$

To make use of Lemma 3.2 to (3.19) and (3.20), respectively, we have

$$Y_{21} \|\tilde{V}_1^T \bar{V}_2\|_F^2 \leq \|\tilde{V}_1^T (I_n - D^{-1}) \bar{V}_2\|_F^2 + \|\tilde{V}_1^H (I_n - D^T) V_2\|_F^2 \quad (3.21)$$

and

$$Y_{22} \|\tilde{V}_2^T \bar{V}_2\|_F^2 \leq \|\tilde{V}_2^T (I_n - D^{-1}) \bar{V}_2\|_F^2 + \|\tilde{V}_2^H (I_n - D^T) V_2\|_F^2. \quad (3.22)$$

Note that $\|\tilde{V}_1^T \bar{V}_2\|_F^2 = \|\tilde{V}_2^T \bar{V}_1\|_F^2$,

$$\|\tilde{V}_1^T (I_n - D^{-1}) \bar{V}_2\|_F^2 + \|\tilde{V}_1^T (I_n - D^{-1}) \bar{V}_1\|_F^2 + \|\tilde{V}_2^T (I_n - D^{-1}) \bar{V}_1\|_F^2 + \|\tilde{V}_2^T (I_n - D^{-1}) \bar{V}_2\|_F^2 = \|(I_n - D^{-1})\|_F^2,$$

and

$$\|\tilde{V}_1^T (I_n - D^T) \bar{V}_2\|_F^2 + \|\tilde{V}_1^T (I_n - D^T) \bar{V}_1\|_F^2 + \|\tilde{V}_2^T (I_n - D^T) \bar{V}_1\|_F^2 + \|\tilde{V}_2^T (I_n - D^T) \bar{V}_2\|_F^2 = \|(I_n - D^T)\|_F^2,$$

which together with (3.7), (3.8), (3.21) and (3.22) lead to

$$(Y_{12}^2 + Y_{21}^2) \|\tilde{V}_1^T \bar{V}_2\|_F^2 + Y_{11}^2 \|\tilde{V}_1^T \bar{V}_1\|_F^2 + Y_{22}^2 \|\tilde{V}_2^T \bar{V}_2\|_F^2 \leq \|(I_n - D^{-1})\|_F^2 + \|(I_n - D^T)\|_F^2. \quad (3.23)$$

Simply,

$$r = \|\bar{V}_1\|_F^2 = \|\tilde{V}^T \bar{V}_1\|_F^2 = \|\tilde{V}_1^T \bar{V}_1\|_F^2 + \|\tilde{V}_2^T \bar{V}_1\|_F^2.$$

So, $\|\tilde{V}_1^T \bar{V}_1\|_F^2 = r - \|\tilde{V}_2^T \bar{V}_1\|_F^2$. Also, $\|\tilde{V}_2^T \bar{V}_2\|_F^2 = n - r - \|\tilde{V}_2^T \bar{V}_1\|_F^2$, which combining with (3.23) leads to the required bound (3.18). $\square$

Using $\|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F^2 \leq \min\{r, n - r\}$, by Theorem 3.2, it is easy to prove the following corollary.

Corollary 3.6. *Under the same assumption as in Theorem 3.3 we have*

$$\begin{aligned} \|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F &\leq \frac{1}{\sqrt{Y_{12}^2 + Y_{21}^2}} \sqrt{\|(I_n - D^{-1})\|_F^2 + \|(I_n - D^T)\|_F^2 - \min\{Y_{11}^2, Y_{22}^2\} |n - 2r|} \\ &\leq \frac{1}{\sqrt{Y_{12}^2 + Y_{21}^2}} \sqrt{\|(I_n - D^{-1})\|_F^2 + \|(I_n - D^T)\|_F^2}. \end{aligned}$$

Remark 3.7. *The results obtained in this section are very analogous to the ones given in [25] for spectral decomposition. The main differences are that our bounds are for Takagi factorization and they consist of Takagi values and Takagi vector matrix.*

4 Numerical Examples

In this section, we will give two simple examples for our theoretical results got in the above sections. To get the Takagi factorization, we will use divide and conquer method from [36]. All calculations are done in MATLAB R2016a, with accuracy 2.22×10^{-16} .

Example 4.1. Let the test matrix A and its perturbed matrix ΔA be

$$A = \begin{bmatrix} 2+i & 1-i & 0 & 0 \\ 1-i & 2i & 3i & 0 \\ 0 & 3i & 1-3i & 1-i \\ 0 & 0 & 1-i & 4+5i \end{bmatrix}, \quad \Delta A = 10^{-7} \times \begin{bmatrix} 8+i & 4-5i & 0 & 0 \\ 4-5i & 2+4i & 4i & 0 \\ 0 & 4i & 3+5i & 3-5i \\ 0 & 0 & 3-5i & 9 \end{bmatrix}.$$

By algorithm from [36], we have the Takagi factorization of A as follows:

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix},$$

where

$$V_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.201306588699844 - 0.238135233820210i & -0.802886932540193 + 0.508074641480586i \\ 0.861440272774566 + 0.400858982937276i & 0.230197143347006 + 0.210337841187582i \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.754344479484571 + 0.312459714103782i & 0.561905566057725 - 0.132647910525122i \\ -0.220942382690395 - 0.533402096794177i & 0.694553476582847 + 0.429257655532603i \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\Sigma = \text{diag}(6.836803449339018, 2.873694241756794, 2.828427124746190, 2.236067977499789)$, and the Takagi factorization of $A + \Delta A$ as follows:

$$\tilde{V} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix},$$

where

$$\tilde{V}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.201306585104707 - 0.238135253628679i & -0.802886988999223 + 0.508074544401161i \\ 0.861440292633261 + 0.400858930299187i & 0.230197129899046 + 0.210337874890748i \end{bmatrix},$$

$$\tilde{V}_2 = \begin{bmatrix} 0.754344625915864 + 0.312459656921694i & 0.561905366641230 - 0.132648057234239i \\ -0.220942205747094 - 0.533401996497803i & 0.694553563032545 + 0.429257731357937i \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\tilde{\Sigma} = \text{diag}(6.836804205565012, 2.873693834556914, 2.828428067555274, 2.236068707948664)$. Simple computation gives

$$\|\tilde{V} - V\|_F = 3.976090651187170e - 07, \quad \frac{\|\tilde{V} - V\|_F}{\|V\|_F} = 1.988045325593585e - 7.$$

Taking $\nu = \omega = 1$ in Theorem 2.1, we acquire the following absolute perturbation bound

$$h_v(\epsilon) = 2.182476863812960e - 05.$$

Taking $\nu = \|A\|_F$, $\omega = \|V\|_F$ in Theorem 2.2, we acquire the following relative perturbation bound

$$h_v(\epsilon) = 3.128265760674018e - 04.$$

The numerical outcome demonstrates that the upper bound (2.9) is correct.

Example 4.2. Let the Hankel test matrix A and its multiplicative perturbed matrix D be

$$A = \begin{bmatrix} 2+i & 0 & 1+i & 2+4i \\ 0 & 1+i & 2+4i & 1-3i \\ 1+i & 2+4i & 1-3i & -3-2i \\ 2+4i & 1-3i & -3-2i & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1.9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.1 \end{bmatrix}.$$

The Takagi factorization of A as follows:

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix},$$

where

$$V_1 = \begin{bmatrix} -0.081556436048372 - 0.226665344186142i & 0.351297055191155 + 0.028837892464748i \\ -0.166515153373528 - 0.320977546400311i & 0.774301156021931 - 0.143939188320727i \\ -0.700473459039316 + 0.327093203620493i & -0.098233908066737 + 0.285112467879928i \\ 0.239542770051662 + 0.395200858463957i & 0.287629919221060 - 0.286055928718713i \end{bmatrix},$$

$$V_2 = \begin{bmatrix} -0.620826081122612 - 0.076016675443944i & -0.137315904059124 - 0.638491118968067i \\ 0.448303096452921 + 0.094781960273007i & 0.185612153961115 + 0.067632760345430i \\ 0.039587768697195 + 0.324138939396012i & 0.390383715255341 - 0.228856419382055i \\ -0.502057749924142 + 0.200354505933439i & 0.463334875692298 + 0.339106854338816i \end{bmatrix}$$

and $\Sigma = \text{diag}(10.237473814812493, 6.698923725621110, 4.608093116935691, 0.289876930764161)$ and the Takagi factorization of $\tilde{A} = D^T A D$ as follows:

$$\tilde{V} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix},$$

where

$$\tilde{V}_1 = \begin{bmatrix} 0.680872090585938 + 0.277660455966341i & -0.444177577093778 + 0.173138639560008i \\ 0.197890858218465 + 0.099627730860347i & -0.021031436371808 + 0.313472887420342i \\ 0.363404477804955 - 0.143375241447721i & 0.538505686105767 - 0.283335320391104i \\ 0.154321579733052 - 0.483525559418203i & -0.530183654449579 - 0.150531504476704i \end{bmatrix},$$

$$\tilde{V}_2 = \begin{bmatrix} 0.188423955067938 + 0.025757394741940i & 0.084770944658876 + 0.434389244418428i \\ -0.888214736027497 + 0.039915499932574i & -0.230538429772975 - 0.092408435655071i \\ 0.032827894625369 - 0.384316633385813i & -0.493886542974962 + 0.290538440154562i \\ 0.119287113419004 + 0.101528048100126i & -0.526519598681413 - 0.369963774386065i \end{bmatrix}$$

and $\tilde{\Sigma} = \text{diag}(13.695728700551077, 10.788788076580641, 5.886297772715807, 0.460071440599145)$. Like the additive perturbation bounds, the following numerical calculations for the perturbation bounds in Theorem 3.3, Corollary 3.4, Theorem 3.5 and Corollary 3.6 are also correct.

$$Y_{12}^2 \|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F^2 + Y_{11}^2 \|\cos \Theta(\bar{V}_1, \tilde{V}_1)\|_F^2 = 0.012418478920268,$$

$$Y_{12}^2 \|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F^2 + (1 - Y_{12}^2 \|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F^2) \sum_{i=1}^r \frac{|\sigma_i - \tilde{\sigma}_{\zeta_i}|}{|\tilde{\sigma}_{\zeta_i}|^2 + |\sigma_i|^2} = 0.021540277595979,$$

$$(Y_{12}^2 - Y_{11}^2) \|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F^2 + r Y_{11}^2 = 0.012418478920268,$$

$$\|(I_n - D^{-1})V_1\|_F^2 + \|(I_n - D^T)V_1\|_F^2 = 0.195441939258419,$$

$$(Y_{12}^2 + Y_{21}^2 - Y_{11}^2 - Y_{22}^2) \|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F^2 + r Y_{11}^2 + (n - r) Y_{22}^2 = 0.012418478920268,$$

$$\|(I_n - D^{-1})\|_F^2 + \|(I_n - D^T)\|_F^2 = 1.052641194111857,$$

$$\|\sin \Theta(\bar{V}_1, \tilde{V}_1)\|_F = 1.181297600561688,$$

$$\frac{1}{\sqrt{Y_{12}^2 + Y_{21}^2}} \sqrt{\|(I_n - D^{-1})\|_F^2 + \|(I_n - D^T)\|_F^2} = 11.258974118849668.$$

5 Conclusion

In this article we studied the additive and multiplicative perturbation for the Takagi vector matrix. Splitting operators and Lyapunov majorants method were used to obtain the additive perturbation for the Takagi vector matrix. Numerical results depicts the accuracy of our theoretical results. Similar to the generalized SVD we will try to generalize the Takagi factorization and study its perturbation bounds in future.

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